HMMT March 2021 Integration Bee Finals

Sponsored by Five Rings Capital

March 6, 2021

A Message from our Sponsor, Five Rings Capital

Our Contestants (in alphabetical order)

- Adithya Balachandran

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- Daniel Chen

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- Gopal Goel

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- Daniel Chen
- Gopal Goel
- Zhening Li

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  - If 0 or 4 people get the integral - no points for anyone
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  - If 2 people get the integral, then $+2$ for them, and -2 for others
  - If 3 people get the integral, then $+1$ for them, and -3 for the one person that did not.

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- If there is a tie at the end, we will have a tie-breaking integral estimation question.

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- For the spectators - feel free to follow along and try these integrals as well!

Any questions before we start?

Problem 1

Evaluate the following Integral (in terms of $\alpha, \beta \neq 0 \in \mathbb{R}$):

$$\int_0^1 \frac{1}{(\alpha x + \beta(1-x))^2}\, dx$$

Solution 1

$$\frac{1}{\alpha\beta}$$

Problem 2

Evaluate the following Integral:

$$\int e^{2x} \tan^{-1}(e^x)\, dx$$

Solution 2

$$\frac{1}{2}((e^{2x}+1)\tan^{-1}(e^x)-e^x)+C$$

Problem 3

Evaluate the following Integral:

$$\int \frac{x^2 + 2020 \cdot 2021}{(x \sin x + 2021 \cos x)^2} \, dx$$

Solution 3

$$\frac{2021 \sin x - x \cos x}{x \sin x + 2021 \cos x} + C$$

Problem 4

Evaluate the following Limit:

$$\lim_{n\to\infty} \frac{\int_0^\infty (x-1)^n e^{-x}\,dx}{\int_0^\infty x^n e^{-x}\,dx}$$

Solution 4

$$\frac{1}{e}$$

Problem 5

Evaluate the following Integral:

$$\int_0^6 \sqrt[3]{x+\sqrt[3]{x+\sqrt[3]{x+\cdots}}}\,dx$$

Solution 5

$$\frac{39}{4}$$

Problem 6

Evaluate the following Integral:

$$\int \frac{1}{x^2 - x\sqrt{x^2 - 1}}\, dx$$

Solution 6

$$\sqrt{x^2-1}-\tan^{-1}(\sqrt{x^2-1})+x+C$$

Problem 7

Evaluate the following Integral:

$$\int_0^\infty \frac{\arctan(x) - \arctan(\pi x)}{x}\, dx$$

Solution 7

$$-\frac{1}{2}\pi \log \pi$$

Problem 8

Evaluate the following Integral:

$$\int \sqrt{x} \log(x+1)\, dx$$

Solution 8

$$\frac{2}{3}x^{3/2}\log(x+1) - \frac{4}{9}x^{3/2} + \frac{4}{3}\sqrt{x} - \frac{4}{3}\arctan\sqrt{x} + C$$

Problem 9

Evaluate the following Integral:

$$\int_0^{\pi} \frac{x \sin^{2020}(x)}{\cos^{2020}(x) + \sin^{2020}(x)} dx$$

Solution 9

$$\frac{\pi^2}{4}$$

Problem 10

Evaluate the following Integral (in terms of $0 < \theta < \frac{\pi}{2}$):

$$\int_{-\infty}^{0} \frac{\cos(\theta x)}{\cos^{x}(\theta)}\, dx$$

Solution 10

$$-\frac{\log\cos\theta}{\log^2\cos\theta+\theta^2}$$

Problem 11

Estimate the following Integral:

$$\int_0^1 e^{-x^3}\,dx$$

Solution 11

$$\approx 0.807511$$