UNDISCOUNTED/NONCONTRACTIVE

DP PROBLEMS

LECTURE OUTLINE

- Undiscounted total cost problems
- Positive and negative cost problems
- Deterministic optimal cost problems
- Adaptive (linear quadratic) DP
- Affine monotonic and risk sensitive problems

Reference:

Updated Chapter 4 of Vol. II of the text:

Noncontractive Total Cost Problems

On-line at:

http://web.mit.edu/dimitrib/www/dpchapter.html

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CONTRACTIVE/SEMICONTRACTIVE PROBLEMS

- Infinite horizon total cost DP theory divides in
  - "Easy" problems where the results one expects (uniqueness of solution of Bellman Eq., convergence of PI and VI, etc) hold
  - "Difficult" problems where one of more of these results do not hold
- "Easy" problems are characterized by the presence of strong contraction properties in the associated algorithmic maps T and T_μ
- A typical example of an "easy" problem is discounted problems with bounded cost per stage (Chs. 1 and 2 of Voll. II) and some with unbounded cost per stage (Section 1.5 of Voll. II)
- Another is semicontractive problems, where T_μ is a contraction for some μ but is not for other μ, and assumptions are imposed that exclude the "ill-behaved" μ from optimality
- A typical example is SSP where the improper policies are assumed to have infinite cost for some initial states (Chapter 3 of Vol. II)
- In this lecture we go into "difficult" problems

UNDISCOUNTED TOTAL COST PROBLEMS

- Beyond problems with strong contraction properties. One or more of the following hold:
  - No termination state assumed
  - Infinite state and control spaces
  - Either no discounting, or discounting and unbounded cost per stage
  - Risk-sensitivity/exotic cost functions (e.g., SSP problems with exponentiated cost)
- Important classes of problems
  - SSP under weak conditions
  - Positive cost problems (control/regulation, robotics, inventory control)
  - Negative cost problems (maximization of positive rewards - investment, gambling, finance)
  - Deterministic positive cost problems - Adaptive DP
  - A variety of infinite-state problems in queueing, optimal stopping, etc
  - Affine monotonic and risk-sensitive problems (a generalization of SSP)

POS. AND NEG. COST - FORMULATION

- System $x_{k+1} = f(x_k, u_k, w_k)$ and cost

$$J_\pi(x_0) = \lim_{N\to\infty} \mathop{E}_{\substack{w_k \\ k=0,1,\ldots}} \left\{ \sum_{k=0}^{N-1} \alpha^k g\big(x_k, \mu_k(x_k), w_k\big) \right\}$$

Discount factor $\alpha \in (0,1]$, but g may be unbounded

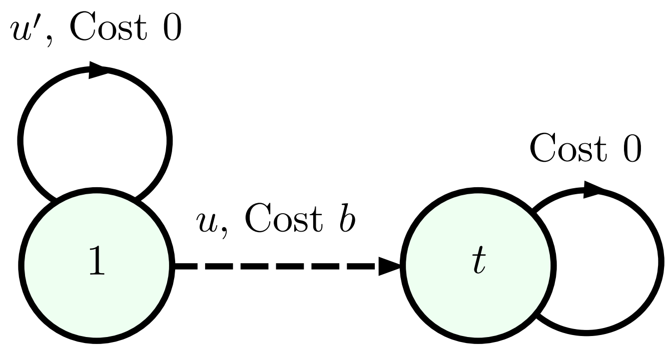
- Case P: $g(x,u,w) \geq 0$ for all (x,u,w)
- Case N: $g(x,u,w) \leq 0$ for all (x,u,w)
- Summary of analytical results:
  - Many of the strong results for discounted and SSP problems fail
  - Analysis more complex; need to allow for J_π and J^* to take values $+\infty$ (under P) or $-\infty$ (under N)
  - However, J^* is a solution of Bellman's Eq. (typically nonunique)
  - Opt. conditions: μ is optimal if and only if $T_\mu J^* = TJ^*$ (P) or if $T_\mu J_\mu = TJ_\mu$ (N)

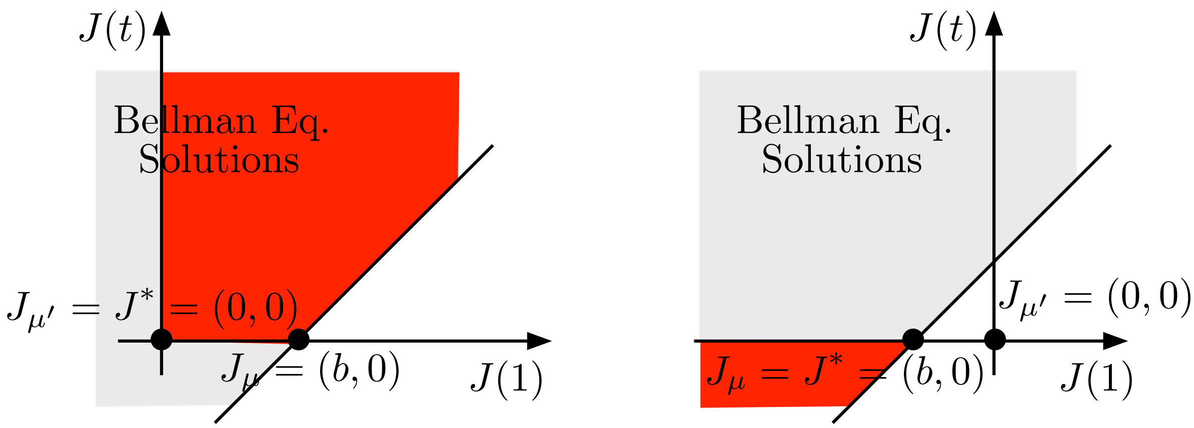
SUMMARY OF ALGORITHMIC RESULTS

- Neither VI nor PI are guaranteed to work
- Behavior of VI
  - P: $T^k J \to J^*$ for all J with $0 \leq J \leq J^*$, if $U(x)$ is finite (or compact plus more conditions - see the text)
  - N: $T^k J \to J^*$ for all J with $J^* \leq J \leq 0$
- Behavior of PI
  - P: J_{μ^k} is monotonically nonincreasing but may get stuck at a nonoptimal policy
  - N: J_{μ^k} may oscillate (but an optimistic form of PI converges to J^* - see the text)
- These anomalies may be mitigated to a greater or lesser extent by exploiting special structure, e.g.
  - Presence of a termination state
  - Proper/improper policy structure in SSP
- Finite-state problems under P can be transformed to equivalent SSP problems by merging all states x with $J^*(x) = 0$ into a termination state. They can then be solved using the powerful SSP methodology (see Ch. 4, Section 4.1.4)

EXAMPLE FROM THE PREVIOUS LECTURE

- This is essentially a shortest path example with termination state t





Case P
VI fails starting from
$J(1) \neq 0$, $J(t) = 0$
PI stops at μ

Case N
VI fails starting from
$J(1) < J^*(1)$, $J(t) = 0$
PI oscilllates between μ and μ'

- Bellman Equation:

$$J(1) = \min\big[J(1), b + J(t)\big], \qquad J(t) = J(t)$$

DETERM. OPT. CONTROL - FORMULATION

- System: $x_{k+1} = f(x_k, u_k)$, arbitrary state and control spaces X and U
- Cost positivity: $0 \le g(x, u)$, $\forall\, x \in X,\ u \in U(x)$
- No discounting:

$$J_\pi(x_0) = \lim_{N\to\infty} \sum_{k=0}^{N-1} g\big(x_k, \mu_k(x_k)\big)$$

- "Goal set of states" X_0
  - All $x \in X_0$ are cost-free and absorbing
- A shortest path-type problem, but with possibly infinite number of states
- A common formulation of control/regulation and planning/robotics problems
- Example: Linear system, quadratic cost (possibly with state and control constraints), $X_0 = \{0\}$ or X_0 is a small set around 0
- Strong analytical and computational results

DETERM. OPT. CONTROL - ANALYSIS

- Bellman's Eq. holds (for not only this problem, but also all deterministic total cost problems)

$$J^*(x) = \min_{u \in U(x)} \big\{ g(x,u) + J^*\big(f(x,u)\big) \big\}, \quad \forall\, x \in X$$

- Definition: A policy π terminates starting from x if the state sequence $\{x_k\}$ generated starting from $x_0 = x$ and using π reaches X_0 in finite time, i.e., satisfies $x_{\bar{k}} \in X_0$ for some index $\bar{k}$

- Assumptions: The cost structure is such that
  - $J^*(x) > 0$, $\forall\, x \notin X_0$ (termination incentive)
  - For every x with $J^*(x) < \infty$ and every $\epsilon > 0$, there exists a policy π that terminates starting from x and satisfies $J_\pi(x) \le J^*(x) + \epsilon$.

- Uniqueness of solution of Bellman's Eq.: J^* is the unique solution within the set

$$\mathcal{J} = \big\{ J \mid 0 \le J(x) \le \infty, \forall\, x \in X,\ J(x) = 0, \forall\, x \in X_0 \big\}$$

- Counterexamples: Earlier SP problem. Also linear quadratic problems where the Riccati equation has two solutions (observability not satisfied).

DET. OPT. CONTROL - VI/PI CONVERGENCE

- The sequence $\{T^k J\}$ generated by VI starting from a $J \in \mathcal{J}$ with $J \geq J^*$ converges to J^*
- If in addition $U(x)$ is finite (or compact plus more conditions - see the text), the sequence $\{T^k J\}$ generated by VI starting from any function $J \in \mathcal{J}$ converges to J^*
- A sequence $\{J_{\mu^k}\}$ generated by PI satisfies $J_{\mu^k}(x) \downarrow J^*(x)$ for all $x \in X$
- PI counterexample: The earlier SP example
- Optimistic PI algorithm: Generates pairs $\{J_k, \mu^k\}$ as follows: Given J_k, we generate μ^k according to

$$\mu^k(x) = \arg \min_{u \in U(x)} \big\{ g(x,u) + J_k\big(f(x,u)\big) \big\}, \quad x \in X$$

and obtain J_{k+1} with $m_k \geq 1$ VIs using μ^k:

$$J_{k+1}(x_0) = J_k(x_{m_k}) + \sum_{t=0}^{m_k-1} g\big(x_t, \mu^k(x_t)\big), \quad x_0 \in X$$

If $J_0 \in \mathcal{J}$ and $J_0 \geq TJ_0$, we have $J_k \downarrow J^*$.

- Rollout with terminating heuristic (e.g., MPC).

LINEAR-QUADRATIC ADAPTIVE CONTROL

- System: $x_{k+1} = Ax_k + Bu_k, \;\; x_k \in \Re^n, u_k \in \Re^m$
- Cost: $\sum_{k=0}^{\infty}(x_k' Q x_k + u_k' R u_k)$, $Q \geq 0$, $R > 0$
- Optimal policy is linear: $\mu^*(x) = Lx$
- The Q-factor of each linear policy μ is quadratic:

$$Q_\mu(x,u) = \begin{pmatrix} x' & u' \end{pmatrix} K_\mu \begin{pmatrix} x \\ u \end{pmatrix} \qquad (*)$$

- We will consider A and B unknown
- We use as basis functions all the quadratic functions involving state and control components

$$x^i x^j, \qquad u^i u^j, \qquad x^i u^j, \qquad \forall\, i,j$$

These form the "rows" $\phi(x,u)'$ of a matrix Φ

- The Q-factor Q_μ of a linear policy μ can be exactly represented within the subspace spanned by the basis functions:

$$Q_\mu(x,u) = \phi(x,u)' r_\mu$$

where r_μ consists of the components of K_μ in (*)

- Key point: Compute r_μ by simulation of μ (Q-factor evaluation by simulation, in a PI scheme)

PI FOR LINEAR-QUADRATIC PROBLEM

- Policy evaluation: r_μ is found (exactly) by least squares minimization

$$\min_r \sum_{(x_k,u_k)} \left| \phi(x_k,u_k)'r - \Big(x_k'Qx_k + u_k'Ru_k + \phi\big(x_{k+1},\mu(x_{k+1})\big)'r\Big) \right|^2$$

where (x_k, u_k, x_{k+1}) are "enough" samples generated by the system or a simulator of the system.

- Policy improvement:

$$\overline{\mu}(x) \in \arg\min_u \big(\phi(x,u)'r_\mu\big)$$

- Knowledge of A and B is not required
- If the policy evaluation is done exactly, this becomes exact PI, and convergence to an optimal policy can be shown
- The basic idea of this example has been generalized and forms the starting point of the field of adaptive DP
- This field deals with adaptive control of continuous-space (possibly nonlinear) dynamic systems, in both discrete and continuous time

FINITE-STATE AFFINE MONOTONIC PROBLEMS

- Generalization of positive cost finite-state stochastic total cost problems where:
  - In place of a transition prob. matrix P_μ, we have a general matrix $A_\mu \geq 0$
  - In place of 0 terminal cost function, we have a more general terminal cost function $\bar{J} \geq 0$
- Mappings

$$T_\mu J = b_\mu + A_\mu J, \qquad (TJ)(i) = \min_{\mu \in \mathcal{M}} (T_\mu J)(i)$$

- Cost function of $\pi = \{\mu_0, \mu_1, \dots\}$

$$J_\pi(i) = \limsup_{N \to \infty} (T_{\mu_0} \cdots T_{\mu_{N-1}} \bar{J})(i), \quad i = 1, \dots, n$$

- Special case: An SSP with an exponential risk-sensitive cost, where for all i and $u \in U(i)$

$$A_{ij}(u) = p_{ij}(u) e^{g(i,u,j)}, \quad b(i,u) = p_{it}(u) e^{g(i,u,t)}$$

- Interpretation:

$$J_\pi(i) = E\{e^{(\text{length of path of } \pi \text{ starting from } i)}\}$$

AFFINE MONOTONIC PROBLEMS: ANALYSIS

- The analysis follows the lines of analysis of SSP
- Key notion (generalizes the notion of a proper policy in SSP): A policy μ is stable if $A_\mu^k \to 0$; else it is called unstable
- We have

$$T_\mu^N J = A_\mu^N J + \sum_{k=0}^{N-1} A_\mu^k b_\mu, \quad \forall\, J \in \Re^n,\ N = 1, 2, \ldots,$$

- For a stable policy μ, we have for all $J \in \Re^n$

$$J_\mu = \limsup_{N\to\infty} T_\mu^N J = \limsup_{N\to\infty} \sum_{k=0}^{\infty} A_\mu^k b_\mu = (I - A_\mu)^{-1} b_\mu$$

- Consider the following assumptions:
  (1) There exists at least one stable policy
  (2) For every unstable policy μ, at least one component of $\sum_{k=0}^{\infty} A_\mu^k b_\mu$ is equal to ∞
- Under (1) and (2) the strong SSP analytical and algorithmic theory generalizes
- Under just (1) the weak SSP theory generalizes.