Problem 1
Define an such that \( a_1 = \sqrt{3} \) and for all integers \( i \), \( a_{i+1} = a_i^2 - 2 \). What is \( a_{2016} \)?

We can find the first three terms through the recursive definition: \( a_1 = \sqrt{3}, a_2 = 1, a_3 = -1 \). Note that since \( -1 = (-1)^2 - 2 \), we have that \( a_n = a_3 = -1 \) for \( n \geq 3 \). Hence \( a_{2016} = -1 \).
**Problem 2**

Jennifer wants to do origami, and she has a square of side length 1. However, she would prefer to use a regular octagon for her origami, so she decides to cut the four corners of the square to get a regular octagon. Once she does so, what will be the side length of the octagon Jennifer obtains?

First, draw a diagram and label the diagram, as shown above. The interior angle of a polygon with \( n \) sides is \( \frac{180(n-2)}{n} \). Therefore, the interior angle of a polygon with 8 sides is \( \frac{180(8-2)}{8} = 135 \) degrees. This means that the triangles in the corner have angles of \( 180 - 135 = 45 \) degrees, making them \( 45^\circ - 45^\circ - 90^\circ \) triangles.

Furthermore, a regular octagon has equal side lengths. Call this side length \( x \). As seen from the diagram, the hypotenuse of one of the corner triangles is a side length of the octagon. From the \( 45^\circ - 45^\circ - 90^\circ \) triangle proportionality, we know that the ratio of the triangle’s leg to the triangle’s hypotenuse if 1 to \( \sqrt{2} \). Therefore, if the hypotenuse has length \( x \), the leg has length \( \frac{x}{\sqrt{2}} \).

From the diagram, it is easy to see that the side length of the square is made up of the legs of two of the corner triangles and one side of the octagon.

\[
\text{side length of square} = 2 \cdot (\text{leg of corner triangle}) + (\text{side length of the octagon})
\]

Substituting in the expressions we derived for the leg of a corner triangle and the side length of the octagon,

\[
1 = 2 \cdot \frac{x}{\sqrt{2}} + x
\]

\[
1 = \sqrt{2}x + x
\]

\[
x = \frac{1}{1 + \sqrt{2}}
\]

Therefore, the side length of the octagon is \( \frac{1}{1 + \sqrt{2}} \).
Problem 3

A little boy takes a 12 in long strip of paper and makes a Mobius strip out of it by tapping the ends together after adding a half twist. He then takes a 1 inch long train model and runs it along the center of the strip at a speed of 12 inches per minute. How long does it take the train model to make two full complete loops around the Mobius strip? A complete loop is one that results in the train returning to its starting point.

A Mobius strip is a strip of paper where you twist one end and connect it to the other, making a loop with a twist in the middle. Thus, it technically has only one side; if you drag your pencil along a line on the strip, you would make draw on both sides of the paper before returning to the start of your line. Thus, if you have a 12 in. strip making a Mobius strip, it takes 24 inches (one loop takes you to the other side of the paper, the second loop brings you back to the start) to make a loop. Two loops is thus 48 inches, and moving at 12 inches per minute means it takes you four minutes to finish your two loops.
Problem 4

How many graphs are there on 6 vertices with degrees 1,1,2,3,4,5?

Suppose we let the 6 vertices of the graphs be $ABCDEF$, and let $A$ be the vertex with degree of 5, thus connecting $A$ to each of $BCDEF$, making them each with a degree of 1. Now we assume $B$ to be the vertex with degree of 5, connecting three of the remaining four vertices, making there a maximum of one vertex with a degree of one, making the requirement of two degree ones impossible, rendering there to be zero solutions.
**Problem 5**

Let $ABC$ be a right triangle with $AB = BC = 2$. Let $ACD$ be a right triangle with angle $DAC = 30$ degrees and angle $DCA = 60$ degrees. Given that $ABC$ and $ACD$ do not overlap, what is the area of triangle $BCD$?

We can use properties of 30-60-90 triangles and 45-45-90 triangles to immediately find that $AC = 2\sqrt{2}, CD = \sqrt{2}, AD = \sqrt{6}$. Since $\angle BCD = 45^\circ + 60^\circ = 105^\circ$, we can use the sine area formula to obtain the area of the triangle as 

$$\frac{1}{2} \cdot \sqrt{2} \cdot 2 \cdot \sin(105^\circ) = \sqrt{2} \cdot \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{\sqrt{3} + 1}{2}.$$
Problem 6

How many integers less than 400 have exactly 3 factors that are perfect squares?

Our first instinct is to look at all the perfect squares that are less than 400, which go from 1 to 361. We can check that only 16 and 81 satisfy the condition, so, we then look for multiples of 16 and 81 that work.

Let $S = \{1, 2, 3\}$ and $T = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23\}$.

We see that $\{81 \cdot x \mid x \in S\} \cup \{16 \cdot y \mid y \in T\}$, are the only numbers that satisfy the condition simply by manually verifying (a euphemism for bashing). Hence there are $3 + 16 = 19$ solutions.
**Problem 7**

Suppose $f(x, y)$ is a function that takes in two integers and outputs a real number, such that it satisfies

$$f(x, y) = \frac{f(x, y + 1) + f(x, y - 1)}{2}$$

$$f(x, y) = \frac{f(x + 1, y) + f(x - 1, y)}{2}$$

What is the minimum number of pairs $(x, y)$ we need to evaluate to be able to uniquely determine $f$?

This one is a little hard to describe, but imagine placing two points vertically on the graph. This allows you to determine the value of any point on that vertical line. Now, pick a horizontal point with a separate $y$ value than either of the aforementioned points, and using the vertical points find the value of the point between them that shares a $y$-value with the new point. Now you can determine any point on that horizontal line. Finally choose a point that shares no $x$ or $y$ values with either point predetermined. Using similar techniques, you can find the scale for a new vertical and a new horizontal line. Once you have two vertical lines and two horizontal lines, you can determine the value of any vertical or horizontal line on the graph and thus can find any point. This means that the number of pairs we need is $4$. 
Problem 8

How many ways are there to divide 10 candies between 3 Berkeley students and 4 Stanford students, if each Berkeley student must get at least one candy? All students are distinguishable from each other; all candies are indistinguishable.

There is a restriction that each Berkeley student must receive at least one piece of candy; to take care of this restriction, first distribute each of the 3 Berkeley students 1 piece of candy. There are 10 - 3 = 7 pieces of candy left. Now, the problem becomes: distribute 7 candies amongst 7 distinguishable people, where it is acceptable for a person to receive zero pieces of candy. Notice that the condition in which 3 students attend Berkeley and 4 students attend Stanford is irrelevant now.

To find the number of ways to distribute 7 candies to 7 distinguishable people, where it is acceptable for a person to receive zero pieces of candy, we utilize Starts and Bars. Let the candies be represented by stars. Since we want to partition the candy into 7 spaces, where a space represents a person. We need 6 bars, as 6 bars creates 7 spaces.

Now, our problem becomes: how many ways are there to rearrange 13 objects (7 stars and 6 bars)? There are 13 spaces for the 7 stars and 6 bars; choose 6 spaces for the bars, and place the stars in the empty spaces. Therefore, there are $\binom{13}{6} = 1716$ ways to distribute the candies.
Problem 9
How many subsets (including the empty-set) of \{1, 2, ..., 6\} do not have three consecutive integers?

To do this, we will use complementary counting. First, we look for the total number of subsets of the set \{1, 2, 3, 4, 5, 6\}. Each number in the set has two choices: appear in the chosen subset, not appear in the chosen subset. Therefore, there are \(2^6\) subsets in total. Now, we count the number of subsets of \{1, 2, 3, 4, 5, 6\} which do have three consecutive integers using case work. Notice, obviously, that it is impossible for the empty subset or a subset with 1 or 2 integers to have three consecutive integers.

**Case 1:** 3 integer subsets which contain three consecutive integers (4 total):
\{1, 2, 3\} \{2, 3, 4\} \{3, 4, 5\} \{4, 5, 6\}

**Case 2:** 4 integer subsets which contain three consecutive integers (9 total):
\{1, 2, 3, 4\} \{1, 2, 3, 5\} \{1, 2, 3, 6\} \{1, 3, 4, 5\} \{1, 4, 5, 6\} \{2, 3, 4, 5\} \{2, 3, 4, 6\} \{2, 4, 5, 6\} \{3, 4, 5, 6\}

**Case 3:** 5 integer subsets which contain three consecutive integers:
All of the 5 integer subsets will contain three consecutive integers, so we just need to find the number of 5 integer subsets. We can do this by choosing one integers out of our set to not include: \(_6^1\). Therefore, there are \(_6^1\) = 6 integer subsets which contain three consecutive integers.

**Case 4:** 6 integer subsets which contain three consecutive integers (1 total):
\{1, 2, 3, 4, 5, 6\}.

Adding all these cases, there are 20 subsets which contain three consecutive integers. We are looking for the number of subsets which do not contain three consecutive integers: \(2^6 - 20 = 44\).
Problem 10

What is the smallest possible perimeter of a triangle with integer coordinate vertices, area \( \frac{1}{2} \), and no side parallel to an axis?

According to Pick’s Theorem, for a triangle with integer coordinate vertices, the area given is
\[
A = I + \frac{B}{2} - 1,
\]
with \( I \) being the amount of interior points and \( B \) being the amount of points on the border. Given that this is a triangle with area \( \frac{1}{2} \), we can substitute the formula as
\[
\frac{1}{2} = I + \frac{3 + B_1}{2} - 1,
\]
with \( B_1 \) being extra vertices on the border or the triangle. We then get
\[
1 = 2I + 3 + B_1 - 2,
\]
\[
0 = 2I + B_1,
\]
and since \( I \) and \( B_1 \) are both non-negative integers, both \( I \) and \( B_1 \) is 0, implying that there will be no interior or border points in the triangle. Since all points in the area \([0, 2] \times [0, 2]\) won’t make a triangle that either fits the given requirement of no parallel to an axis or the requirement above, the smallest triangle that fits both conditions is the triangle with vertices of \((0, 0), (1, 1), (2, 3)\) with side length of \( \sqrt{2}, \sqrt{5}, \) and \( \sqrt{13} \), giving the perimeter of \( \sqrt{2} + \sqrt{5} + \sqrt{13} \).
Problem 11

Circles $C_1$ and $C_2$ intersect at points $X$ and $Y$. Point $A$ is a point on $C_1$ such that the tangent line with respect to $C_1$ passing through $A$ intersects $C_2$ at $B$ and $C$, with $A$ closer to $B$ than $C$, such that $2016 \cdot AB = BC$. Line $XY$ intersects line $AC$ at $D$. If circles $C_1$ and $C_2$ have radii of 20 and 16, respectively, find the ratio of $\sqrt{1 + BC/BD}$. 
Problem 12
Consider a solid hemisphere of radius 1. Find the distance from its center of mass to the base.

We can describe the hemisphere as a solid of revolution of the function $y = \sqrt{1-x^2}$ with $0 \leq x \leq 1$. The center of mass is then given by:

$$\frac{\int_0^1 x \pi y^2 \, dx}{\int_0^1 \pi y^2 \, dx} = \frac{\int_0^1 x(1-x^2) \, dx}{\int_0^1 1-x^2 \, dx} = \frac{\frac{1}{2} - \frac{1}{4}}{1 - \frac{1}{3}} = \frac{3}{8}$$
Problem 13

Consider an urn containing 51 white and 50 black balls. Every turn, we randomly pick a ball, record the color of the ball, and then we put the ball back into the urn. We stop picking when we have recorded \( n \) black balls, where \( n \) is an integer randomly chosen from \{1, 2, ..., 100\}. What is the expected number of turns?

To do this question, let us first find the expected value to pick just one black ball \((n = 1)\). The expected value can be written as:

\[
EV = \frac{50}{101} \cdot \sum_{t=0}^{\infty} (t+1) \left( \frac{51}{101} \right)^t = \frac{50}{101} \left( 1 + 2 \cdot \left( \frac{51}{101} \right) + 3 \cdot \left( \frac{51}{101} \right)^2 + 4 \cdot \left( \frac{51}{101} \right)^3 + \cdots \right)
\]

This is kind of similar to a geometric series, where \( r = \frac{51}{101} \), but it is much more difficult because each term is also multiplied by a consecutive integer. Well, we know:

\[
\sum_{t=0}^{\infty} r^t = 1 + r + r^2 + \cdots = \frac{1}{1 - r}
\]

If you take the derivative of both sides, you get:

\[
1 + 2r + 3r^2 + \cdots = \frac{1}{(1 - r)^2}
\]

Using this formula for our series, the sum of the series is:

\[
\frac{50}{101} \cdot \frac{1}{(1 - \frac{51}{101})^2} = \frac{101}{50}
\]

Now, if you know the expected value of turns to do something once, the expected value to do it \( n \) amount of times is just the expected value of doing it once times \( n \). The average value between 1 and 100 is \( \frac{101}{2} \), so the expected value for all \( n \) from 1 to 100 is \( \frac{101}{50} \cdot \frac{101}{2} = \frac{102.01}{2} \).
Problem 14
Consider the set of axis-aligned boxes in $\mathbb{R}^d$, $B(a, b) = \{x \in \mathbb{R}^d : \forall i, a_i \leq x_i \leq b_i\}$ where $a, b \in \mathbb{R}^d$. In terms of $d$, what is the maximum number $n$, such that there exists a set of $n$ points $S = \{x_1, ..., x_n\}$ such that no matter how one partition $S = P \cup Q$ with $P, Q$ disjoint and $P, Q$ can possibly be empty, there exists a box $B$ such that all the points in $P$ are contained in $B$, and all the points in $Q$ are outside $B$?
Problem 15

Let $s_1, s_2, s_3$ be the three roots of $x^3 + x^2 + \frac{9}{2}x + 9$.

$$\prod_{i=1}^{3}(4s_i^4 + 81)$$

can be written as $2^a3^b5^c$. Find $a + b + c$.

Vieta’s formula comes to mind: we know $s_1 + s_2 + s_3 = -1$, $s_1s_2 + s_1s_3 + s_2s_3 = \frac{9}{2}$, and $s_1s_2s_3 = -9$. However, this does not help us much, yet.

The quartics look messy, so we replace $s_i^4$ with $a_i$. Therefore, we are now looking for:

$$\prod_{i=1}^{3}(4s_i^4 + 81) = (4s_1^4 + 81)(4s_2^4 + 81)(4s_3^4 + 81)$$

$$= (4a_1 + 81)(4a_2 + 81)(4a_3 + 81)$$

$$= 64(a_1a_2a_3) + 16 \cdot 81(a_1a_2 + a_1a_3 + a_2a_3) + 4 \cdot 81^2(a_1 + a_2 + a_3) + 81^3$$

We know that $a_1a_2a_3 = (s_1s_2s_3)^4$, which we can easily find. By Vieta’s formula, $s_1s_2s_3 = -9$, so $a_1a_2a_3 = (-9)^4 = 9^4$.

In a similar manner, we can find $a_1a_2 + a_1a_3 + a_2a_3$.

$$a_1a_2 + a_1a_3 + a_2a_3 = s_1^4s_2^4 + s_1^4s_3^4 + s_2^4s_3^4$$

$$= (s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2)^2 - 2(s_1^2s_2^2s_3^2)(s_1^2 + s_2^2 + s_3^2)$$

However, we cannot find the values of $s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2$ or $s_1^2 + s_2^2 + s_3^2$ easily through Vieta’s formula. Thus, more algebra manipulation is needed first.

$$s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2 = (s_1s_2 + s_1s_3 + s_2s_3)^2 - 2s_1s_2s_3(s_1 + s_2 + s_3)$$

$$= \left(\frac{9}{2}\right)^2 - 2(-9)(-1)$$

$$= \frac{9}{4}$$

Similarly, we can find $s_1^2 + s_2^2 + s_3^2$.

$$s_1^2 + s_2^2 + s_3^2 = (s_1 + s_2 + s_3)^2 - 2(s_1s_2 + s_1s_3 + s_2s_3)$$

$$= (-1)^2 - 2\left(\frac{9}{2}\right)$$

$$= -8.$$
Now, we can go back to our original goal of finding $a_1a_2 + a_1a_3 + a_2a_3$.

$$a_1a_2 + a_1a_3 + a_2a_3 = s_1^4s_2^4 + s_1^4s_3^4 + s_2^4s_3^4$$

$$= (s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2)^2 - 2(s_1^2s_2^2s_3^2)(s_1^2 + s_2^2 + s_3^2)$$

$$= \left(\frac{9}{4}\right)^2 - 2 \cdot (-9)^2 \cdot (-8)$$

$$= \frac{81}{16} + 81 \cdot 16$$

$$= \frac{81(16^2 + 1)}{16}$$

Finally, we need to look for $a_1 + a_2 + a_3$.

$$a_1 + a_2 + a_3 = s_1^4 + s_2^4 + s_3^4$$

$$= (s_1^2 + s_2^2 + s_3^2)^2 - 2(s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2)$$

Previously, we have already found the values of $s_1^2 + s_2^2 + s_3^2$ and $s_1^2s_2^2 + s_1^2s_3^2 + s_2^2s_3^2$, so we can just plug them in.

$$a_1 + a_2 + a_3 = s_1^4 + s_2^4 + s_3^4$$

$$= (-8)^2 - 2 \cdot \left(\frac{9}{4}\right)$$

$$= \frac{119}{2}$$

Now, we can finally go back to our original initiative, which was to find

$$\prod_{i=1}^{3}(4a_i + 81) = (4a_1 + 81)(4a_2 + 81)(4a_3 + 81)$$

Plugging in the values we have found for $a_1a_2a_3, a_1a_2 + a_1a_3 + a_2a_3$ and $a_1 + a_2 + a_3$:

$$\prod_{i=1}^{3}(4a_i + 81) = (4a_1 + 81)(4a_2 + 81)(4a_3 + 81)$$

$$= 64(a_1a_2a_3) + 16 \cdot 81(a_1a_2 + a_1a_3 + a_2a_3) + 4 \cdot 81^2(a_1 + a_2 + a_3) + 81^3$$

$$= 64(9^4) + 16 \cdot 81\left(\frac{81(16^2 + 1)}{16}\right) + 4 \cdot 81^2 \left(\frac{119}{2}\right) + 81^3$$

$$= 3^8(2^6 + 16^2 + 1 + 2 \cdot 119 + 81)$$

$$= 3^8(640)$$

$$= 3^8 \cdot 2^7 \cdot 5$$

$$= 2^7 \cdot 3^8 \cdot 5$$

Therefore, $\prod_{i=1}^{3}(4s_i^4 + 81) = 2^7 \cdot 3^8 \cdot 5$, and the sum of the powers is $7 + 8 + 1 = 16$.