

# Statistical Multiplexing of Multiple Time-Scale Markov Streams

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**Abstract**—We study the problem of statistical multiplexing of cell streams that have correlations at multiple time-scales. Each stream is modeled by a *singularly perturbed* Markov-modulated process with some state transitions occurring much less frequently than others. One motivation of this model comes from variable-rate compressed video, where the fast time-scale dynamics may correspond to correlations between adjacent frames, while the slow time-scale dynamics may correspond to correlations within the same scene of a video sequence. We develop a set of large deviations results to estimate the buffer overflow probabilities in various asymptotic regimes in the buffer size, rare transition probabilities, and the number of streams. Using these results, we characterize the multiplexing gain in both the channel capacity and the buffering requirements and highlight the impact of the slow time-scale of the streams.

## I. INTRODUCTION

A KEY concept behind the emerging asynchronous transfer mode (ATM) broadband integrated service networks is the efficient sharing of link capacities through statistical multiplexing of variable-rate traffic streams. Buffering is required at the network nodes to absorb traffic fluctuations when the instantaneous rate of the aggregate incoming stream exceeds the capacity of the outgoing link. To be able to provide quality-of-service guarantees to users of the network, it is necessary to estimate the cell loss probabilities due to buffer overflows when these traffic streams interact. A better understanding of this problem is essential for dealing with higher-level network management issues such as call admissions, call routing, bandwidth and buffer allocation, and congestion control. The problem is particularly challenging because the traffic streams can belong to different classes of services with very different statistical characteristics.

In this paper, we will focus on the simple scenario when multiple streams are multiplexed onto a common fixed-capacity link at a buffered switch. While this is a well-studied problem in the literature, the novelty here is our model of the traffic streams as *singularly perturbed multiple time-scale* Markov processes. The key characteristic of this class of models is that some state transitions in the Markov chain occur much more infrequently than others, and this allows the

modeling of correlations in the traffic stream arrival rate at different time-scales.

The motivation for considering this model is three-fold. First, experimental studies of variable-rate compressed video traffic (e.g. [15], [20]) have demonstrated that statistical correlations in the bit-rate typically exist at several time-scales, such as intraframe correlations, correlations between adjacent frames, and long-range correlations associated with phenomena such as scene changes when the coder adapts to the characteristics of different scenes. Second, this model enables us to study situations when dynamics of *different* traffic streams occur at different time-scales. Because these networks will carry traffic from very different classes of services, this is expected to be a common phenomenon. Third, traffic management based on a separation between cell level and burst level congestion has been proposed for ATM networks (see [18]). This approach can be viewed as a specific example of a multiple time-scale decomposition of the traffic process, and our general results here provides a better understanding of when such a separation is valid.

Recently, the single-link statistical multiplexing problem has received a lot of attention. In particular, there has been a line of work suggesting that the nature of the interaction of the streams is such that, for a given cell loss probability  $p$ , it is possible to assign an *effective bandwidth* to a traffic stream depending only on  $p$  and the statistics of the stream, with the property that the loss probability requirement is *approximately* satisfied if and only if the sum of the effective bandwidths of the incoming streams is less than the capacity of the outgoing link (Kelly [13], Guerin *et al.* [9], Gibbens and Hunt [11], Elwalid and Mitra [8], Kesidis *et al.* [14], Whitt [24], Chang *et al.* [1], [2]).

While these works differ in the stochastic models for the traffic streams, they are all essentially based on *large deviations* estimates of the loss probability in the asymptotic regime of *large* buffers. In this sense, effective bandwidth is only an approximate notion for finite buffers. One difficulty of applying these results in practice is that there is little intuition regarding how large buffers have to be for the asymptotic estimates to be accurate. A thesis of this paper is that it is crucial to take into consideration the correlation time-scales of the streams, particularly in relation to the size of buffer. By explicitly incorporating this information in our multiple time-scale models and in our asymptotic estimates, we will show that while the effective bandwidth concept generalizes naturally to multiple time-scale streams in some parameter regimes, in other regimes it breaks down and has to be replaced

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by better estimates. Moreover, the correlation time scales have a major impact on the amount of statistical multiplexing gain that can be achieved. Numerical work demonstrating the inaccuracy of the effective bandwidth approximation in some regimes has also been reported in Choudhury *et al.* [10].

Our approach is based on a set of large deviations results for the cell loss probability in various *joint asymptotic* regimes in the buffer size, the correlation time-scale parameter, and the number of traffic streams sharing the link. The results not only yield estimates for the loss probability but, equally importantly, provides insights on the *typical* bursting behavior of the traffic streams that leads to the cell losses. They show that, depending on the parameter regimes, the dynamics at the fast time-scale or at the slow time-scale play the major role in the overflow behavior.

The paper is organized as followed. In Section II, we review the basic large deviation result underlying the effective bandwidth concept for single time-scale Markov-modulated processes. In Section III, we give a derivation of that result using martingale techniques to clarify the role of the correlation time-scale in the accuracy of the approximation. We introduce the multiple time-scale Markov models in Section IV, and present large deviations results for a single multiple time-scale stream. Using these results, we obtain an expression for the effective bandwidth of multiple time-scale streams in Section V. In Section VI, we obtain large deviations results for the loss probability when a large number of independent and statistically similar multiple time-scale streams are multiplexed together, and show that the effective bandwidth approximation can be overly conservative in this regime. Finally, in Section VII, we apply these large deviations results to quantify the statistical multiplexing gain in terms of both link capacity and buffer requirements. Due to space constraints, we only sketch proofs of the results. Refer to [21] for the full proofs.

## II. EFFECTIVE BANDWIDTH OF SINGLE TIME-SCALE MARKOV STREAM

Consider a time-slotted model with  $X_t$  being the number of cells<sup>1</sup> arriving at the multiplexer in time slot  $t$ . The multiplexer is served by a fixed-rate channel of capacity  $c$  cells per time slot. Cells that cannot be immediately transmitted on the outgoing channel are queued up in a buffer of size  $B$ . Excess cells arriving at a full buffer are considered lost.

We consider a Markov-modulated model for the arrival stream. Specifically let  $\{H_t\}$  be a discrete-time, finite-state, irreducible, stationary Markov chain with state space  $\mathcal{S}$ , and let  $(p_{ij})$  be its transition matrix. We shall call  $\{H_t\}$  the *source state* process. The arrival stream  $\{X_t\}$  is *modulated* by the Markov chain  $\{H_t\}$ , such that the distribution of  $X_t$  at time  $t$  depends only on the source state  $H_t$  at time  $t$ , and given a realization of the chain  $\{H_t\}$ , the  $X_t$ 's are independent. Since  $\{H_t\}$  is stationary, so is the cell arrival process  $\{X_t\}$ . The source state  $H_t$  can be thought of as modeling the burstiness of the stream at time  $t$ ; the Markov structure models the correlation in the cell arrival statistics over time. For stability, we assume that the average number of cells arriving per time

<sup>1</sup> $X_t$  could equally be the number of bits, bytes, etc.

slot is less than the channel capacity,  $E(X_1) < c$ . We will also assume there is a peak rate constraint  $P_0$  on the cell arrival rate, i.e.,  $X_1 < P_0$ .

We are interested in the regime where cell loss is a rare event (of the order  $10^{-4}$  to  $10^{-9}$ ). The *effective bandwidth* of the arrival stream is based on certain *large deviations* approximation of the cell loss probability  $p(B)$  when the buffer size  $B$  is large. Let  $g_i(r) \equiv E[\exp(rX_t)|H_t = i]$  be the generating function (which we assume to exist and be differentiable for all  $r$ ) of the conditional distribution of  $X_t$  given the source state  $H_t = i$ . Consider the matrix  $A(r)$  whose entries are  $a_{ij}(r) = p_{ij}g_i(r)$ ,  $i, j \in \mathcal{S}$ . Since the given chain is irreducible, the matrix  $A(r)$  is also irreducible for any  $r$ . By the Perron-Frobenius theorem, the matrix  $A(r)$  has a largest positive simple eigenvalue  $\rho(r)$  (the spectral radius of  $A(r)$ ) with a strictly positive right eigenvector  $\eta_r$ , unique up to scaling

$$A(r)\eta_r = \rho(r)\eta_r. \quad (1)$$

It can be shown (see for example [7]) that the *log spectral radius* function  $\Lambda(r) \equiv \log \rho(r)$  is convex and differentiable for all  $r \in \mathfrak{R}$ , and that  $\Lambda(0) = 0$  and  $\Lambda'(0) = E(X_1)$ . It follows that if  $E(X_1) < c$ , then the equation

$$\Lambda(r) - cr = 0$$

has a unique positive root  $r^* > 0$ .<sup>2</sup> The key result underlying the effective bandwidth concept is that in the asymptotic regime of large buffer sizes, the loss probability decays exponentially with  $B$ , with the exponent given by  $r^*$ ; i.e.

$$\lim_{B \rightarrow \infty} \frac{1}{B} \log p(B) = -r^*. \quad (2)$$

Less formally, this result means that for large buffer size  $B$ , the loss probability is approximately

$$p(B) \approx \exp(-r^*B). \quad (3)$$

The result (2) is essentially a consequence of a more general theorem of de Veciana and Walrand [6], which makes precise a more heuristic argument presented earlier by Kesidis, Walrand and Chang [14]. A similar large deviations result was proved by Dembo and Karlin [5] but in a slightly different setting. Analogous results for continuous-time Markov fluid models were obtained by Gibbens and Hunt [11] and by Elwalid and Mitra [8] via spectral expansions.

Now consider the situation when the channel is shared by  $N$  independent Markov modulated streams with possibly different statistics. Let  $\Lambda_j$  be the log spectral radius function of the  $j$ th stream, as defined earlier. Note that the aggregate arrival stream is also a Markov-modulated process, and by direct computation, its log spectral radius function is simply  $\sum_{j=1}^N \Lambda_j$ . Using result (2) and the convexity of the log spectral radius function, we see that if the loss probability requirement is such that for some prescribed  $\delta > 0$ , the loss probability satisfies

$$\lim_{B \rightarrow \infty} \frac{1}{B} \log p(B) \leq -\delta$$

<sup>2</sup>We take  $r^* = \infty$  when there are no roots. This only happens in the uninteresting case when the overflow probability is 0.

then a necessary and sufficient condition for meeting this requirement is that

$$\sum_{j=1}^N \Lambda_j(\delta) - c\delta \leq 0.$$

Hence, one can assign an effective bandwidth

$$e_j(\delta) \equiv \frac{\Lambda_j(\delta)}{\delta} \quad (4)$$

to each stream, and the loss probability requirement is satisfied iff  $\sum_j e_j(\delta) \leq c$ .

### III. HOW LARGE IS LARGE?

The validity of the effective bandwidth formula (4) depends critically on the accuracy of the large deviations approximation (3) of the loss probability. It is imperative therefore to have a better understanding of what the assumption of "large buffers" really means in terms of the statistics of the arrival stream (i.e., large with respect to what?). This is not only helpful in having a better sense of when this effective bandwidth formula is applicable in practice, but also motivates the multiple time-scale models to be introduced in the sequel. Here, we will use martingale techniques to shed some insight on this issue.

Let us first consider the simplest case where the time-slotted arrival stream  $\{X_t\}$  is an i.i.d. process. Using a reasonably standard reduction by means of renewal theory (see for example [17]), the cell loss probability in this case can be well approximated by the probability that, starting from an empty buffer, the buffer becomes full before emptying it out again. This is a barrier-hitting probability for the negative-drift random walk  $S_0 = 0, S_t = \sum_{n=1}^t (X_n - c)$ , and can be analysed using Wald's martingale  $M_t(r) = \exp\{rS_t - t(\Lambda(r) - cr)\}$ , where  $\Lambda(r) = \log E[\exp(rX_1)]$  is the log moment generating function of the number of cells arriving per time slot. (For details on this approach, see for example [19].) Defining the stopping time  $T \equiv \min\{t \geq 1 : S_t \geq B \text{ or } S_t \leq 0\}$ , and applying the optional stopping theorem<sup>3</sup> on the martingale  $M_t(r^*)$  (where  $\Lambda(r^*) - cr^* = 0, r^* > 0$ ), we get

$$E[\exp(r^* S_T)] = 1. \quad (5)$$

(This is also known as Wald's Identity.) Note that when the random walk  $\{S_t\}$  first hits a barrier, there may be overshoots. However, if we make the assumption that the overshoots are small compared to the buffer size, then (5) implies that

$$\mathcal{P}(S_T \geq B) \approx \exp(-r^* B).$$

Thus in this i.i.d. case, one should expect the large deviations loss probability approximation (3) to be accurate when the fluctuations of the net number of cell arrivals per time slot are small relative to the buffer size  $B$ . Using the martingale approach, one can also show that, conditional on filling the

<sup>3</sup>The optional stopping theorem says that under appropriate regularity condition,  $E(M_T) = E(M_0)$  for any martingale  $\{M_t\}$  and random stopping time  $T$ . If  $M_t$  is interpreted as a gambler's fortune after the  $t$ th game, this essentially says that one cannot win in a sequence of fair games without looking ahead.

buffer before emptying out again, the expected time to fill the buffer starting from an empty buffer is proportional to the size of the buffer [19].

For the general case when the arrival process is modulated by a Markov chain  $\{H_t\}$  with log spectral radius function  $\Lambda(r)$ , we can use a martingale generalized from Wald's martingale in the i.i.d. case

$$M_t(r) \equiv \exp\{rS_t - t(\Lambda(r) - cr)\} \frac{\eta_r(H_t)}{\eta_r(H_0)} \quad (6)$$

where  $\eta_r$  is a right eigenvector defined in (1). A more insightful approach, however, is to essentially transform the problem back to the i.i.d. case by imposing a *regenerative structure* on the process  $\{S_t\}$ . (This approach is used by Ney and Nummelin [16] in their study of general large deviations properties of Markov additive processes.) Specifically, let  $\tau_0 = 0$  and  $\tau_i$  ( $i > 1$ ) be the  $i$ th time the chain  $\{H_t\}$  returns to the state  $H_0$ . Define  $Y_0 = 0, Y_i = S_{\tau_i} - S_{\tau_{i-1}} - c(\tau_i - \tau_{i-1})$  for  $i > 1$ . Note that  $\{Y_t\}$  is an i.i.d. process, and if we define a stopping time  $T$  as the smallest  $i$  such that the first barrier crossing occurs in the interval  $(\tau_{i-1}, \tau_i]$ , then

$$E[\exp(r_Y S_T)] = 1$$

where  $r_Y$  is the unique positive root of the equation  $\Lambda_Y(r) = 0$ ,  $\Lambda_Y(r) \equiv \log E[\exp(rY_1)]$ . The parameter  $r_Y$  can be expressed in terms of the statistics of the original Markov-modulated process. It can be shown that the optional stopping theorem is also applicable to the martingale  $M_t$  in (6) and the return time  $\tau_1$ . This yields

$$E[\exp\{rY_1 - \tau_1(\Lambda(r) - cr)\}] = 1.$$

Hence, the unique positive root  $r^*$  of the equation  $\Lambda(r) - cr = 0$  also satisfies the equation  $\Lambda_Y(r) \equiv \log E[\exp(rY_1)] = 0$ . By the uniqueness of  $r_Y$ , this implies that  $r_Y$  is in fact the unique positive root of the equation  $\Lambda(r) - rc = 0$ .

Hence we can reinterpret the estimate (3) of the loss probability for Markov-modulated arrivals as that of the loss probability of the embedded i.i.d. arrival process  $\{S_{\tau_{i+1}} - S_{\tau_i}\}$  with a channel of varying capacity  $\{(\tau_{i+1} - \tau_i)c\}$ . In this sense, the large deviations behavior underlying the result for Markov-modulated arrivals is qualitatively not too different from the large deviations behavior for i.i.d. arrivals. In the regime where one can expect the estimate to be accurate, the typical route to buffer overflow would be through a large number of regeneration epochs (proportional to the size of the buffer), with the overshoot small compared to the buffer size. Here, the overshoot is the net number of cells that arrive in the period between the time the buffer first overflows and the next regeneration epoch. (See Fig. 1.) This holds when the fluctuations in the number of cells arrived in a regeneration period is small compared to the buffer size. When the calculations are done using the martingale (6), the term involving the right eigenvector accounts for this overshoot.

### IV. LARGE DEVIATIONS OF MULTIPLE TIME-SCALE MARKOV STREAMS

When the traffic stream has only fast time-scale correlation, the time it takes to return to any one source state is small, and

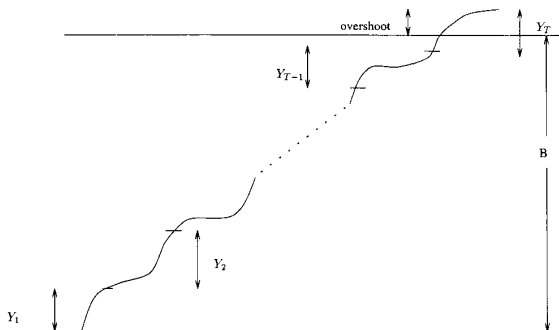


Fig. 1. Regenerative structure and overflow path.

the above qualitative picture for the large deviations behavior holds rather accurately. On the other hand, when the stream has slow time-scale dynamics as well, the qualitative picture is not so appropriate since the regeneration time has a much wider fluctuation. Thus, the large deviations approximation may not be justified. In the next few sections, we will explicitly model the multiple time-scale dynamics and derive a set of large deviations results for this model, together with qualitative pictures of the typical manner in which buffer overflows occur in the multiple time-scale setting. These large deviations results will form a basis for understanding bandwidth and buffer allocation issues for multiple time-scale streams, as well as the statistical multiplexing gain achievable. We will discuss these issues in Section VII.

As before, the traffic stream is modeled by a stationary Markov-modulated process  $(H_t, X_t)$ , with the underlying Markov chain having a finite state space  $\mathcal{S}$ . The multiple time-scale aspect is modeled by imposing an *asymptotic structure* on the underlying Markov chain. Specifically, we index the process by a small parameter  $\alpha$  and let  $(p_{ij}(\alpha))$  be the transition matrix of the modulating chain, where the transition probabilities are assumed to be differentiable functions of the parameter  $\alpha$ . For  $\alpha \neq 0$ , the Markov chain is irreducible. As  $\alpha \rightarrow 0$ , the probabilities of some of the transitions go to zero. These are the rare transitions which govern the slow time-scale dynamics. Let  $\mathcal{R} = \{(i, j) : p_{ij}(0) = 0\}$  be the set of such rare transitions. When  $\alpha = 0$ , the chain is decomposed into  $K$  irreducible component subchains with state spaces  $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K$ . They model the fast time-scale dynamics in the different regimes of the source. Thus, for small  $\alpha > 0$ , the source would typically spend a long time in a subchain, and then occasionally jump to a different subchain through a transition in  $\mathcal{R}$ .

This multiple time-scale model is similar to the singular-perturbed or  $\epsilon$ -decomposable Markov models (see for example [3]), although here there is no restriction that the rare transition probabilities must be integral powers of  $\alpha$ .

We introduce the asymptotic structure on the rare transition probabilities to model the fact that the process remains in each subchain for a long time before switching to another subchain. However, we assume that the process spends a significant fraction of time in each subchain. More precisely, for each  $\alpha \neq 0$ , let  $\pi_\alpha$  be the steady-state distribution of the entire ergodic chain. We make the assumption that for

each subchain  $k$

$$\lim_{\alpha \rightarrow 0} \pi_\alpha(\mathcal{S}_k) \equiv \pi(\mathcal{S}_k) > 0.$$

Also, let  $\mu_i \equiv E[X_t | H_t = i]$  be the number of arrivals per time slot (arrival rate) when the chain is in state  $i$ , and let

$$\bar{\mu}_k = \sum_{i \in \mathcal{S}_k} \frac{\pi(i)}{\pi(\mathcal{S}_k)} \mu_i$$

be the average arrival rate (per time slot) conditional on being in subchain  $k$  (in the limit as  $\alpha \rightarrow 0$ ). For brevity, we will refer to  $\bar{\mu}_k$  as the average rate of subchain  $k$ .

We also assume that the transition probabilities *within* each subchain remain fixed, independent of  $\alpha$ . More formally, for all  $1 \leq k \leq K$  and for all  $i, j \in \mathcal{S}_k$ , the probability of a transition from state  $i$  to  $j$  conditional on the process staying within subchain  $k$

$$p_{ij|k}(\alpha) \equiv \frac{p_{ij}(\alpha)}{\sum_{s \in \mathcal{S}_k} p_{is}(\alpha)} = p_{ij}(0)$$

is fixed independent of  $\alpha$ .<sup>4</sup>

For the  $k$ th subchain of the stream ( $k = 1, 2, \dots, K$ ), we can compute the spectral radius function  $\rho_k(r)$ , and the log spectral radius function  $\Lambda_k(r) \equiv \log \rho_k(r)$ , of the matrix

$$A_k(r) \equiv (p_{ij}(0)g_i(r))_{\{i,j \in \mathcal{S}_k\}} \quad (7)$$

where  $g_i(r) \equiv E(\exp(rX_1) | H_1 = i)$ .

Also, let  $[\eta_r^k(1), \dots, \eta_r^k(|\mathcal{S}_k|)]^t$  be a (positive) right eigenvector of the matrix  $A_k(r)$  corresponding to the eigenvalue  $\rho_k(r)$ .

We define

$$\rho_{\max}(r) \equiv \max_{1 \leq k \leq K} \rho_k(r), \quad \Lambda_{\max}(r) \equiv \max_{1 \leq k \leq K} \Lambda_k(r).$$

We are interested in obtaining estimates of the loss probability in the regime where the buffer size is large with respect to the fast time-scale dynamics, but not necessarily with respect to the slow time-scale dynamics. To capture this mathematically, we take the joint limit as both the buffer size  $B$  becomes large and the slow time-scale parameter  $\alpha$  goes to zero.<sup>5</sup>

We use a combination of martingale techniques and large deviations theory to obtain our results. The main result we need from large deviations theory is the Gärtner-Ellis Theorem. Here is a special case which suffices for our purposes. For a proof of the general result, see [7].

*Theorem 1:* Let  $Z_1, Z_2, \dots$  be a sequence of real-valued random variables, possibly dependent, and define for each  $n$

$$\Lambda_n(r) \equiv \frac{1}{n} \log E[\exp(rnZ_n)]$$

Suppose that the  $Z_n$ 's have asymptotically the same mean,  $\lim_{n \rightarrow \infty} E(Z_n) = \bar{\mu}$ , and the *asymptotic log moment generating function*, defined as  $\Lambda(r) \equiv \lim_{n \rightarrow \infty} \Lambda_n(r)$ , exists and is

<sup>4</sup>This assumption is not necessary, and in general all our asymptotic results can instead be expressed in terms of the limiting conditional probabilities (as  $\alpha \rightarrow 0$ ). The assumption simplifies the notations in the paper.

<sup>5</sup>From now on, we will denote the loss probability by  $p(B, \alpha)$  to emphasize its dependence on the slow time-scale parameter  $\alpha$ .

differentiable for all  $r$ . Then, for all  $\mu > \bar{\mu}$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{P}(Z_n > \mu) = -\Lambda^*(\mu)$$

where  $\Lambda^*$  is the Legendre transform of  $\Lambda$ , defined by

$$\Lambda^*(\mu) \equiv \sup_{r \geq 0} [\mu r - \Lambda(r)]. \quad (8)$$

In the main case of interest, the random variables  $Z_n$ 's are given by  $Z_n = \frac{1}{n} \sum_{t=1}^n X_t$ , where  $\{X_t\}$  is a Markov-modulated process. It can be shown that in this case, the asymptotic log moment generating function is in fact equal to the log spectral radius function [7]

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log E[\exp(r \sum_{t=1}^n X_t)] = \log \rho(r) \quad (9)$$

where  $\rho(r)$  is the spectral radius function for the process  $\{X_t\}$ .

The following is the first major result of the section.

*Theorem 2:* Suppose the multiple time-scale stream  $\{X_t\}$  is served by a channel of constant rate  $c$ , and let  $B$  be the buffer size. If the average rate  $\bar{\mu}_k$  of each subchain is less than  $c$ , then the steady-state loss probability  $p(B, \alpha)$  in the regime of large buffers  $B$  and small  $\alpha$  is

$$\lim_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{1}{B} \log p(B, \alpha) = -r^*$$

where  $r^*$  is the unique positive root of

$$\Lambda_{\max}(r) - rc = 0.$$

Alternatively,  $r^* = \min\{r_1^*, \dots, r_K^*\}$ , where  $r_k^*$  is the unique positive root of  $\Lambda_k(r) - rc = 0$ .

That the two characterizations of  $r^*$  are equivalent follows from the convexity of the functions  $\Lambda_k$ 's.

We have seen that if the stream is a *single* time-scale Markov-modulated process with log spectral radius function  $\Lambda(r)$ , then the exponent for the loss probability is just the positive root of the equation  $\Lambda(r) - rc = 0$ . Hence, Theorem 2 is a generalization of this result to multiple time-scale processes. The theorem essentially says that the loss probability is determined by the "worst" subchain, the one with the largest loss probability when regarded as a single time-scale process. Note that  $\max_k \Lambda_k(r)$  is also a convex function, we can view this as a *generalized* log spectral radius function.

It should be emphasized that the limit for the loss probability is *uniform* in the two asymptotic parameters, and is independent of the relative rates of approach of  $B$  and  $\alpha$  to their respective limits. In particular, it shows that the approximation is valid in the regime when  $\frac{1}{\alpha} \gg B \gg 1$ , i.e., the buffer is large with respect to the fast time-scale dynamics but small with respect to the slow time-scale dynamics.

*Sketch of Proof:*

*Upper bound:* Suppose the system is in steady-state. Let  $\beta > 0$ . We first upper bound the probability of the event  $E_\beta$  that the buffer is full at time 0 and the buffer was last empty at time  $-\beta B$ . A necessary condition for this event to happen is that more than  $\beta Bc + B$  cells have arrived in the time interval  $[-\beta B, 0]$ . Let  $N_\beta$  be the number of rare transitions (jumps from one subchain to another) in this time interval. We show

that with high probability,  $N_t = o(B)$ ,<sup>6</sup> and that when there are that few transitions, we can get a good large deviations upper bound on  $\mathcal{P}(E_\beta)$ . Specifically

$$\begin{aligned} \mathcal{P}(E_\beta) &\leq \mathcal{P}\left(\sum_{t=-\beta B}^0 (X_t - c) \geq B\right) \\ &\leq \mathcal{P}\left(\sum_{t=-\beta B}^0 (X_t - c) \geq B \quad \& \quad N_\beta \leq m\right) \\ &\quad + \mathcal{P}(N_\beta > m) \end{aligned} \quad (10)$$

for any  $m \leq \beta B$ . It can be shown that

$$\mathcal{P}(N_\beta > m) \leq \left(\frac{\beta B d(\alpha) e}{m}\right)^m$$

where  $d(\alpha) \equiv \max_{i \in \mathcal{S}} \sum_{(i,j) \in \mathcal{R}} p_{ij}(\alpha)$  and  $d(\alpha) \rightarrow 0$  as  $\alpha \rightarrow 0$ . This follows from a standard combinatorial bound on the sum of Bernoulli random variables and a coupling argument.

We can upper bound the second term in (10) using Chernoff's bound: for any  $r > 0$

$$\begin{aligned} \mathcal{P}\left(\sum_{t=-\beta B}^0 (X_t - c) \geq B \quad \& \quad N_\beta \leq m\right) \\ \leq E\left[\exp\left\{r \sum_{t=-\beta B}^0 (X_t - c)\right\} \cdot I_{\{N_\beta \leq m\}}\right] \exp\{-rB\} \end{aligned}$$

where  $I_A$  is the characteristic function on a set  $A$ . To bound the expectation term above, consider first the simple case when each subchain has only one state (i.e., an i.i.d. process). Then, since conditional on each source state sequence,  $\{X_t\}$  is an i.i.d. process

$$\begin{aligned} E\left[\exp\left\{r \sum_{t=-\beta B}^0 (X_t - c)\right\} \cdot I_{\{N_\beta \leq m\}}\right] \\ \leq \left[\max_k g_k(r) \exp\{-cr\}\right]^{\beta B} \end{aligned}$$

where  $g_k$  is the generating function of the  $k$ th subchain. To tackle the general case when each subchain has multiple states, we use the idea discussed in Section III on the reduction of Markov-modulated processes to i.i.d. processes using regenerative constructions. Specifically, if it were true that for each subchain  $k$ , there is a state  $s_k$  such that the process always enters and leaves the subchain through  $s_k$ , then we can just "sample" the process at the time instants it is in one of the  $s_1, s_2, \dots, s_K$ , and we are essentially back in the case when each subchain has a single state with corresponding generating function  $\rho_i(\theta)$ . However, if the process enters and leaves a subchain in different states, then we have to introduce a "correction term" each time this happens, analogous to the overshoot term described earlier. One can implement this idea

<sup>6</sup>  $a_n = o(b_n)$  if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$

by making use of martingales of the type (6), and it can be shown that

$$E \left[ \exp \left\{ r \sum_{t=-\beta B}^0 (X_t - c) \right\} \cdot I_{\{N_\beta \leq m\}} \right] \leq \left[ \max_k \rho_k(r) \exp\{-cr\} \right]^{\beta B} h(r)^{m+1}$$

where  $h(r) > 0$  is an upper bound on the overshoot terms ( $h(r)$  can be obtained terms of the components of the right eigenvectors corresponding to the spectral radius functions of the subchains.) Putting all this together in (10), we have

$$\mathcal{P}(E_\beta) \leq \rho_{\max}(r)^{\beta B} h(r)^{m+1} \cdot \exp\{-rB(1+c\beta)\} + \left( \frac{\beta B d(\alpha) e}{m} \right)^m$$

where  $\rho_{\max}(r) = \max_k \rho_k(r)$ . One can choose  $m$  as a function of  $B$  and  $\alpha$  such that as  $B \rightarrow \infty$  and  $\alpha \rightarrow 0$ ,  $\lim_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{m(B, \alpha)}{B} = 0$  and the second term above goes to zero faster than exponentially in  $B$ . Hence

$$\limsup_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{1}{B} \log \mathcal{P}(E_\beta) \leq \beta \Lambda_{\max}(r) - r(1+c\beta).$$

Optimizing over all  $r > 0$  to get the tightest bound, we get

$$\limsup_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{1}{B} \log \mathcal{P}(E_\beta) \leq -\beta \Lambda_{\max}^*(c + \frac{1}{\beta})$$

where  $\Lambda_{\max}^*$  is the Legendre transform of  $\Lambda_{\max}$  as defined in (8). Note that  $\mathcal{P}(\cup_{\beta > 0} E_\beta)$  is the steady-state probability that the buffer is full. This can be related to the steady-state cell loss probability  $p(\alpha, B)$  by noting that

$$E(X_1)p(\alpha, B) \leq P_0 \mathcal{P}(\cup_{\beta > 0} E_\beta)$$

where  $P_0$  is the peak arrival rate. It can be shown, using techniques similar to those in [6] and [4], that in the large deviations regime

$$\limsup_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{1}{B} \log p(B, \alpha) \leq \sup_{\beta > 0} \limsup_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{1}{B} \log \mathcal{P}(E_\beta) \leq -\inf_{\beta > 0} \beta \Lambda_{\max}^*(c + \frac{1}{\beta}). \quad (11)$$

This is essentially an example of *Laplace's principle*, that the probability of a rare event is of the same order of magnitude as the probability of the most likely way that the rare event can happen. The optimization problem in (11) has a nice geometric interpretation (see Fig. 2); solving it gives

$$\limsup_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{1}{B} \log p(B, \alpha) \leq -r^*, \quad \Lambda_{\max}(r^*) - cr^* = 0.$$

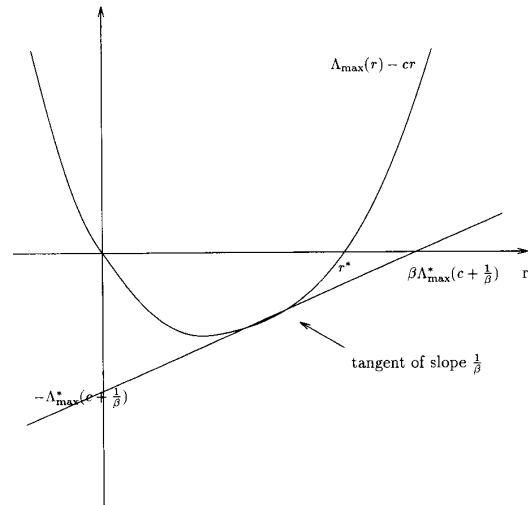


Fig. 2. Geometric interpretation of optimization problem.

*Lower bound:* Fix  $\beta \geq 0$ . A sufficient condition for cells to be lost some time during the interval  $[-\beta B, 0]$  is  $\sum_{t=-\beta B+1}^0 (X_t - c) > B$ . Now

$$E(X_1)p(B, \alpha) \geq \mathcal{P}(\text{cells are lost at time 0}) \geq \frac{\mathcal{P}(\text{cells are lost during the interval}[-\beta B, 0])}{\beta B} \quad (12)$$

$$\geq \frac{1}{\beta B} \mathcal{P}\left(\sum_{t=-\beta B+1}^0 (X_t - c) > B\right). \quad (13)$$

(union bound)

The probability of the event  $F_k$  that the stream stays in the same subchain  $k$  throughout the interval  $[-\beta B, 0]$  is at least  $\pi_\alpha(\mathcal{S}_k)(1 - d(\alpha))^{\beta B}$ . Hence, for each  $k$

$$\mathcal{P}\left(\sum_{t=-\beta B+1}^0 (X_t - c) > B\right) \geq \pi_\alpha(\mathcal{S}_k)(1 - d(\alpha))^{\beta B} \cdot \mathcal{P}\left(\sum_{t=-\beta B+1}^0 (X_t - c) > B | F_k\right). \quad (14)$$

This second probability can be estimated by the Gärtner-Ellis Theorem, applied to a Markov-modulated process corresponding to subchain  $k$

$$\lim_{B \rightarrow \infty} \frac{1}{B} \log \mathcal{P}\left(\sum_{t=-\beta B+1}^0 (X_t - c) > B | F_k\right) = -\beta \Lambda_k^*(c + \frac{1}{\beta}).$$

Using this in (14) and since as  $\alpha \rightarrow 0$  and  $B \rightarrow \infty$ ,  $d(\alpha) \rightarrow 0$  and the term  $(1 - d(\alpha))^{\beta B}$  does not go to zero as fast as exponentially in  $B$ , we get

$$\liminf_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{1}{B} \log p(B, \alpha) \geq -\beta \Lambda_k^*(c + \frac{1}{\beta}).$$

But this holds for all  $\beta \geq 0$  and for all subchains

$$\begin{aligned} \liminf_{B \rightarrow \infty, \alpha \rightarrow 0} \frac{1}{B} \log p(B, \alpha) &\geq - \min_k \inf_{\beta \geq 0} \beta \Lambda_k^* \left( c + \frac{1}{\beta} \right) \\ &= - \min_k r_k^*. \quad \square \end{aligned}$$

The above proof provides not only an estimate of the loss probability but also the typical way that the buffer gets filled. First, the stream enters the subchain with the smallest root  $r_k^*$  (slow time-scale dynamics). Then, starting from an empty buffer, the buffer gets filled due to a burst of arrivals while the stream remains in that subchain (fast time-scale dynamics). Hence, the above result implies a clean *separation of time-scales* in the typical way of overflowing the buffer.

Such separation of time-scale phenomenon needs not hold in all large deviations rare events associated with multiple time-scale processes. See [22] for an example in which there is a tight coupling between the slow and fast time-scale dynamics in the typical way leading to the rare event.

Theorem 2 covers only the case when the average rates of all the subchains are less than the channel capacity. We now turn to the case when some of the subchains have average rates greater than the channel capacity.

In this case, the situation depends critically on the relationship between the slow time-scale and the buffer size. Specifically, if  $p_{ij}(\alpha)B \approx 0$  for all rare transitions  $(i, j)$ , then the probability of cell loss is significant while the traffic stream is in a subchain with average rate greater than the channel rate. This is because with high probability, the time the stream spends in that subchain (proportional to the reciprocal of the rare transition probabilities) will be larger than the time it takes to fill the buffer with the average rate of the subchain (proportional to the buffer size). On the other hand, if  $p_{ij}(\alpha)B \gg 0$  for all rare transitions  $(i, j)$ , then the buffer is large with respect to the slow time scale. In this regime, the loss probability can still be made small, as long as the overall average rate of the stream is less than the channel capacity. Intuitively, the expected time to fill the buffer in this regime is proportional to the buffer size, so that there is still sufficient time to average between the high-rate and low-rate subchains in the path leading to buffer overflow.

The situation in the first regime is made precise by the following theorem.

**Theorem 3:** Suppose there is a subchain  $k$  whose average rate  $\bar{\mu}_k$  is greater than the channel capacity  $c$ . Then

$$\left\{ \begin{array}{l} \liminf_{B \rightarrow \infty} p(B, \alpha) > 0 \\ p_{ij}(\alpha)B \rightarrow 0 \quad \forall (i, j) \in \mathcal{R} \end{array} \right\}$$

To derive an estimate for the loss probability in the regime where  $p_{ij}(\alpha)B \gg 0$ , we need to make a regularity assumption on the rare transition probabilities. We assume that the probabilities of all the rare transitions are linear in the parameter  $\alpha$ , so that they are of the same order of magnitude. We are interested in the regime where the product  $\alpha B$  is large. The key result is that in this regime, the loss probability can be well approximated by the situation when the arrival process is a certain continuous-time, continuous-space Markov fluid and when the buffer size is scaled to be  $\alpha B$ . The states of

the Markov fluid are obtained by appropriately averaging the subchains of the original multiple time-scale process. The fast time-scale dynamics of the individual subchains are irrelevant, and the slow time-scale dynamics essentially take on the role of the fast time-scale dynamics in the scaled picture.

Recall that  $\pi$  is the steady-state distribution of the entire modulating chain, and  $\bar{\mu}_k$  is the average rate of subchain  $k$ .

**Theorem 4:** Assume that the overall average arrival rate (per time-slot) of the multiple time-scale stream is less than the channel rate  $c$ , but at least one of the  $\bar{\mu}_k$ 's is greater than  $c$ . Then the loss probability in the asymptotic regime of small  $\alpha$  and large  $\alpha B$  is given by

$$\lim_{\alpha \rightarrow 0, \alpha B \rightarrow \infty} \frac{1}{\alpha B} \log p(B, \alpha) = -r_f^*$$

where  $r_f^*$  is the unique positive root of the equation  $\Lambda_f(r) - rc = 0$  and  $\Lambda_f(r)$  is the largest eigenvalue of the matrix  $Q + rM$ , where

$$Q \equiv \left[ \sum_{i \in \mathcal{S}_k, j \in \mathcal{S}_l} \pi(i) p'_{ij}(0) \right]_{k, l=1, \dots, K}$$

$$M \equiv \text{diag}(\bar{\mu}_1, \bar{\mu}_2, \dots, \bar{\mu}_K).$$

( $p'_{ij}(0)$  is the derivative of  $p_{ij}(\alpha)$  with respect to  $\alpha$  evaluated at  $\alpha = 0$ .)

The exponent  $r_f^*$  is the same as the large deviations exponent of the loss probability when the buffer size is  $\alpha B$  and the arrival stream is a continuous-time Markov fluid with infinitesimal generator  $Q$  and rate matrix  $M$  ([8], [11]). This result shows that the fluid model is appropriate for estimating the large deviations loss probability when the sojourn time in one source state is long compared to the fluctuations of the arrival process about its drift, but small compared to the buffer size. (This is essentially a result on the robustness of the idealized Markov fluid model. While it is quite obvious that the Markov fluid model would be rather robust with respect to *average* quantities such as average delays or average buffer occupancy, it is not *a priori* clear that it is also robust with respect to small large deviations probabilities of rare events such as buffer overflow.)

We note that although the loss probability goes to zero in this asymptotic regime, it only goes to zero exponentially in the product  $\alpha B$ , whereas in the regime where the average rates of *all* subchains are less than the channel capacity, the loss probability goes to zero exponentially in the buffer size  $B$ .

*Sketch of Proof:* As in the proof of Theorem 2, one can show that the loss probability can be estimated by first fixing a  $\beta > 0$  and estimating  $\mathcal{P}(\sum_{t=-\beta B}^0 (X_t - c) > B)$ , and then maximize over all  $\beta$  (à la Laplace's Principle). We estimate this former probability by applying the Gärtner-Ellis Theorem to an appropriately scaled process. Specifically

$$\mathcal{P}\left(\sum_{t=-\beta B}^0 (X_t - c) > B\right) = \mathcal{P}\left(\frac{1}{\alpha\beta B} \sum_{t=-\beta B}^0 \alpha X_t > c + \frac{1}{\beta}\right). \quad (15)$$

Define  $n \equiv \alpha\beta B$  and  $Z_n \equiv \frac{1}{\alpha\beta B} \sum_{t=-\beta B}^0 \alpha X_t$ . To apply the Gärtner-Ellis Theorem to estimate (15), we compute the

asymptotic log moment generating function and find that

$$\lim_{\alpha \rightarrow 0, n \rightarrow \infty} \frac{1}{n} \log E[\exp(rnZ_n)] = \Lambda_f(r).$$

To show this, we make use of the martingale (6) to convert the problem into that of computing limits of the spectral radii and right eigenvectors of irreducible matrices, and then use perturbation techniques together with the properties of the spectral radius of irreducible matrices to compute the limit. Since  $\Lambda_f$  is differentiable for all  $r$

$$\lim_{\alpha \rightarrow 0, n \rightarrow \infty} \frac{1}{n} \log \mathcal{P}(Z_n > c + \frac{1}{\beta}) = -\Lambda_f^*(c + \frac{1}{\beta}).$$

Optimizing over  $\beta > 0$  gives us the desired result.  $\square$

## V. EFFECTIVE BANDWIDTH FOR MULTIPLE TIME-SCALE STREAMS

Let us now consider the multiplexing of  $N$  independent multiple time-scale Markov-modulated streams. Suppose the  $j$ th stream has  $K_j$  subchains, with log spectral radius functions  $\Lambda_{jk}$ ,  $k = 1, \dots, K_j$ . Let  $\Lambda_{j,\max} \equiv \max_k \Lambda_{jk}$  be the generalized spectral radius function of stream  $j$ . It can be seen that the superposition of these streams is also a multiple time-scale Markov-modulated stream, and its generalized spectral radius function is  $\sum_{j=1}^N \Lambda_{j,\max}$ .

One can derive an expression for the effective bandwidth of multiple time-scale streams, in analogy to the formula (4) for single time-scale streams. As before, suppose the requirement on the loss probability is such that

$$\lim_{B \rightarrow \infty} \frac{1}{B} \log p(B) \leq -\delta.$$

It follows from the results of the previous section and the convexity of  $\sum_{j=1}^N \Lambda_{j,\max}(r) - rc$  that this requirement is satisfied if and only if

$$\sum_{j=1}^N \frac{\Lambda_{j,\max}(\delta)}{\delta} \leq c.$$

Thus, one can assign an effective bandwidth

$$e_j(\delta) \equiv \frac{\Lambda_{j,\max}(\delta)}{\delta}$$

to the  $j$ th stream such that the loss probability requirement is satisfied if and only if the sum of the effective bandwidths of the streams does not exceed the shared capacity of the outgoing link. Rewriting this in terms of the log spectral radius functions of the subchains of the  $j$ th stream, we obtain

$$e_j(\delta) = \max_{1 \leq k \leq K_j} \frac{\Lambda_{jk}(\delta)}{\delta}. \quad (16)$$

Thus, the effective bandwidth of a multiple time-scale source is the maximum of the effective bandwidths of its component subchains when regarded as single time-scale streams.

## VI. MULTIPLEXING OF LARGE NUMBER OF STREAMS

The effective bandwidth formula (16) for multiple time-scale streams is based on the large deviations estimate of the loss probability in Theorem 2. The qualitative picture for the typical overflow path associated with this estimate is that, first, the streams enter into a certain worst-case

combination of subchains and, second, there is an unlikely burst of cell arrivals to fill the buffer while the streams remain in this combination of subchains. The asymptotic estimate is essentially an estimate of the probability of the second event while ignoring the probability that the streams are in that worst-case combination of subchains. This is because for a fixed number of streams and fixed steady-state statistics of the streams, this latter probability is constant and therefore gets washed out in the asymptotics of large buffers. However, if the number of streams is large and their statistics are similar, this probability of being in the worst-case combination may in fact be very small and should not be neglected in approximating the loss probability. We will look at the joint asymptotic regime where there are many streams, in addition to a large buffer and rare transition probabilities, and derive better estimates of the loss probability which in turn yields a formula for the effective bandwidth less conservative than (16). The asymptotic regime where there are large number of sources has also been analysed in [23] and [12] for single time-scale sources.

As a base case for comparison, consider first the problem of approximating the loss probability when a large number of statistically identical and independent *single (fast) time-scale* streams are multiplexed together. Each stream  $\{X_t^{(j)}\}$  ( $j = 1, 2, \dots, N$ ) is a stationary Markov-modulated process. Let  $\Lambda(r)$  be the (common) spectral radius function, and  $\bar{\mu}$  be the steady-state average cell arrival rate for each stream. Let  $Nc$  and  $NB$  be the capacity of the out-going link and the buffer size respectively, scaled so that the capacity and buffer per incoming stream remains fixed. Let  $p(B, N)$  be the loss probability, as a function of both the buffer size and the number of streams. We are interested in the asymptotic regime where both the number of streams  $N$  and the buffer per stream are large.

*Theorem 5:* If  $c > \bar{\mu}$  then the loss probability  $p(B, N)$  decays exponentially with the product  $NB$ , with the exponent given by

$$\lim_{B \rightarrow \infty, N \rightarrow \infty} \frac{1}{NB} \log p(B, N) = -r^*$$

where  $r^*$  is the unique positive root of the equation  $\Lambda(r) - rc = 0$ .

The above result says that if the channel capacity is linearly scaled as the number of streams  $N$  increases, the exponential rate of decrease of the loss probability in the total buffer size ( $NB$ ) can be kept fixed at  $r^*$ . This means that the effective bandwidth of fast time-scale streams remains additive even when a large number of them are multiplexed together. There is also no qualitative change in the typical behavior leading to cell loss as the number of streams increase. Namely, typical losses are due to simultaneous bursting of all the streams, each bursting at a rate equal to that leading to typical cell losses in the case when the stream is served alone by a channel of capacity  $c$ .

The situation is more interesting when a large number of *multiple time-scale* streams are multiplexed together. Suppose now each stream consists of  $K$  subchains between which there are rare transitions in the set  $\mathcal{R}$ . Let  $\Lambda_1(r), \dots, \Lambda_K(r)$  and  $\bar{\mu}_1, \dots, \bar{\mu}_K$  be the log spectral radius functions and the average



rates of the subchains, respectively, and let  $\bar{\mu}$  be the overall average rate of each stream. Also, let  $q_1, \dots, q_K$  be the steady-state probabilities that the stream is in each of the subchains, in the limit of small  $\alpha$ .

We focus on the situation when the slow time-scale is long compared to the buffer size. Recall the corresponding situation for a single stream using a dedicated channel. Theorem 3 basically says that when the slow time-scale is long compared to the buffer size,  $\hat{\mu}$  (the maximum of the average rates of the subchains) cannot be greater than the channel capacity if the buffer fullness probability is to be small. However, when we multiplex a large number of sources, the situation is different. We will show that if the number of streams are large, then one can satisfy a small loss probability requirement even with a channel capacity per stream less than  $\hat{\mu}$ .

Let  $Y$  be a random variable which takes on value  $\bar{\mu}_k$  with probability  $q_k$ ,  $k = 1, \dots, K$ . Let  $\Lambda_Y(r)$  be the log moment generating function of  $Y$ . We have the following result.

*Theorem 6:*

1) If  $\hat{\mu} > c > \bar{\mu}$  then the loss probability decays exponentially with  $N$

$$\lim_{N \rightarrow \infty, B \rightarrow \infty} \frac{1}{N} \log p(B, N, \alpha) = -\Lambda_Y^*(c).$$

$p_{ij}(\alpha)B \rightarrow 0, (i, j) \in \mathcal{R}$

2) If  $c > \hat{\mu}$ , then the loss probability decays exponentially with the product  $NB$ , with exponent given by

$$\lim_{N \rightarrow \infty, B \rightarrow \infty} \frac{1}{NB} \log p(B, N, \alpha) = -r^*$$

$\alpha \rightarrow 0$

where  $r^*$  is the unique positive root of  $\Lambda_{\max}(r) - rc = 0$ .

Recall that the effective bandwidth formula for multiple time-scale streams assigns a bandwidth corresponding to the maximum of the effective bandwidths of the subchains of a stream, and this bandwidth is at least  $\hat{\mu}$ . The additivity of the effective bandwidth means that it predicts a required bandwidth of at least  $N\hat{\mu}$  for the aggregate stream. The previous theorem however says that one can in fact get by with a bandwidth of less than  $N\hat{\mu}$ , if the number of streams is sufficiently large. Hence, the effective bandwidth of multiple time-scale streams is actually *subadditive* as the number of streams become large. This is the fundamental difference between multiplexing of fast time-scale streams and multiplexing of streams with a slow time-scale component.

In the first case of Theorem 6, the loss probability is approximated by the Chernoff's estimate for the probability  $\mathcal{P}(\sum_{i=1}^N Y_i > Nc)$ , where  $Y_1, \dots, Y_N$  are i.i.d. copies of  $Y$ . In this case, the typical behavior leading to cell losses is no longer entering into a certain worst-case combination of subchains and then bursting at unusually high cell arrival rates while staying in that worst-case combination. Rather, the typical behavior leading to cell losses is entering into combinations of subchains such that the total average cell arrival rate is greater than the capacity of the out-going link. Once such a combination is entered upon, cell losses will likely occur because the time spent in this combination will with high

probability be significantly longer than the time it takes to fill the buffer. Thus, in this case, the fast time-scale dynamics have little role to play in the typical behavior leading to cell losses. Also, the cell loss probability is approximately the same as that for an unbuffered multiplexing system, as derived by Hui [12].

*Sketch of Proof:* To show the upper bound, we decompose each arrival stream into a superposition of fast and slow time-scale components. For each stream  $j$ , let  $G_t^{(j)}$  be the index of the subchain that stream  $j$  is in at time  $t$ . Fix  $\epsilon > 0$  and define processes

$$U_t^{(j)} \equiv X_t^{(j)} - \bar{\mu}_{G_t^{(j)}} - \epsilon, \quad V_t^{(j)} \equiv \bar{\mu}_{G_t^{(j)}} + \epsilon.$$

Using an argument based on the union bound, it can be shown that

$$p(B, N, \alpha) \leq \frac{P_0}{E(X_1)} [\mathcal{P}(W_t^u > NB) + \mathcal{P}(W_t^v > 0)] \quad (17)$$

where  $\{W_t^u\}$  and  $\{W_t^v\}$  are stationary processes satisfying

$$W_{t+1}^u = (W_t^u + \sum_{j=1}^N U_t^{(j)})^+, \quad W_{t+1}^v = (W_t^v + \sum_{j=1}^N V_t^{(j)} - Nc)^+.$$

$W_t^v$  is the queue length process when the fast-time scale component is removed from the original stream (the slow system).  $W_t^u$  is the queue length process of a fictitious system with arrival process consists of only the fast time-scale component of the original stream but the channel has capacity 0 (but note that the number of "arrivals" in a time slot can be negative!).

Using techniques similar to the proof of Theorem 2, one can show that

$$\lim_{N \rightarrow \infty, B \rightarrow \infty} \frac{1}{NB} \log \mathcal{P}(W_t^u > NB) = -r_\epsilon$$

$\alpha \rightarrow 0$

where  $r_\epsilon$  is the unique positive root of  $\max_{1 \leq k \leq K} [\Lambda_k(r) - \bar{\mu}_k r] - \epsilon r = 0$ .

Consider now the second term. Using Little's Law, one can relate the probability that the slow system is busy to the probability that the instantaneous arrival rate of the slow system is greater than the channel capacity, which can then be estimated by the Chernoff's bound. It can be shown that

$$\limsup_{N \rightarrow \infty, B \rightarrow \infty} \frac{1}{N} \log \mathcal{P}(W_t^v > 0) \leq -\Lambda_Y^*(c - 2\epsilon).$$

$\alpha \rightarrow 0$

Thus the first term decays exponentially in the product  $NB$  and hence is negligible compared to the upper bound for the second term. hence

$$\limsup_{N \rightarrow \infty, B \rightarrow \infty} \frac{1}{N} \log p(B, N, \alpha) \leq -\Lambda_Y^*(c - 2\epsilon).$$

$\alpha \rightarrow 0$

Taking  $\epsilon \rightarrow 0$  gives us the desired upper bound.

To show the lower bound, one argues that as long as the arrival streams enter into a combination of subchains such

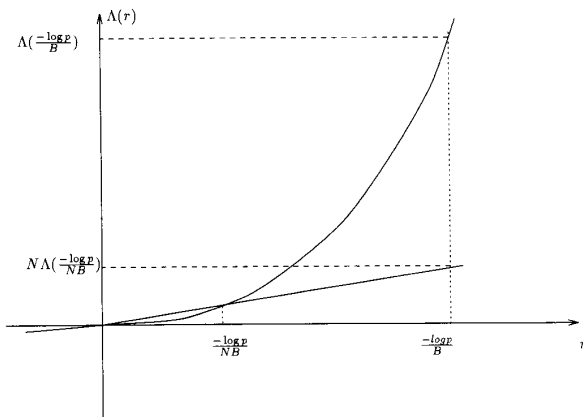


Fig. 3. Graphical illustration of multiplexing gain.

that the total average rates of the subchains is greater than the channel capacity, then with high probability there will be cell losses. This is because the slow time-scale is significantly longer than the time-scale dictated by the buffer size. Hence, the loss probability is essentially lower bounded by the steady-state probability that the instantaneous total average rate of the subchains exceeds the channel capacity, and it is fairly standard to estimate this probability using large deviations techniques.  $\square$

#### VII. STATISTICAL MULTIPLEXING GAIN: BANDWIDTH AND BUFFER REQUIREMENTS

Here, we will look into the effect of the presence of a slow time-scale on the gain achievable by multiplexing a large number of independent and statistically identical Markov streams. We will evaluate the gain in terms of both bandwidth and the buffering required for the same loss probability. Specifically, given the total available buffer space, how much bandwidth can one save by multiplexing rather than allocating a dedicated channel and buffer to each stream? And given the same total channel capacity, how much buffering can be saved by using a shared buffer and channel?

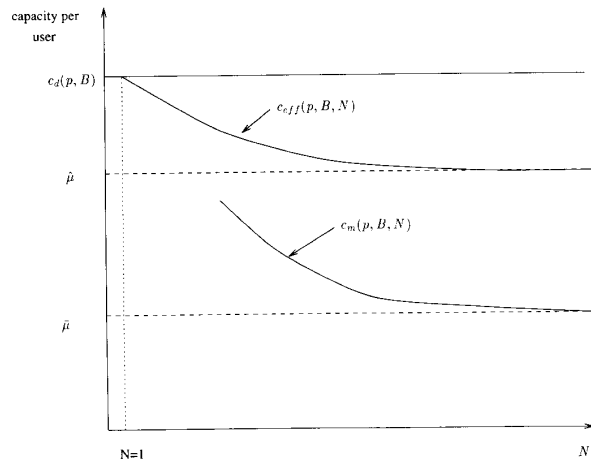
Suppose  $p$  is the desired loss probability. First let us fix the amount of buffer to be  $B$  cells *per stream*. For single fast time-scale streams with log spectral radius function  $\Lambda$ , Theorem 5 tells us that the channel capacity *per stream* we need if the  $N$  streams are multiplexed and share a buffer of size  $NB$  is approximately

$$c_m(p, B, N) = \frac{NB}{-\log p} \Lambda\left(\frac{-\log p}{NB}\right).$$

If each stream is given a dedicated buffer of size  $B$  and a dedicated channel, then the capacity of the dedicated channel needed to achieve the same loss probability is

$$c_d(p, B) = \frac{B}{-\log p} \Lambda\left(\frac{-\log p}{B}\right).$$

It can be seen that since  $\Lambda(0) = 0$  and  $\Lambda(r)$  is increasing and convex for  $r > 0$  (the mean rate of the stream being positive),

Fig. 4. Channel capacity requirements for given loss probability  $p$  and buffer size  $B$  per stream.

there is always a multiplexing gain. (See Fig. 3.) As  $N \rightarrow \infty$ ,  $c_m(p, B, N)$  approaches the mean arrival rate of each stream.

Consider now the analogous calculations for independent and statistically identical multiple time-scale streams, whose slow time-scale is significantly longer than the time-scale dictated by the buffer. Let  $\Lambda_1, \Lambda_2, \dots, \Lambda_K$  be the log spectral radius functions of the subchains of a stream. Using Theorem 2, one can compute the channel capacity required per stream in the dedicated scenario

$$c_d(p, B) = \frac{B}{-\log p} \max_k \Lambda_k\left(\frac{-\log p}{B}\right).$$

In the multiplex scenario, one can use Theorem 6 to compute the channel capacity required per stream

$$c_m(p, B, N) = (\Lambda_Y^*)^{-1}\left(\frac{-\log p}{N}\right).$$

As  $N \rightarrow \infty$ ,  $c_m(p, B, N)$  approaches the mean rate of each stream. Fig. 4 shows  $c_d(p, B)$ ,  $c_m(p, B, N)$  and also the capacity requirement  $c_{eff}(p, B, N)$  predicted by the effective bandwidth formula (16). It can be seen that the latter is overly conservative for large  $N$ .

The differences between multiplexing of fast time-scale streams and multiplexing of multiple time-scale streams become more striking when one looks at the gain in the amount of buffering required for a given capacity  $c$  per stream. Consider first the multiplexing of fast time-scale streams. Using Theorem 5, it can be seen that if  $B$  is the size of the buffer for each stream in the dedicated channel scenario, the amount of buffering needed in a shared buffer would also be approximately  $B$  to achieve the same loss probability. This means there is a multiplexing gain of a factor of  $\frac{1}{N}$  in the buffering needed per stream.

Consider now the case of multiplexing of multiple time-scale streams, where the channel capacity allocated per stream  $c$  is less than  $\hat{\mu}$ , the maximum of the average rates of the subchains, but greater than  $\bar{\mu}$ , the average rate of the entire stream. In the situation when each stream has a dedicated channel of capacity  $c$ , it follows from Theorems 3 and 4 that to

have small loss probability, one needs  $\alpha B$  to be large ( $\alpha$  is the order of the rare transition probabilities) so that the buffer can absorb the slow time-scale variation in the traffic intensity. Specifically, to satisfy a loss probability requirement of  $p$ , Theorem 4 says that we need a buffer of size approximately

$$B_d(p, \alpha) = \frac{-\log p}{r_f^* \alpha}$$

where  $r_f^*$  can be computed from the statistics of the stream and the channel capacity, but is independent of the slow time-scale parameter  $\alpha$ . On the other hand, if the streams are multiplexed together then we need approximately

$$N = \frac{-\log p}{\Lambda_Y^*(c)}$$

streams and a shared buffer of size  $NB_m$  to achieve a loss probability of  $p$ , as long as  $B_m$  is large enough to absorb the fast time-scale fluctuations of an individual stream. We note that in this multiplex scenario, both the number of streams and the buffering per stream needed to achieve the required loss probability is independent of the slow time-scale parameter  $\alpha$ . This means that if  $\alpha$  is very small, the multiplexing gain in terms of buffering is very significant, of the order of  $\frac{1}{\alpha}$ , compared to the multiplexing gain of  $\frac{1}{N}$  for fast time-scale streams.

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