Important Inequalities

Beckman Math Club

The Arithmetic Mean – Geometric Mean (AM-GM) Inequality

$$\frac{x+y}{2} \geq \sqrt{xy} \text{ and in general, } \frac{a_1+a_2+a_3+\cdots+a_n}{n} \geq \sqrt[n]{a_1a_2a_3\cdots a_n}$$

Example 1

Question: What is the minimum value of $\dfrac{18}{n} + \dfrac{n}{2}$ for positive values of n?

Solution: By the AM-GM inequality, we know that $\frac{\frac{18}{n}+\frac{n}{2}}{2} \geq \sqrt{\frac{18}{n}\cdot\frac{n}{2}}$. This implies $\frac{18}{n}+\frac{n}{2} \geq 2\sqrt{\frac{18}{n}\cdot\frac{n}{2}}$, so that $\frac{18}{n}+\frac{n}{2} \geq 2\sqrt{\frac{18}{2}} = 2\sqrt{9} = \boxed{6}$.

Equality in the AM-GM Inequality: Equality in the AM-GM holds if and only if all members of $a_1, a_2, \ldots, a_n$ are equal. In the simple two element case, clearly $\frac{x+x}{2} = \sqrt{x\cdot x}$.

Example 2

Question: In triangle ABC, $2a^2+4b^2+c^2 = 4ab+2ac$. Compute the numerical value of $\cos B$. (Old ARML Indiv.)

Solution: By the AM-GM inequality, we know that $a^2+4b^2 \geq 2\sqrt{4a^2b^2} = 4ab$. Likewise, $a^2+c^2 \geq 2\sqrt{a^2c^2} = 2ac$. Thus, adding the two inequalities, we find that $2a^2+4b^2+c^2 \geq 4ab+2ac$. But in the question, we are given that the quantities are exactly equal. Thus we are in the equality case of AM-GM for both inequalities. This implies that $a^2 = 4b^2$ and $a^2 = c^2 \implies a = c$, since side lengths are positive. Then we can use the law of cosines to find that $\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{4b^2+4b^2-b^2}{2a^2} = \dfrac{7b^2}{8b^2} = \boxed{\dfrac{7}{8}}$.

Example 3

Question: Show that the equilateral triangle has the most area for any triangle with a fixed perimeter.

Solution: Suppose that the triangle has side lengths a, b, c and a fixed perimeter, hence a fixed semipermieter s.

Heron's formula gives us the area of the triangle as $A = \sqrt{s(s-a)(s-b)(s-c)}$. As s is a constant, we wish to maximize A by maximizing the quantity $(s-a)(s-b)(s-c)$.

By the AM-GM inequality, we know that $\frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$.

Doing algebraic manipulations we find that

$$(s-a)(s-b)(s-c) \leq \left(\frac{3s-(a+b+c)}{3}\right)^3 = \frac{s^3}{27}$$

The product $(s-a)(s-b)(s-c)$ has the constant $\frac{s^3}{27}$ as its upper bound, so the maximum value for this product is indeed the above value. This maximum is reached in the equality case, i.e, when $s-a = s-b = s-c$, which happens only when $a = b = c$.

An extension: The AM-GM-HM Inequality

$$\frac{x+y}{2} \geq \sqrt{xy} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}} \text{ and in general,}$$

$$\frac{a_1+a_2+a_3+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \cdots a_n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots++\frac{1}{a_n}}$$

Example 4

Question: Prove Nesbitt's Inequality: $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \geq \frac{3}{2}$ for $a, b, c > 0$.

Solution: We first start by combining the numerators on the fractions, by adding 1 to each of the fractions, yielding the inequality $\frac{a+b+c}{b+c}+\frac{a+b+c}{a+c}+\frac{a+b+c}{a+b} \geq \frac{9}{2}$. Then we can factor: $(2a+2b+2c)\left(\frac{1}{b+c}+\frac{1}{a+b}+\frac{1}{a+c}\right) \geq 9$. Now, dviding both sides of the inequality by $3\left(\frac{1}{b+c}+\frac{1}{a+b}+\frac{1}{a+c}\right)$ we obtain that $\frac{(a+b)+(a+c)+(b+c)}{3} \geq \frac{3}{\frac{1}{b+c}+\frac{1}{a+b}+\frac{1}{a+c}}$, which is simply the AM-HM inequality applied to $a+b, a+c, b+c$. Since each step in the proof above was reversible, we have shown the desired result.

The complete generalization: The Power Mean Inequality

Let $M(p) = \left(\frac{a_1^p + a_2^p + a_3^p + \cdots + a_n^p}{n}\right)^{1/p}$ for positive values $a_1, a_2, a_3, \ldots, a_n$. Then $M(p_2) \geq M(p_1)$ if $p_2 \geq p_1$, with equality when all a_i are equal.

Some special cases:

$p \to -\infty$	$M(p) = \min(a_1, a_2, \ldots, a_n)$
$p = -1$	$M(p) = \text{Harmonic Mean} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}}$
$p \to 0$	$M(p) = \text{Geometric Mean} = \sqrt[n]{a_1 a_2 a_3 \cdots a_n}$
$p = 1$	$M(p) = \text{Arithmetic Mean} = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$
$p \to \infty$	$M(p) = \max(a_1, a_2, \ldots, a_n)$

The Cauchy-Schwarz Inequality

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots a_n^2)(b_1^2 + b_2^2 + \cdots b_n^2)$$

with equality when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Example 5

Question: Prove the power means inequality for $(p_1, p_2) = (1, 2)$. This result is also known as the QM-AM inequality.

Solution: We need to show that $\sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geq \frac{a_1+a_2+\cdots+a_n}{n}$. By applying Cauchy-Schwarz, we can show that $(a_1 + a_2 + \cdots + a_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(1^2 + 1^2 + \cdots + 1^2)$, where there are n 1's in the sum. So the right hand side is equal to $n(a_1^2 + a_2^2 + \cdots + a_n^2)$. Dividing by n^2 and taking the square root yields our result.

Example 6

Question: If x_1, x_2, x_3 are three positive numbers such that $x_1 + 2x_2 + 3x_3 = 60$, what is the smallest possible value of the sum $x_1^2 + x_2^2 + x_3^2$?

Solution: By Cauchy-Schwarz, we know that $(x_1+2x_2+3x_3)^2 \leq (x_1^2+x_2^2+x_3^2)(1^2+2^2+3^2)$ Rearranging we find that $x_1^2 + x_2^2 + x_3^2 \geq \frac{60^2}{14} = \frac{1800}{7}$.

Exercises

1. Show that $x + \frac{1}{x} \geq 2$ for all $x > 0$.

2. Demonstrate that if $a_1 a_2 \cdots a_n = 1$, then $a_1 + a_2 + \cdots + a_n \geq n$.

3. Prove that for $a, b, c > 0$, $(a+b)(a+c)(b+c) \geq 8abc$.

4. Let b and h denote the base of a triangle whose area is 200. Compute the minimum value of $b + h$.

5. Find the minimum value of $\frac{9x^2 \sin^2 x + 4}{x \sin x}$ for $0 < x < \pi$.

6. (Mandlebrot 1998/2) Determine the minimum value of the sum $\frac{a}{2b} + \frac{b}{4c} + \frac{c}{8a}$ for positive a, b, c.

7. Find the minimum value of the function $f(x, y, z) = \frac{x}{y} + \sqrt{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}}$.

8. If a, b, c are positive real numbers, find the minimum value of the quantity

$$\frac{c}{a} + \frac{a}{b+c} + \frac{b}{c}$$

9. Prove Titu's Lemma from the Cauchy-Schwarz inequality:

$$\frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} \leq \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n}$$

10. Let $a, b, c, d > 0$ such that $a + b + c + d = 1$. Prove that

$$\frac{1}{4a+3b+c} + \frac{1}{3a+b+4d} + \frac{1}{a+4c+3d} + \frac{1}{4b+3c+d} \geq 2.$$

11. Let a, b, c be real numbers. Prove the inequality

$$2a^2 + 3b^2 + 6c^2 \geq (a+b+c)^2.$$

12. Let x, y, z be positive real numbers. Prove the inequality

$$\frac{2}{x+y} + \frac{2}{x+z} + \frac{2}{y+z} \geq \frac{9}{x+y+z}$$

Brief Solutions

1. Trivial.
2. Trivial.
3. Apply AM-GM to each term in parentheses on the LHS.
4. Trivial (40).
5. Divide the fraction into two, then AM-GM. (12).
6. Trivial (3/4).
7. Apply AM-GM to x/y, 2 copies of 1/2 sqrt(y/z), and 3 copies of 1/3 cbrt(z/x). $(2^{2/3} * 3^{1/2})$
8. Add and subtract 1. Add the one to the last fraction. Then use AM-GM. (2)
9. Let $u_i = \frac{a_i}{\sqrt{b_i}}, v_i = \sqrt{b_i}$. Then CS to u_i, v_i gives the desired result.
10. AM-HM.
11. CS to (1/2,1/3,1/6) and $(2a^2, 3b^2, 6c^2)$
12. Titu's.