A Thermodynamic Proof of AM-GM

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I first saw this proof on a Math Stackexchange forum.

The AM-GM inequality states that, for positive reals $x_1, x_2, \ldots, x_n$ that

$$\sum_i \frac{x_i}{n} \geq \sqrt[n]{\prod_i x_i}$$

Here we present a proof of this fact based off thermodynamic principles.

Consider n beakers of water, each kept at a temperature T_i. When all the beakers are mixed, then the final temperature of the system will be equal to the arithmetic mean $T = \sum_i \frac{T_i}{n}$ as a consequence of the first law of thermodynamics (no heat is lost).

Now, consider the change in entropy upon mixing the n beakers. We can consider this change as a two-step process, since entropy is a state function: First, changing the temperature from T_i to T, and second, mixing all the beakers. Since all the beakers are then identical after the first step and are at the same temperature, then the second step does not contribute to the entropy change of the system. The entropy change of the first step is given by $\Delta S = C \log\left(\frac{T}{T_i}\right)$, where C is the heat capacity of the water. Then, the total entropy change is given by

$$\Delta S_{tot} = C \sum_i \log\left(\frac{T}{T_i}\right) = C \log\left(\frac{T^n}{\prod_i T_i}\right).$$

By the second law of thermodynamics, this quantity is greater than or equal to zero, and thus we have that

$$T^n \geq \prod_i T_i \implies T = \sum_i \frac{T_i}{n} \geq \sqrt[n]{\prod_i T_i}$$

with equality iff there is no entropy change, i.e every beaker started off at the same temperature T.