HMMT February 2020 Integration Bee Finals

Edward Jin

February 15, 2020
Sponsored by Five Rings Capital

Problem 0

Evaluate the following Integral:

$$\int_0^1 1 \, dx$$

Solution 0

Answer:

$$1$$

Problem 1

Evaluate the following Integral:

$$\int_0^{\pi/2} \cos(x) \sin^{-1}(\cos(x))\, dx$$

Solution 1

Answer:

$$1$$

Problem 2

Evaluate the following Integral:

$$\int_0^{\pi/4} \frac{\tan(x)\sec^2(x)\,dx}{\sqrt{2-\tan^2 x}}\,dx$$

Solution 2

Answer:

$$\sqrt{2} - 1$$

Problem 3

Evaluate the following Integral:

$$\int \frac{\tan^{-1}(x)}{x^2}\,dx$$

Solution 3

Answer:

$$-\frac{1}{2}\log(x^2+1)+\log x-\frac{\tan^{-1}(x)}{x}$$

Problem 4

Evaluate the following Integral:

$$\int_0^1 \sin^{-1}(\sqrt{x})\, dx$$

Solution 4

Answer:

$$\frac{\pi}{4}$$

Problem 5

Evaluate the following Integral:

$$\int_0^1 \sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt[3]{\cdots}}}}\,dx$$

Solution 5

Answer:

$$\frac{2}{3}$$

Problem 6

Evaluate the following Integral:

$$\int_0^{2\pi} \cos(x)\cos(2x)\cos(3x)\cos(4x)\cos(5x)\cos(6x)\,dx$$

Solution 6

Answer:

$$0$$

Problem 7

Evaluate the following Integral:

$$\int \sin^x(x)\left(\log \sin x + x \cot x\right)\, dx$$

Solution 7

Answer:

$$\sin^x(x) + C$$

Problem 8

Evaluate the following Integral:

$$\int \frac{\sin(x)e^{\sec(x)}}{\cos^2(x)}\,dx$$

Solution 8

Answer:

$$e^{\sec x} + C$$

Problem 9

Evaluate the following Integral:

$$\int_0^{2\pi} \left(\frac{\sin(3x)}{\sin(x)}\right)^3 \, dx$$

Solution 9

Answer:

$$14\pi$$

Problem 10

Evaluate the following Integral:

$$\int \sinh^2 x \, dx$$

Solution 10

Answer:

$$\frac{1}{4}\sinh(2x) - \frac{x}{2} + C$$

Problem 11

Evaluate the following Limit:

$$\lim_{N\to\infty}\int_0^{\pi/2}\frac{\sin(Nx)}{\sin x}\,dx$$

Solution 11

Answer:

$$\frac{\pi}{2}$$

Problem 12

Evaluate the following Integral:

$$\int \log(1 + x^2)\, dx$$

Solution 12

Answer:

$$x \log(1 + x^2) - 2x + 2\tan^{-1}(x) + C$$

Problem 13

Evaluate the following Integral:

$$\int \sec(x)\cosh(x)(\cosh(x)\tan(x) + 2\sinh(x))\, dx$$

Solution 13

Answer:

$$\sec x \cosh^2 x + C$$

Problem 14

Evaluate the following Integral:

$$\int_0^e W(x)\, dx$$

where $W(x)$ is the *Lambert-W function*, defined as the inverse of $f(x) = xe^x$ (i.e. $W(x)e^{W(x)} = x$).

Solution 14

Answer:

$$e - 1$$

Problem 15

Evaluate the following Integral:

$$\int e^x x^{e^x} \left(\log x + \frac{1}{x} \right) \, dx$$

Solution 15

Answer:

$$x^{e^x} + C$$

Problem 16

Evaluate the following Integral:

$$\int \frac{\sin(1/x)}{x^3}\,dx$$

Solution 16

Answer:

$$\frac{\cos(1/x)}{x} - \sin(1/x) + C$$

Problem 17

Evaluate the following Integral:

$$\int \sin^4 x + \cos^4 x \, dx$$

Solution 17

Answer:

$$\frac{3}{4}x + \frac{1}{16}\sin(4x) + C$$

Problem 18

Evaluate the following Integral:

$$\int_0^{2\pi} \cos^{10} x \, dx$$

Solution 18

Answer:

$$\frac{63}{128}\pi$$

Problem 19

Evaluate the following Integral:

$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

Solution 19

Answer:

$$\sqrt{2}\tanh^{-1}\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}\log\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$$

Problem 20

Evaluate the following Integral:

$$\int \sqrt{1+x^2}\,dx$$

Solution 20

Answer:

$$\frac{1}{2}\sinh^{-1}(x)+\frac{1}{2}x\sqrt{1+x^2}+C$$

or

$$\frac{1}{2}\log(x+\sqrt{1+x^2})+\frac{1}{2}x\sqrt{1+x^2}+C$$

Problem 21

Evaluate the following Integral:

$$\int_0^1 \left(\frac{1}{2} + \frac{x}{3} + \frac{x^2}{8} + \frac{x^3}{40} + \cdots + \frac{x^n}{n!(n+2)} + \cdots\right) dx$$

where the sum is infinite.

Solution 21

Answer:

$$e - 2$$

Problem 22

Define the Gamma function as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

The Gamma function additionally satisfies the property

$$\Gamma(x)\Gamma(1-x) = \pi \csc(\pi z).$$

Given the above information, evaluate the following Integral:

$$\int_0^1 \log \Gamma(x) dx$$

Solution 22

Answer:

$$\frac{1}{2}\log(2\pi)$$

Problem 23

Evaluate the following Integral:

$$\int \frac{dx}{x^{2/3} + x^{4/3}}$$

Solution 23

Answer:

$$3\tan^{-1}(\sqrt[3]{x}) + C$$

Problem 24

Evaluate the following Integral:

$$\int_0^\infty \frac{x}{(x^2+1)(a^2x^2+1)}\,dx$$

for positive a.

Solution 24

Answer:

$$\frac{\log a}{a^2 - 1}$$