Regular Policies in Stochastic Optimal Control and Abstract Dynamic Programming

Dimitri P. Bertsekas

Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology

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**Classical Total Cost Stochastic Optimal Control (SOC)**

**System:**  
\[ x_{k+1} = f(x_k, u_k, w_k) \]

- \( x_k \): State at time \( k \), from some space \( X \)
- \( u_k \): Control at time \( k \), from some space \( U \)
- \( w_k \): Random “disturbance” at time \( k \), from a countable space \( W \), with \( p(w_k \mid x_k, u_k) \) given

**Policies:**  
\( \pi = \{ \mu_0, \mu_1, \ldots \} \)

- Each \( \mu_k \) maps states \( x_k \) to controls \( u_k = \mu_k(x_k) \in U(x_k) \) (a constraint set)
- Cost of \( \pi \) starting at \( x_0 \), with discount factor \( \alpha \in (0, 1] \):

\[
J_{\pi}(x_0) = \limsup_{N \to \infty} E \left\{ \sum_{k=0}^{N} \alpha^k g(x_k, \mu_k(x_k), w_k) \right\}
\]

- Optimal cost starting at \( x_0 \): \( J^*(x_0) = \inf_{\pi} J_{\pi}(x_0) \)
- Optimal policy \( \pi^* \): Satisfies \( J_{\pi^*}(x) = J^*(x) \) for all \( x \in X \)

**Bellman’s (Optimality) Equation:**

\[
J^*(x) = \inf_{u \in U(x)} E \left\{ g(x, u, w) + \alpha J^*(f(x, u, w)) \right\}, \quad \forall x \in X
\]
### Three Main Classes of Total Cost SOC Problems

<table>
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<tr>
<th>Discounted:</th>
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<tbody>
<tr>
<td>$\alpha &lt; 1$ and bounded $g$</td>
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<tr>
<td>Dates to 50s (Bellman, Shapley)</td>
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<tr>
<td>Nicest results; key fact is contraction property in Bellman’s equation</td>
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<tr>
<th>Undiscounted ($g \leq 0$ or $g \geq 0$):</th>
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<tr>
<td>$N$-step horizon costs are going ↓ or ↑ with $N$</td>
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<tr>
<td>Dates to 60s (Blackwell, Strauch); positive and negative DP</td>
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<td>Not nearly as powerful results compared with the discounted case</td>
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<th>Stochastic Shortest Path (SSP):</th>
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<tr>
<td>Dates to 60s (Eaton-Zadeh, Derman, Pallu de la Barriere)</td>
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<td>Also known as first passage or transient programming</td>
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<td>Aim is to reach a special termination state at min expected cost</td>
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<tr>
<td>Under favorable assumptions (including finite state space), results are almost as strong as for the discounted case (some contraction properties)</td>
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<tr>
<td>In general, very complex behavior is possible</td>
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A deterministic shortest path problem

Bellman’s equation: \( J(1) = \min \{ b + J(0), J(1) \} \), \( J(0) = J(0) \)
Solutions with \( J(0) = 0 \): All \( J(1) \leq b \)

Value iteration (VI) starting from any \( J_0 \) with \( J_0(0) = 0 \)

- VI for the terminating policy: \( J_{\mu, k}(1) = b \) (works)
- VI for the nonterminating policy: \( J_{\mu', k+1}(1) = J_{\mu', k}(1) \) (fails)
- VI for the entire problem: \( J_{k+1}(1) = \min \{ b, J_k(1) \} \)
- If \( b < 0 \): \( J_k(1) \to J^*(1) \) starting with \( J_0(1) \geq b \) (works depending on \( J_0 \))
- If \( b > 0 \): \( J_k(1) \to J^*(1) \) only if \( J_0(1) = 0 \); starting from \( J_0(1) \geq b \), \( J_k(1) \to J_{\mu}(1) \)

Policy iteration (PI) starting from \( \mu \)

- If \( b < 0 \): Oscillates between \( \mu \) and \( \mu' \). If \( b > 0 \): Converges to suboptimal \( \mu \)
A stochastic shortest path problem (from Bertsekas and Yu, 2015)

For $p = 1$: $J_p(1) = J_p(2) = 1$

For $p = 0$: $J_p(1) = J_p(5) = 1$

For $p = 1/2$ (which is optimal): $J_p(2) = J_p(5) = 1$ BUT $J_p(1) = 0$

- The Bellman Eq. is violated at 1 for $p = 1/2$: $J_p(1) \neq pJ_p(2) + (1 - p)J_p(5)$
- Mathematically, the difficulty is that $\limsup E\{\cdot\} \neq E\{\limsup \{\cdot\}\}$

Consider the deterministic problem that chooses either $p = 1$ or $p = 0$

- Bellman’s equation $J^*(1) = \min\{J^*(2), J^*(5)\}$ is satisfied
- Introducing randomization
  - Lowers the optimal cost and invalidates Bellman’s equation
  - VI fails to converge to $J^*$ from any initial condition
What is the Root of the Anomalies?

A (partial) answer

The presence of policies that are not well-behaved in terms of VI (e.g., involve zero length cycles)

We call these policies “irregular” and we investigate

- What problems can they cause?
- Under what assumptions are they “harmless”?
D. P. Bertsekas, Abstract Dynamic Programming, Athena Scientific, 2013. (Regularity introduced in the context of semicontractive models, i.e., models where some policies involve contraction-like properties, and some do not.)


Outline

1. Regularity of Policy-State Pairs
2. Applications to Nonnegative Cost Optimal Control
3. S-Regular Stationary Policies - Policy Iteration
4. Applications to Stochastic Shortest Path (SSP) Problems
5. Abstract DP Formulation
**S-Regular stationary policy** \( \mu \) (\( S \) is a set of “value” functions on \( X \))

\( \mu \) is \( S \)-regular if it behaves well with respect to VI when started from \( S \), i.e., if VI using \( \mu \) converges to \( J_\mu \) starting from all \( J \in S \)

**Extension: \( S \)-Regular set of policy-state pairs**

A set \( C \) of policy-state pairs \((\pi, x)\) is \( S \)-regular if for all \((\pi, x) \in C\), VI using \( \pi \) and starting from \( x \) converges to \( J_\pi(x) \) starting from all \( J \in S \)

**Key idea: Exclude the irregular pairs** (i.e., optimize over the \( S \)-regular set)

- The (restricted) optimal cost function,
  
  \[
  J_C^*(x) = \inf_{(\pi, x) \in C} J_\pi(x),
  \]

  may be the unique solution of Bellman’s equation within \( S \), while \( J^* \) may not be!

- This is an interesting and (possibly) better-behaved problem

- Also \( J_C^* \) may be obtained by VI starting from within \( S \)
**Definition:** For a set of functions $S \subset E(X)$ (the set of extended real-valued functions on $X$), we say that a collection $C$ of policy-state pairs $(\pi, x_0)$ is $S$-regular if

$$J_\pi(x_0) = \limsup_{N \to \infty} E \left\{ \alpha^N J(x_N) + \sum_{k=0}^{N-1} \alpha^k g(x_k, \mu_k(x_k), w_k) \right\}, \quad \forall (\pi, x_0) \in C, \ J \in S$$

**Notes:**

- **Interpretation:** Addition of a terminal cost function $J \in S$ does not matter in the definition of $J_\pi(x_0)$
- **Example:** $\alpha = 1$ and $J \in S$ are s.t. $J(x_k) \to 0$ for generated $\{x_k\}$ under $\pi$
- **Example:** $\alpha < 1$ and $J \in S$ are s.t. $\{J(x_k)\}$: bounded for generated $\{x_k\}$ under $\pi$
- For $(\mu, x) \in C$ with $\mu$ stationary: $J_\mu(x)$ is obtained by VI starting with any $J \in S$
- A set $C$ of policy-state pairs $(\pi, x)$ may be $S$-regular for many different sets $S$

**Optimal cost function over regular collections**

$$J_C^*(x) = \inf_{\{\pi \mid (\pi, x) \in C\}} J_\pi(x), \quad x \in X$$
Abstract Notation - Connection with Abstract DP

- **Mapping of a stationary policy \( \mu \):** For any control function \( \mu \), with \( \mu(x) \in U(x) \) for all \( x \), and \( J \in E(X) \) define the mapping \( T_\mu : E(X) \mapsto E(X) \) by

\[
(T_\mu J)(x) = E\{g(x, \mu(x), w) + \alpha J(f(x, \mu(x), w))\}, \quad x \in X
\]

- **Value Iteration mapping:** For any \( J \in E(X) \) define the mapping \( T : E(X) \mapsto E(X) \)

\[
(TJ)(x) = \inf_{u \in U(x)} E\{g(x, u, w) + \alpha J(f(x, u, w))\}, \quad x \in X
\]

- **Note that Bellman’s equation is** \( J = TJ \) **and VI starting from** \( J \) **is** \( T^k J, k = 0, 1, \ldots \)

Abstract notation relating to regularity

- **We have**

\[
(T_{\mu_0} \cdots T_{\mu_{N-1}} J)(x_0) = E\left\{ \alpha^N J(x_N) + \sum_{k=0}^{N-1} \alpha^k g(x_k, \mu_k(x_k), w_k) \right\}
\]

- **\( \mathcal{C} \) is \( S \)-regular if**

\[
J_\pi(x) = \limsup_{N \to \infty} (T_{\mu_0} \cdots T_{\mu_N} J)(x), \quad \forall (\pi, x) \in \mathcal{C}, \; J \in S
\]
Let $\mathcal{C}$ be an $S$-Regular Collection

- For all fixed points $J'$ of $T$, and all $J \in E(X)$ such that $J' \leq J \leq \hat{J}$ for some $\hat{J} \in S$,

\[ J' \leq \liminf_{k \to \infty} T^k J \leq \limsup_{k \to \infty} T^k J \leq J^*_\mathcal{C} \]

- If in addition $J^*_\mathcal{C}$ is a fixed point of $T$ (a common case), then $J^*_\mathcal{C}$ is the largest fixed point
Characterizing VI Convergence

VI-Related Properties

- If $J_C^*$ is a fixed point of $T$, then VI converges to $J_C^*$ starting from any $J \in E(X)$ such that $J_C^* \leq J \leq \hat{J}$ for some $\hat{J} \in S$
- $J^*$ does not enter the picture! It is possible that VI converges to $J_C^*$ and not to $J^*$ (which may not even be a fixed point of $T$)
- When $J^*$ is a fixed point of $T$, a useful analytical strategy is to choose $C$ such that $J_C^* = J^*$. Then a VI convergence result is obtained
Cost nonnegativity, $g \geq 0$, provides a favorable structure (Strauch 1966)

- $J^*$ is the smallest fixed point of $T$ within $E^+(X)$
- VI converges to $J^*$ starting from 0 under some mild compactness conditions

Regularity-based analytical approach

- Define a collection $\mathcal{C}$ such that $J^*_\mathcal{C} = J^*$
- Define a set $S \subset E^+(X)$ such that $\mathcal{C}$ is $S$-regular
- Use the main result in conjunction with the fixed point property of $J^*$ to show that $J^*$ is the unique fixed point of $T$ within $S$
- Use the main result to show that the VI algorithm converges to $J^*$ starting from $J$ within the set $\{J \in S \mid J \geq J^*\}$
- Enlarge the set of functions starting from which VI converges to $J^*$ using a compactness condition

We use this approach in three major applications
Application to Nonnegative Cost Deterministic Optimal Control

Classic problem of regulation to a terminal set

- System: \( x_{k+1} = f(x_k, u_k) \). Cost per stage: \( g(x_k, u_k) \geq 0 \)
- Cost-free and absorbing terminal set of states \( X_s \) that we aim to reach or approach asymptotically at minimum cost

Assumptions

- \( J^*(x) > 0 \) for all \( x \notin X_s \)
- Controllability: For all \( x \) with \( J^*(x) < \infty \) and \( \epsilon > 0 \), there exists a policy \( \pi \) that reaches (in a finite number of steps) \( X_s \) starting from \( x \) with cost \( J_\pi(x) \leq J^*(x) + \epsilon \)

Define

- \( \mathcal{C} = \{ (\pi, x) \mid J^*(x) < \infty, \ \pi \text{ reaches } X_s \text{ starting from } x \} \)
- \( S = \{ J \in E^+(X) \mid J(x) = 0, \ \forall x \in X_s \} \)

Results

- \( J^* \) is the unique solution of Bellman’s equation within \( S \)
- VI converges to \( J^* \) starting from any \( J_0 \in S \) with \( J_0 \geq J^* \) (and for any \( J_0 \in S \) under a compactness condition)
Application to Nonnegative Cost Stochastic Optimal Control

Problem

- System: \( x_{k+1} = f(x_k, u_k, w_k) \)
- Cost per stage: \( g(x_k, u_k, w_k) \geq 0 \)

Define

- \( C = \{ (\pi, x) \mid J_\pi(x) < \infty \} \); so \( J^*_C = J^* \)
- \( S = \{ J \in E^+(X) \mid E_{x_0}^\pi \{ J(x_k) \} \to 0, \forall (\pi, x_0) \in C \} \)

Results

- \( J^* \) is the unique solution of Bellman’s equation within \( S \)
- VI converges to \( J^* \) starting from any \( J_0 \in S \) with \( J_0 \geq J^* \) (and for any \( J_0 \in S \) under a compactness condition)

An interesting consequence (Yu and Bertsekas, 2013)

If a function \( J \in E^+(X) \) satisfies \( J^* \leq J \leq cJ^* \) for some \( c \geq 1 \), VI converges to \( J^* \) starting from \( J \)
The problem with discount factor $\alpha < 1$

**Terminology and definitions**

- $X_f = \{ x \in X \mid J^*(x) < \infty \}$
- $\pi$ is stable from $x_0 \in X_f$ if there is a bounded subset of $X_f$ such that the sequence $\{x_k\}$ generated starting from $x_0$ and using $\pi$ lies with probability 1 within that subset.
- $C = \{ (\pi, x) \mid x \in X_f, \pi \text{ is stable from } x \}$
- $J \in E^+(X)$ is bounded on bounded subsets of $X_f$ if for every bounded subset $\tilde{X} \subset X_f$ there is a scalar $b$ such that $J(x) \leq b$ for all $x \in \tilde{X}$.
- $S = \{ J \in E^+(X) \mid J \text{ is bounded on bounded subsets of } X_f \}$

**Assumption**

$C$ is nonempty, $J^* \in S$, and for every $x \in X_f$ and $\epsilon > 0$, there exists a policy $\pi$ that is stable from $x$ and satisfies $J_\pi(x) \leq J^*(x) + \epsilon$.

**Results**

- $J^*$ is the unique solution of Bellman’s equation within $S$.
- VI converges to $J^*$ starting from any $J_0 \in S$ with $J_0 \geq J^*$ (and for any $J_0 \in S$ under a compactness condition).
S-Regular Collections Involving Stationary Policies

Definitions: For a nonempty set of functions $S \subset E(X)$

- We say that a stationary policy $\mu$ is $S$-regular if $T^k_\mu J \to J_\mu$ for all $J \in S$
- Equivalently, $\mu$ is $S$-regular if the set $C = \{(\mu, x) \mid x \in X\}$ is $S$-regular
- Let $\mathcal{M}_S$ be the set of policies that are $S$-regular, and define

$$J^*_S(x) = \inf_{\mu \in \mathcal{M}_S} J_\mu(x), \quad \forall x \in X$$

- Equivalently, $J^*_S = J^*_C$ when $C = \mathcal{M}_S \times X$

VI Convergence Result

Given a set $S \subset E(X)$, assume that

- There exists at least one $S$-regular policy
- $J^*_S$ is a fixed point of $T$

Then $T^k J \to J^*_S$ for every $J \in E(X)$ such that $J^*_S \leq J \leq \hat{J}$ for some $\hat{J} \in S$. 
Policy Iteration

Definitions:
- **Standard PI**: $T_{\mu^{k+1}} J_{\mu^k} = TJ_{\mu^k}$
- **Optimistic PI**: $T_{\mu^k} J_k = TJ_k$, $J_{k+1} = T^{m_k}_{\mu^k} J_k$ (evaluation of the current policy is approximate, using $m_k$ iterations of VI)

Convergence of standard PI, assuming $J^* \geq 0$
- The sequence $\{\mu^k\}$ satisfies $J_{\mu^k} \downarrow J_\infty$, where $J_\infty$ is a fixed point of $T$ with $J_\infty \geq J^*$
- If for a set $S \subset E(X)$, the policies $\mu^k$ generated are $S$-regular and we have $J_{\mu^k} \in S$ for all $k$, then $J_{\mu^k} \downarrow J^*_S$ and $J^*_S$ is a fixed point of $T$

Convergence of optimistic PI
- The sequence $\{J_k\}$ satisfies $J_k \downarrow J_\infty$, where $J_\infty$ is a fixed point of $T$
- If for a set $S \subset E(X)$, the policies $\mu^k$ generated are $S$-regular and we have $J_{\mu^k} \in S$ for all $k$, then $J_k \downarrow J^*_S$ and $J^*_S$ is a fixed point of $T$

With more analysis and conditions, we can show that $J_\infty = J^*$. This is true for the deterministic and stochastic nonnegative cost problems.
### Problem Formulation
- Finite state space $X = \{0, 1, \ldots, n\}$ with 0 being a cost-free and absorbing state
- Transition probabilities $p_{xy}(u)$
- $U(x)$ is finite for all $x \in X$
- No discounting ($\alpha = 1$)

### Proper policies
- $\mu$ is **proper** if the terminal state $t$ is reached w.p.1 under $\mu$ (is **improper** otherwise)
- Let $S = \mathbb{R}^n$. Then $\mu$ is $S$-regular if and only if it is proper. (The idea of an $S$-regular policy evolved as a generalization of a proper policy.)

### Contraction properties
- The mapping $T_\mu$ of a policy $\mu$ is a weighted sup-norm contraction iff $\mu$ proper
- If all stationary policies are proper, then $T$ is a sup-norm contraction, and the problem behaves like a discounted problem
- SSP is a prime example of a **semicontractive model** (some policies correspond to contractions while others do not)
Case where improper policies have infinite cost

If there exists a proper policy and for every improper $\mu$, $J_\mu(x) = \infty$ for some $x$, then:

- $J^*$ is the unique fixed point of $T$ in $\mathbb{R}^n$
- VI converges to $J^*$ starting from every $J \in \mathbb{R}^n$
- PI converges to an optimal proper policy, if started with a proper policy

Case where improper policies have finite cost (due to zero length “cycles”)

Let $\hat{J}$ be the optimal cost function over proper stationary policies only, and assume that $\hat{J}$ and $J^*$ are real-valued. Then:

- $\hat{J}$ is the unique fixed point of $T$ in the set $\{J \in \mathbb{R}^n \mid J \geq \hat{J}\}$
- VI converges to $\hat{J}$ starting from any $J \geq \hat{J}$
- PI need not converge to an optimal policy even if started with a proper policy
- A “perturbed” version of PI (add a $\delta_k > 0$ to $g$, with $\delta_k \downarrow 0$) converges to an optimal policy within the class of proper policies, if started with a proper policy
- An improper policy may be (overall) optimal, while $J^*$ need not be a fixed point of $T$
Main Objective

- **Unification** of the core theory and algorithms of total cost DP
- Simultaneous treatment of a variety of problems: MDP, sequential games, sequential minimax, multiplicative cost, risk-sensitive, etc

Main Idea

- Define a DP problem by its "mathematical signature": an abstract monotone mapping $H : X \times U \times E(X) \mapsto [-\infty, \infty]$

$$J \leq J' \implies H(x, u, J) \leq H(x, u, J'), \quad \forall x, u$$

where $E(X)$ is the set of functions $J : X \mapsto [-\infty, \infty]$

- Stochastic optimal control example: $H(x, u, J) = E\{g(x, u, w) + \alpha J(f(x, u, w))\}$
- Minimax example: $H(x, u, J) = \sup_{w \in W} \{g(x, u, w) + \alpha J(f(x, u, w))\}$
Abstract DP Mappings

- State and control spaces: \( X, U \)
- Control constraint: \( u \in U(x) \)
- Stationary policies: \( \mu : X \mapsto U, \text{ with } \mu(x) \in U(x) \text{ for all } x \)

Monotone Mappings

- Abstract monotone mapping \( H : X \times U \times E(X) \mapsto \mathbb{R} \)

\[
J \leq J' \implies H(x, u, J) \leq H(x, u, J'), \quad \forall x, u
\]

where \( E(X) \) is the set of functions \( J : X \mapsto [-\infty, \infty] \)

- For a stationary policy \( \mu \)

\[
(T_{\mu} J)(x) = H(x, \mu(x), J), \quad \forall x \in X, J \in E(X)
\]

and for VI

\[
(TJ)(x) = \inf_{u \in U(x)} H(x, u, J), \quad \forall x \in X, J \in E(X)
\]
Abstract Problem Formulation

Abstract Optimization Problem

- Given an initial function $\bar{J} \in E(X)$ and policy $\pi = \{\mu_0, \mu_1, \ldots\}$, define
  \[ J_\pi(x) = \limsup_{N \to \infty} (T_{\mu_0} \cdots T_{\mu_N} \bar{J})(x), \quad x \in X \]

- Find $J^*(x) = \inf_\pi J_\pi(x)$ and an optimal $\pi$ attaining the infimum

Notes

- Theory revolves around fixed point properties of mappings $T_\mu$ and $T$:
  \[ J_\mu = T_\mu J_\mu, \quad J^* = TJ^* \]
  These are generalized forms of Bellman’s equation

- Algorithms are special cases of fixed point algorithms
**Principal Abstract Models**

**Contractive:**
- Patterned after discounted
- The DP mappings $T_\mu$ are weighted sup-norm contractions (Denardo 1967)

**Monotone Increasing/Decreasing:**
- Patterned after positive and negative DP
- No reliance on contraction properties, just monotonicity of $T_\mu$ (Bertsekas 1977, Bertsekas and Shreve 1978)

**Semicontractive:**
- Patterned after stochastic shortest path
- Some policies $\mu$ are “regular” ($T_\mu$ is contractive-like); others are not, but focus is on optimization over “regular” policies
Let $\mathcal{C}$ be a collection of policy-state pairs $(\pi, x)$ that is $S$-regular. For all fixed points $J'$ of $T$, and all $J \in E(X)$ such that $J' \leq J \leq \hat{J}$ for some $\hat{J} \in S$, we have

$$J' \leq \liminf_{k \to \infty} T^k J \leq \limsup_{k \to \infty} T^k J \leq J^*_C$$

- If $J^*_C$ is a fixed point of $T$, then $\text{VI}$ converges to $J^*_C$ starting from any $J \in E(X)$ such that $J^*_C \leq J \leq \hat{J}$ for some $\hat{J} \in S$
- When $J^*$ is a fixed point of $T$, a useful analytical strategy is to choose $\mathcal{C}$ such that $J^*_C = J^*$. Then a $\text{VI}$ convergence result is obtained
Diverse Applications to Various Types of DP Problems

Bellman equation, VI, and PI analysis

- To **minimax** problems (also zero sum games); e.g.,

  \[ H(x, u, J) = \sup_{w \in W} \left\{ g(x, u, w) + \alpha J(f(x, u, w)) \right\}, \quad \bar{J}(x) \equiv 0 \]

- To **robust shortest path** planning (minimax with a termination state)

- To **multiplicative and risk-sensitive** cost functions

  \[ H(x, u, J) = E \left\{ g(x, u, w)J(f(x, u, w)) \right\}, \quad \bar{J}(x) \equiv 1 \]

  or

  \[ H(x, u, J) = E \left\{ e^{g(x, u, w)}J(f(x, u, w)) \right\}, \quad \bar{J}(x) \equiv 1 \]

- More ... see the references
Thank you!