A Physical Proof of the Fermat Point

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This proof was communicated to me by Colin Tang.

Consider a triangle ABC such that we want to choose a point P that minimizes the sum of distances from P to A, B, C, or in other words, minimizes $PA + PB + PC$. This point P is called the Fermat Point. It is well known mathematically that $\angle APB = \angle BPC = \angle CPA = 120°$. Here we will make an argument based on physical principles.

Construct the triangle ABC on a horizontal frictionless plane and put holes at each vertex A, B, C. Now, consider a system of three separate equal masses, each connected by long massless strings to a pivot point. Have one string pass through one of the holes, such that there is a mass under each vertex, and let the system come to equilibrium.

Lemma. At equilibrium, the pivot point is at the Fermat point of triangle ABC.
Proof. A system always seeks to minimize energy, and in this case, can do so by minimizing gravitational potential energy. In order to do so, the three weights must be closest to the ground, and this happens when the length of string below the plane is longer, i.e when the length of string on the triangle is shorter. A minimum is reached when the length of string on the triangle is minimized, which is the definition of the Fermat point.

Corollary. The Fermat point satisfies $\angle APB = \angle BPC = \angle CPA = 120°$.
Proof. Consider the forces acting on the pivot point. The vector sum of the three forces acting on the pivot point is equal to zero. Since each mass is equal, this means that the forces must be exactly $120°$ away from each other.