Topics in Reinforcement Learning: Rollout and Approximate Policy Iteration

ASU, CSE 691, Spring 2020

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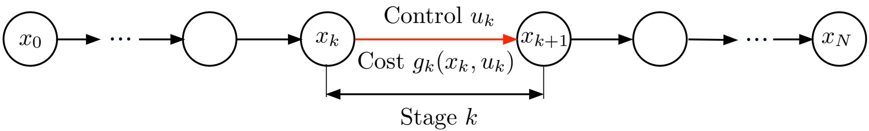
Lecture 2

Outline

Finite Horizon Deterministic Problem



- System

$$x_{k+1} = f_k(x_k, u_k), \qquad k = 0, 1, \ldots, N-1$$

where x_k: State, u_k: Control chosen from some set $U_k(x_k)$

- Cost function:

$$g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k)$$

- For given initial state x_0, minimize over control sequences $\{u_0, \ldots, u_{N-1}\}$

$$J(x_0; u_0, \ldots, u_{N-1}) = g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k)$$

- Optimal cost function $J^*(x_0) = \min_{\substack{u_k \in U_k(x_k) \\ k=0,\ldots,N-1}} J(x_0; u_0, \ldots, u_{N-1})$

DP Algorithm: Solving Progressively Longer Tail Subproblems

Go backward to compute the optimal costs $J_k^*(x_k)$ of the x_k-tail subproblems

Start with

$$J_N^*(x_N) = g_N(x_N), \qquad \text{for all } x_N,$$

and for $k = 0, \ldots, N-1$, let

$$J_k^*(x_k) = \min_{u_k \in U_k(x_k)} \Big[g_k(x_k, u_k) + J_{k+1}^*\big(f_k(x_k, u_k)\big) \Big], \qquad \text{for all } x_k.$$

Then optimal cost $J^*(x_0)$ is obtained at the last step: $J_0^*(x_0) = J^*(x_0)$.

Go forward to construct optimal control sequence $\{u_0^*, \ldots, u_{N-1}^*\}$

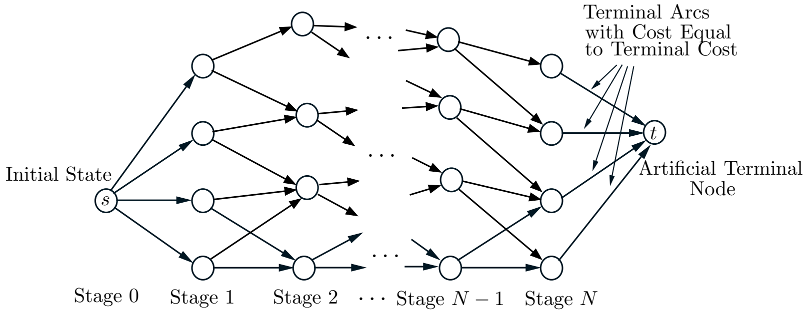
Start with

$$u_0^* \in \arg \min_{u_0 \in U_0(x_0)} \Big[g_0(x_0, u_0) + J_1^*\big(f_0(x_0, u_0)\big) \Big], \qquad x_1^* = f_0(x_0, u_0^*).$$

Sequentially, going forward, for $k = 1, 2, \ldots, N-1$, set

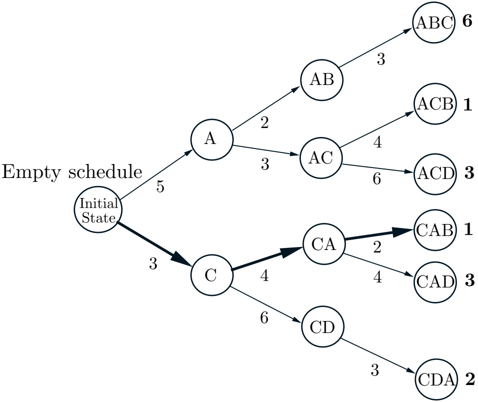
$$u_k^* \in \arg \min_{u_k \in U_k(x_k^*)} \Big[g_k(x_k^*, u_k) + J_{k+1}^*\big(f_k(x_k^*, u_k)\big) \Big], \qquad x_{k+1}^* = f_k(x_k^*, u_k^*).$$

Approximation in value space approach: We replace J_k^* with an approximation $\tilde{J}_k$.



- Nodes correspond to states x_k
- Arcs correspond to state-control pairs (x_k, u_k)
- An arc (x_k, u_k) has start and end nodes x_k and $x_{k+1} = f_k(x_k, u_k)$
- An arc (x_k, u_k) has a cost $g_k(x_k, u_k)$. The cost to optimize is the sum of the arc costs from the initial node s to the terminal node t.
- The problem is equivalent to finding a minimum cost/shortest path from s to t.

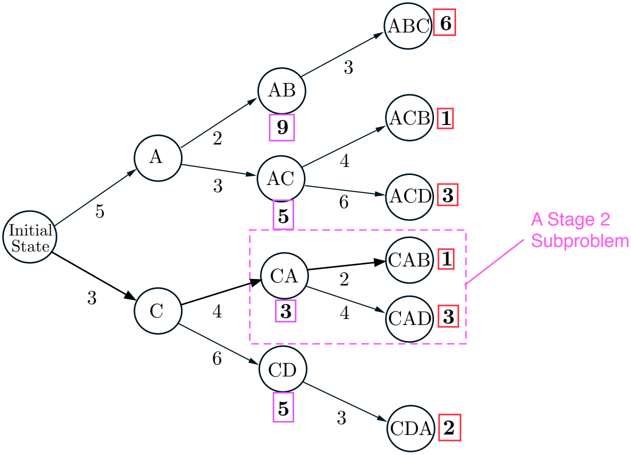
Discrete-State Deterministic Scheduling Example



Find optimal sequence of operations A, B, C, D (A must precede B and C must precede D)

DP Problem Formulation

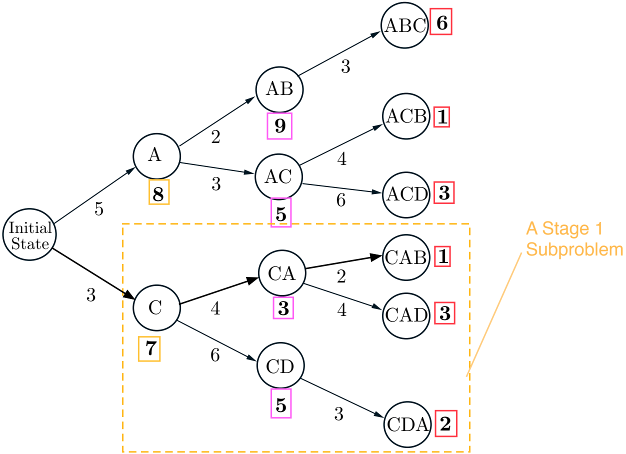
- States: Partial schedules; Controls: Stage 0, 1, and 2 decisions; Cost data shown along the arcs
- Recall the DP idea: Break down the problem into smaller pieces (tail subproblems)
- Start from the last decision and go backwards



Solve the stage 2 subproblems (using the terminal costs - in red)

At each state of stage 2, we record the optimal cost-to-go and the optimal decision

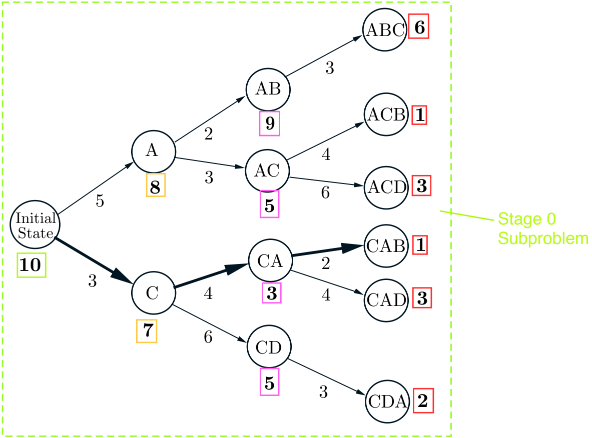
DP Algorithm: Stage 1 Tail Subproblems



Solve the stage 1 subproblems (using the optimal costs of stage 2 subproblems - in purple)

At each state of stage 1, we record the optimal cost-to-go and the optimal decision

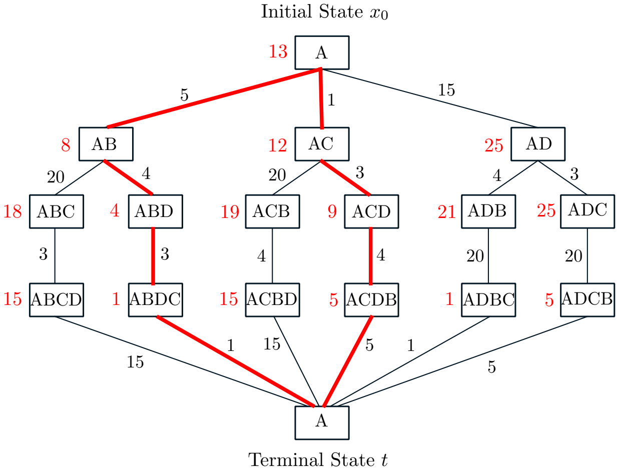
DP Algorithm: Stage 0 Tail Subproblems



Solve the stage 0 subproblem (using the optimal costs of stage 1 subproblems - in orange)

- The stage 0 subproblem is the entire problem
- The optimal value of the stage 0 subproblem is the optimal cost J^*(initial state)
- Construct the optimal sequence going forward

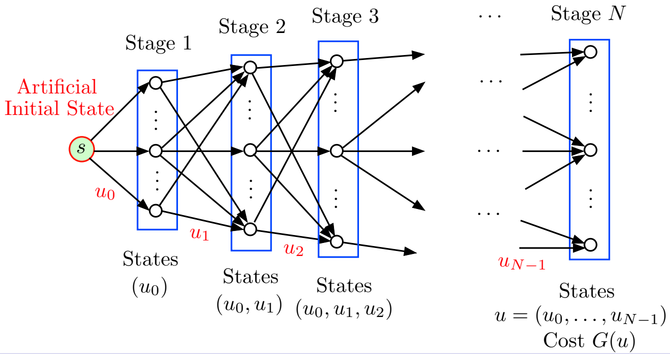
Combinatorial Optimization: Traveling Salesman Example



Matrix of Intercity Travel Costs

	5	1	15
5		20	4
1	20		**3**
15	4	3	

General Discrete Optimization

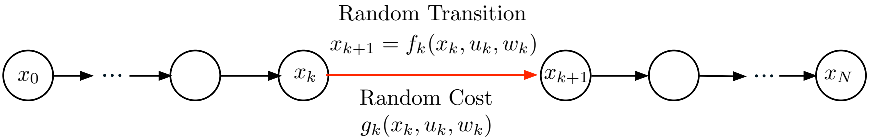


Minimize $G(u)$ subject to $u \in U$

- Assume that each solution u has N components: $u = (u_0, \ldots, u_{N-1})$
- View the components as the controls of N stages
- Define $x_k = (u_0, \ldots, u_{k-1})$, $k = 1, \ldots, N$, and introduce artificial start state $x_0 = s$
- Define just terminal cost as $G(u)$; all other costs are 0

This formulation typically makes little sense for exact DP, but often makes a lot of sense for approximate DP/approximation in value space

Stochastic DP Problems - Perfect State Observation



- System $x_{k+1} = f_k(x_k, u_k, w_k)$ with random "disturbance" w_k (e.g., physical noise, market uncertainties, demand for inventory, unpredictable breakdowns, etc)
- Cost function:

$$E\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)\right\}$$

- Policies $\pi = \{\mu_0, \dots, \mu_{N-1}\}$, where μ_k is a "closed-loop control law" or "feedback policy"/a function of x_k. Specifies control $u_k = \mu_k(x_k)$ to apply when at x_k.
- For given initial state x_0, minimize over all $\pi = \{\mu_0, \dots, \mu_{N-1}\}$ the cost

$$J_\pi(x_0) = E\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k\big(x_k, \mu_k(x_k), w_k\big)\right\}$$

- Optimal cost function $J^*(x_0) = \min_\pi J_\pi(x_0)$

The Stochastic DP Algorithm

Produces the optimal costs $J_k^*(x_k)$ of the tail subproblems that start at x_k

Start with $J_N^*(x_N) = g_N(x_N)$, and for $k = 0, \ldots, N-1$, let

$$J_k^*(x_k) = \min_{u_k \in U_k(x_k)} E\Big\{ g_k(x_k, u_k, w_k) + J_{k+1}^*\big(f_k(x_k, u_k, w_k)\big)\Big\}, \qquad \text{for all } x_k.$$

- The optimal cost $J^*(x_0)$ is obtained at the last step: $J_0^*(x_0) = J^*(x_0)$.
- The optimal control function μ_k^* is constructed simultaneously with J_k^*, and consists of the minimizing $u_k^* = \mu_k^*(x_k)$ above.

Online implementation of the optimal policy, given $J_1^*, \ldots, J_{N-1}^*$

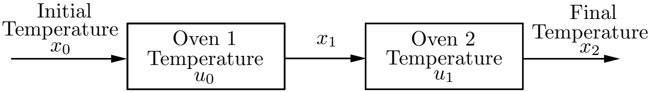
Sequentially, going forward, for $k = 0, 1, \ldots, N-1$, observe x_k and apply

$$u_k^* \in \arg \min_{u_k \in U_k(x_k)} E\Big\{ g_k(x_k, u_k, w_k) + J_{k+1}^*\big(f_k(x_k, u_k, w_k)\big)\Big\}.$$

Issues: Need to compute J_{k+1}^* (possibly off-line), compute expectation for each u_k, minimize over all u_k

Approximation in value space: Use $\tilde{J}_k$ in place of J_k^*; approximate $E\{\cdot\}$ and $\min_{u_k}$.

Linear-quadratic problems involve: multidimensional linear system, quadratic cost, unconstrained controls, independent disturbances



- System: $x_{k+1} = (1 - a)x_k + au_k + w_k$ (w_k are random, independent, and 0-mean)
- Cost: $E\{r(x_N - T)^2 + \sum_{k=0}^{N-1} u_k^2\}$
- A very favorable structure: The optimal policy $\mu_k^*(x_k)$ is a linear function of x_k; it is the same as if w_1 and w_0 were set to their expected values ($= 0$). Can be computed by exact DP
- This is called certainty equivalence
- Certainty equivalence is a common approximation idea for other problems (replace the original stochastic problem with a deterministic version)

DP Algorithm for Q-Factors

- Optimal Q-factors are given by

$$Q_k^*(x_k,u_k) = E\Big\{g_k(x_k,u_k,w_k) + J_{k+1}^*\big(f_k(x_k,u_k,w_k)\big)\Big\}$$

They define optimal policies and optimal cost-to-go functions by

$$\mu_k^*(x_k) \in \arg\min_{u_k\in U_k(x_k)} Q_k^*(x_k,u_k), \qquad J_k^*(x_k) = \min_{u_k\in U_k(x_k)} Q_k^*(x_k,u_k)$$

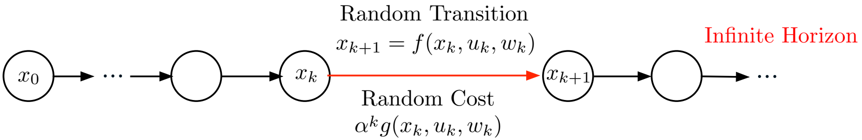
- DP algorithm can be written in terms of Q-factors

$$Q_k^*(x_k,u_k) = E\Big\{g_k(x_k,u_k,w_k) + \min_{u_{k+1}} Q_{k+1}^*\big(f_k(x_k,u_k,w_k),u_{k+1}\big)\Big\}$$

Some math magic: With $E\{\cdot\}$ outside the min, the right side can be approximated by sampling and simulation. (Can be exploited in stochastic iterative algorithms called Q-learning.)

- Approximately optimal Q-factors $\tilde{Q}_k(x_k,u_k)$, define suboptimal policies and suboptimal cost-to-go functions by

$$\tilde{\mu}_k(x_k) \in \arg\min_{u_k\in U_k(x_k)} \tilde{Q}_k(x_k,u_k) \qquad \tilde{J}_k(x_k) = \min_{u_k\in U_k(x_k)} \tilde{Q}_k(x_k,u_k)$$



Infinite number of stages, and stationary system and cost

- System $x_{k+1} = f(x_k, u_k, w_k)$ with state, control, and random disturbance
- Policies $\pi = \{\mu_0, \mu_1, \ldots\}$ with $\mu_k(x) \in U(x)$ for all x and k
- Special scalar α with $0 < \alpha \leq 1$. If $\alpha < 1$ the problem is called discounted
- Cost of stage k: $\alpha^k g\big(x_k, \mu_k(x_k), w_k\big)$
- Cost of a policy $\pi = \{\mu_0, \mu_1, \ldots\}$

$$J_\pi(x_0) = \lim_{N\to\infty} E_{w_k} \left\{ \sum_{k=0}^{N-1} \alpha^k g\big(x_k, \mu_k(x_k), w_k\big) \right\}$$

- Optimal cost function $J^*(x_0) = \min_\pi J_\pi(x_0)$
- If $\alpha = 1$ we assume a special cost-free termination state t. The objective is to reach t at minimum expected cost. The problem is called stochastic shortest path (SSP) problem

Value iteration (VI): Fix horizon N, let terminal cost be 0

- Let $V_{N-k}(x)$ be the optimal cost starting at x with k stages to go, so

$$V_{N-k}(x) = \min_{u \in U(x)} E_w \Big\{ \alpha^{N-k} g(x,u,w) + V_{N-k+1}\big(f(x,u,w)\big) \Big\}$$

- Reverse the time index and divide with α^{N-k}: Define $J_k(x) = V_{N-k}(x)/\alpha^{N-k}$

$$J_k(x) = \min_{u \in U(x)} E_w \Big\{ g(x,u,w) + \alpha J_{k-1}\big(f(x,u,w)\big) \Big\} \qquad \text{(VI)}$$

- $J_N(x)$ is equal to $V_0(x)$, which is the N-stages optimal cost starting from x
- Hence, intuitively, VI converges to J^*:

$$J^*(x) = \lim_{N \to \infty} J_N(x), \qquad \text{for all states } x \quad (??)$$

The following Bellman equation holds: Take the limit in Eq. (VI)

$$J^*(x) = \min_{u \in U(x)} E_w \Big\{ g(x,u,w) + \alpha J^*\big(f(x,u,w)\big) \Big\}, \qquad \text{for all states } x \quad (??)$$

Optimality condition: Let $\mu(x)$ attain the min in the Bellman equation for all x

The policy $\{\mu, \mu, \ldots\}$ is optimal (??). (This type of policy is called stationary.)

How do we Formulate DP Problems?

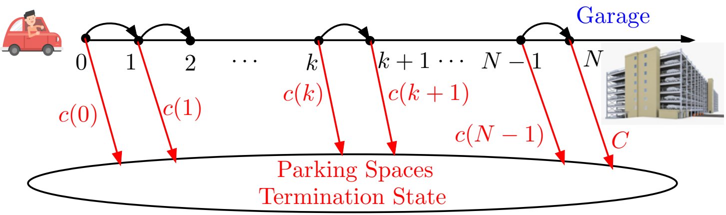
An informal recipe: First define the stages and then the states

Define as state x_k something that summarizes the past for purposes of future optimization, i.e., as long as we know x_k, all past information is irrelevant.

Some examples

- In the traveling salesman problem, we need to include all the info (past cities visited) in the state.
- In the linear quadratic problem, when we select the oven temperature u_k, the total info available is everything we have seen so far, i.e., the material and oven temperatures $x_0, u_0, x_1, u_1, \ldots, u_{k-1}, x_k$. However, all the useful information at time k is summarized in just x_k.
- In partial or imperfect information problems, we use “noisy" measurements for control of some quantity of interest y_k that evolves over time (e.g., the position/velocity vector of a moving object). If I_k is the collection of all measurements up to time k, it is correct to use I_k as state.
- It may also be correct to use alternative states; e.g., the conditional probability distribution $P_k(y_k \mid I_k)$. This is called belief state, and subsumes all the information that is useful for the purposes of control choice.

Problems with a Terminal State: A Parking Example



- Start at spot 0; either park at spot k with cost $c(k)$ (if free) or continue; park at garage at cost C if not earlier.
- Spot k is free with a priori probability $p(k)$, and its status is observed upon reaching it.
- How do we formulate the problem as a DP problem?

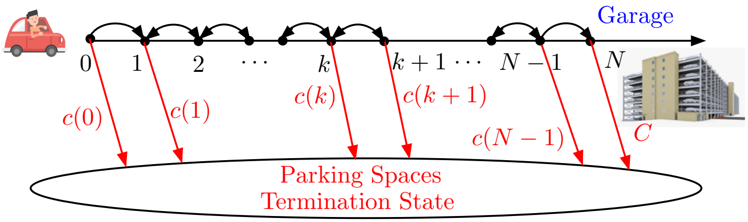
We have three states. F: current spot is free, $\overline{F}$: current spot is taken, parked state

$$J^*_{N-1}(F) = \min\big[c(N-1),\, C\big], \qquad J^*_{N-1}(\overline{F}) = C$$

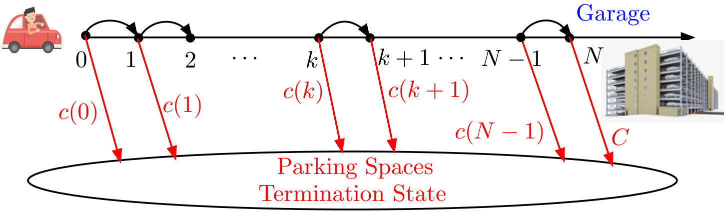
$$J^*_k(F) = \min\Big[c(k),\, p(k+1)J^*_{k+1}(F) + \big(1 - p(k+1)\big)J^*_{k+1}(\overline{F})\Big], \qquad \text{for } k = 0, \ldots, N-2$$

$$J^*_k(\overline{F}) = p(k+1)J^*_{k+1}(F) + \big(1 - p(k+1)\big)J^*_{k+1}(\overline{F}), \qquad \text{for } k = 0, \ldots, N-2$$

More Complex Parking Problems



- Bidirectional parking: We can go back to parking spots we have visited at a cost
- More complicated parking lot topologies
- Multiagent versions: Multiple drivers/autonomous vehicles, “searchers", etc
- “Relatively easy" cases: The status of already seen spots stays unchanged.



- A more complex type of parking example, where taken or free parking spots may free up or get taken, respectively, at the next time step with some probability
- The free/taken state of the spots is "estimated" in a "probabilistic sense" based on the observations (the free/taken status of the spots visited ... when visited)
- What should the "state" be? It should summarize all the info needed for the purpose of future optimization
- First candidate for state: The set of all observations so far. Another candidate: The "belief state", i.e., the conditional probabilities of the free/taken status of all the spots: $p(0), p(1), \ldots, p(N-1)$
- Generally, partial observation problems (POMDP) can be "solved" by DP with state being the belief state: $P(x_k \mid \text{set of observations up to time } k)$

About the Next Lecture

We will cover:

- General principles of approximation in value and policy space
- Brief discussion of the problem approximation approach
- Introduction to rollout

CHAPTER 2 OF THE CLASS NOTES POSTED
PLEASE READ AS MUCH OF SECTIONS 2.1, 2.2 AS YOU CAN

1ST HOMEWORK (DUE IN 2 WEEKS) TO BE ANNOUNCED SHORTLY