A Physical Proof of the Fermat Point

Edward Jin

This proof was communicated to me by Colin Tang.

Consider a triangle $ABC$ such that we want to choose a point $P$ that mini-
mizes the sum of distances from $P$ to $A, B, C$, or in other words, minimizes
$PA + PB + PC$. This point $P$ is called the Fermat Point. It is well known
mathematically that $\angle APB = \angle BPC = \angle CPA = 120^\circ$. Here we will make an
argument based on physical principles.

Construct the triangle $ABC$ on a horizontal frictionless plane and put holes
at each vertex $A, B, C$. Now, consider a system of three separate equal masses,
each connected by long massless strings to a pivot point. Have one string pass
through one of the holes, such that there is a mass under each vertex, and let
the system come to equilibrium.

Lemma. At equilibrium, the pivot point is at the Fermat point of triangle
$ABC$.

Proof. A system always seeks to minimize energy, and in this case, can do
so by minimizing gravitational potential energy. In order to do so, the three
weights must be closest to the ground, and this happens when the length of
string below the plane is longer, i.e when the length of string on the triangle
is shorter. A minimum is reached when the length of string on the triangle is
minimized, which is the definition of the Fermat point.

Corollary. The Fermat point satisfies $\angle APB = \angle BPC = \angle CPA = 120^\circ$.

Proof. Consider the forces acting on the pivot point. The vector sum of the
three forces acting on the pivot point is equal to zero. Since each mass is equal,
this means that the forces must be exactly $120^\circ$ away from each other.