computing it for any given point in the state space by the developed algorithm. Moreover, in this case, the state space is not partitioned in the usual sense, since the whole state space coincides with a particular subregion determined as the result of this maximization.

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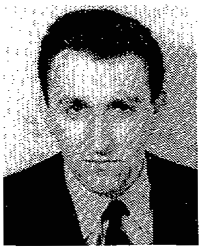

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Infinite-Time Reachability of State-Space Regions by Using Feedback Control

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Abstract—In this paper we consider some aspects of the problem of feedback control of a time-invariant uncertain system subject to state constraints over an infinite-time interval. The central question that we investigate is under what conditions can the state of the uncertain system be forced to stay in a specified region of the state space for all times by using feedback control. At the same time we study the behavior of the region of n-step reachability as n tends to infinity. It is shown that in general this region may exhibit instability as we pass to the limit, and that under a compactness assumption this region converges to a steady state.

A special case involving a linear finite-dimensional system is examined in more detail. It is shown that there exist ellipsoidal regions in state space where the state can be confined by making use of a linear time-invariant control law, provided that the system is stabilizable. Such control laws can be calculated efficiently through the solution of a recursive matrix equation of the Riccati type.

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I. General Remarks

THIS PAPER is concerned with the problem of keeping the state of a discrete-time dynamic system in a specified region of the state space in the presence of uncertainty over an infinite-time interval by using feedback control. The system equation contains uncertain parameters that take values in a given set. The current state of the system is assumed to be known and available to the feedback controller. It is assumed that there are constraints on the value of control that can be applied, and it is required that the state of the system stay in the specified region for all possible values of the uncertain quantities. When a stochastic description of the uncertain quantities is available, a possible formulation of the problem would be to require that the state constraints are satisfied at each time with probability one. However, for the problem considered in this paper, a detailed stochastic description of the uncertain quantities is unnecessary. Only the set of all possible values of the uncertain quantities is required and is assumed given.

One possible way of viewing the problem of this paper is within the framework of regulation problems. In such problems, one is concerned with finding a control law that will keep the state of a system "close" to some reference point or equivalently in a sufficiently "small" region containing that point. In optimal regulation theory, an indirect approach is adopted toward achieving the regulation objective whereby one seeks a control law that minimizes a cost functional in which deviations from the desired reference point are appropriately penalized. The problem considered in this paper may be viewed as a more direct formulation of the regulation problem whereby a subset of the state space containing the reference point is specified and a control law is sought that will keep the state within this subset for all possible values of the uncertain quantities and for all times. If one wishes to adopt an optimal control viewpoint, this is equivalent to specifying a cost functional that takes the value zero whenever the state and control satisfy the given constraints for all times and takes the value of, say, infinity whenever some constraint is violated at some time. Thus, in the problem of this paper, "large" deviations from the reference point are equally penalized with "small" deviations, a feature that may be undesirable in some situations. On the other hand, as will be seen later, the solution of the problem yields not just a single control law, but rather a collection of control laws that can keep the state within the specified region. In some cases it is then possible to select a control law from this collection that has other desirable features such as linearity and stability; or, it is possible to select a control law that somehow takes into account the desirability of keeping the state as close as possible to the reference point. An example of this situation is the case of a linear finite-dimensional system examined in the last section of this paper. The results of this paper also can be used in several other ways in mathematical system studies. For example, in a stochastic or minimax dynamic optimization problem over an infinite-time interval, with state and control constraints, the results of the paper can be used for clarifying the issues associated with the steady-state behavior of the state-space region of feasibility, i.e., the set of initial states starting from which there exists a control law resulting in satisfaction of all the state and control constraints [13]. Another area where the results of this paper can be useful is the area of pursuit-evasion games by using a similar formulation as the one described in [2].

The problem of this paper has been considered earlier in [1], [2], [5], [9], [10] for some special cases under the assumption that the control time interval is finite and it has been called the problem of reachability of a target tube. The purpose of this paper is to consider the same problem when the control-time interval is infinite and to examine the question of convergence to a steady state of various algorithms given earlier.

In Section II the problem is formulated and two different notions of infinite-time reachability are introduced. In Section III we obtain necessary conditions and sufficient conditions for infinite-time reachability. We demonstrate that if reachability can be accomplished by a time-varying control law it can also be accomplished by a time-invariant control law. We also consider a recursive set algorithm for calculating the region of n-step reachability and show that the algorithm can exhibit instability when passing to the limit. In Section IV we derive conditions under which this recursive set algorithm converges to the region of infinite-time reachability. This is accomplished by introducing topological structure on the spaces of definition of the system and by assuming compactness of some sets related to the algorithm. Some important special cases for which the compactness assumption is satisfied are also discussed. The results presented in Sections III and IV are applicable to a very general class of systems for which the state space need not be a linear vector space. It was chosen to formulate the problem in a more general setting, since no less effort is required and no further results can be obtained when the class of systems under consideration is narrowed. On the other hand, the reader will lose little insight into the structure of the solution by considering the system to be defined over a Euclidean space. In Section V we consider in more detail the important special case of a linear finite-dimensional system driven by an input disturbance under the assumption that the admissible sets for the control and the disturbances are ellipsoids. For this case it is shown that there exist ellipsoidal regions in the state space in which the state can be confined by making use of a suitable control law provided the system is stabilizable. Furthermore, it is shown that there exists a linear time-invariant control law that can achieve reachability and, in addition, can make the closed-loop system asymptotically stable. This control law can be obtained from the solution of a recursive matrix Riccati equation, which can be efficiently calculated in a digital computer.

II. Problem Formulation

It is assumed that we are given a time-invariant discrete-time dynamic system

$$x_{k+1} = f(x_k,u_k,w_k), \qquad k = 0,1,2,\cdots \tag{1}$$

where x_k and u_k denote for all k the state and control of the system and w_k denotes some uncertain parameter, which shall be referred to as the "disturbance." The quantities x_k, u_k, and w_k are elements of spaces S_x, S_u, S_w, respectively, and the function $f: S_x \times S_u \times S_w \rightarrow S_x$ is given. It is assumed that the control u_k is constrained to take values from a subset $U(x_k)$ of S_u which may depend on the current state x_k. It is also assumed that the disturbance w_k can take values from a subset $W(x_k,u_k)$ of S_w, which may again depend both on the current state x_k and the applied control u_k. The set $W(x_k,u_k)$ does not depend, however, on previous values w_i, $i < k$, of the disturbance. Thus we have

$$u_k \in U(x_k) \subset S_u, \quad w_k \in W(x_k,u_k) \subset S_w. \tag{2}$$

Given any nonempty subset X of S_x, the question of interest is under what conditions does there exist a control law $\{\mu_0,\mu_1,\cdots\}$ with

$$\mu_k : X \rightarrow S_u, \quad \mu_k(x) \in U(x), \quad \forall x \in X, \quad k = 0,1,\cdots \tag{3}$$

and such that the state x_k of the closed-loop system

$$x_{k+1} = f[x_k,\mu_k(x_k),w_k], \qquad k = 0,1,\cdots \tag{4}$$

belongs to the set X for all k and all possible values $w_k \in W[x_k,\mu_k(x_k)]$

$$x_k \in X, \quad \forall w_k \in W[x_k,\mu_k(x_k)], \qquad k = 0,1,\cdots. \tag{5}$$

We will consider two different questions of interest depending upon the freedom that we have in choosing the initial state of the system. Let us consider the following definitions.

Definition 1: The set X is said to be *infinitely reachable* if there exists a control law $\{\mu_0,\mu_1,\cdots\}$ and some initial state $x_0 \in X$ for which relations (3), (4), and (5) are satisfied.

Definition 2: The set X is said to be *strongly reachable* if there exists a control law $\{\mu_0,\mu_1,\cdots\}$ such that for all initial states $x_0 \in X$ the relations (3), (4), and (5) are satisfied.

It is evident from Definitions 1 and 2 that the requirement of infinite reachability is much weaker than the requirement of strong reachability, since for the former we require that the state remains within the set X starting from at least one $x_0 \in X$, while for the latter we require that this occurs starting from every initial state $x_0 \in X$. In the next section we shall give necessary and sufficient conditions for both infinite and strong reachability of a set X. We shall, furthermore, demonstrate that the two notions are closely related in that a set is infinitely reachable if and only if it contains a strongly reachable set.

III. Necessary and Sufficient Conditions for Reachability

In order to analyze the problem of infinite-time reachability, which was described in the previous section, it is helpful to consider the problem of reachability of a subset X of the state space S_x for one, and more generally, for a finite number of stages. This problem has been considered earlier [1], [2], [9], [10] in a somewhat less general form.

Let us assume that at some time instant k the state of the system is $x_k \in X$. Then in order to guarantee that at the next time instant $(k + 1)$ the state x_{k+1} will belong to X, it is necessary to apply control $u_k \in U(x_k)$ for which $x_{k+1} = f(x_k,u_k,w_k) \in X$ for all $w_k \in W(x_k,u_k)$. Thus, we must have $x_k \in \bar{X}$ where the set $\bar{X}$ is defined by

$$\bar{X} = \{x \mid \exists\, u \in U(x) \text{ s.t. } f[x,u,W(u,x)] \subset X\} \cap X. \tag{6}$$

In (6) the symbol $\exists$ denotes "there exists," the initials s.t. stand for "such that," $f[x,u,W(u,x)]$ denotes the set $\{f(x,u,w) \mid w \in W(u,x)\}$, and the symbols $\subset$, $\cap$ denote set inclusion and set intersection.

If we define a function R mapping subsets of S_x into subsets of S_x by the equation

$$R(Z) = \{x \mid \exists\, u \in U(x) \text{ s.t. } f[x,u,W(u,x)] \subset Z\} \cap Z \tag{7}$$

then from (6) and (7) we have $\bar{X} = R(X)$. It is clear from the above discussion and Definition 2 that a necessary and sufficient condition for strong reachability of a set X is that $\bar{X} = R(X) = X$, i.e., that the set X is a fixed point of the function R.

Proposition 1: The set X is strongly reachable if and only if

$$R(X) = X. \tag{8}$$

We proceed now to investigate conditions for infinite reachability. Given a nonempty set X, consider the set $R^*(X)$ defined as follows: $x_0 \in R^*(X)$ iff $x_0 \in X$ and there exists a control law $\{\mu_0,\mu_1,\cdots\}$ s.t. (3), (4), and (5) are satisfied when x_0 is taken as the initial state of the system.

By Definition 1 we see that X is infinitely reachable if and only if $R^*(X) \neq \emptyset$. Furthermore, we have the following proposition.

Proposition 2: A set X is infinitely reachable if and only if it contains a strongly reachable set. The largest such strongly reachable set is $R^*(X)$ in the sense that $R^*(X)$ is strongly reachable whenever nonempty and if $\tilde{X} \subset X$ is another strongly reachable set then $\tilde{X} \subset R^*(X)$.

Proof: From Definitions 1 and 2 it is clear that if X contains a strongly reachable set it is infinitely reachable. Now let X be infinitely reachable, $\bar{x}$ be any point in the nonempty set $R^*(X)$, and $\{\bar{\mu}_0,\bar{\mu}_1,\cdots\}$ be any control law such that (3), (4), and (5) are satisfied when $\bar{x}$ is taken as the initial state of the system. In order to prove that $R^*(X)$ is strongly reachable by Proposition 1 it will be sufficient to prove that

$$f[\bar{x},\bar{\mu}_0(\bar{x}),W[\bar{x},\bar{\mu}_0(\bar{x})]] \subset R^*(X).$$

Indeed, if there existed some $\tilde{x} \in f[\bar{x},\bar{\mu}_0(\bar{x}),W[\bar{x},\bar{\mu}_0(\bar{x})]]$ such that $\tilde{x} \notin R^*(X)$, then by the definition of $R^*(X)$ the state x_{k+1} of the closed-loop system

$$x_{k+1} = f[x_k,\bar{\mu}_{k+1}(x_k),w_k], \qquad k = 0,1,\cdots$$

with initial state $x_0 = \tilde{x}$ will not belong to X for some k and a feasible choice of $w_0,w_1,\cdots,w_k$. This violates the property of the control law $\{\bar{\mu}_0,\bar{\mu}_1,\cdots\}$ and provides the contradiction.

The fact that $R^*(X)$ is the largest strongly reachable subset of X follows easily from its definition. Q.E.D.

An important observation from the proof of Proposition 2 is that if X is an infinitely reachable set, then if the state of the system is to be guaranteed to stay for all times within X, the initial state must be chosen within the set $R^*(X)$ and the control law must keep the state for all subsequent times within the set $R^*(X)$.

A further interesting observation from the proof of Proposition 2 is that, since $R^*(X)$ is strongly reachable whenever it is nonempty, if reachability can be accomplished by a time-varying control law, it can also be accomplished by a time-invariant control law μ. This control law can be defined as follows: If $x \not\in R^*(X)$, $\mu(x)$ is any element of $U(x)$. If $x \in R^*(X)$, consider any control law $\{\mu_0,\mu_1,\cdots\}$, which is such that (3), (4), and (5) are satisfied when the starting state is x and let $\mu(x) = \mu_0(x)$. When this control law is used and the initial state belongs to $R^*(X)$, then all subsequent states are guaranteed to belong to $R^*(X)$. We summarize the above discussion in the following corollary.

Corollary 1: If X is infinitely reachable there exists a stationary control law μ: $X \to S_x$ with $\mu(x) \in U(x)$ $\forall\, x \in X$ such that if the initial state belongs to $R^*(X)$, all subsequent states of the resulting closed-loop system are guaranteed to belong to $R^*(X)$.

We turn now to the question of characterizing the set $R^*(X)$ which, as the previous discussion shows, is of central importance in the problem of infinite-time reachability of the set X. Since the set $R^*(X)$ can be considered as the "region of infinite-time reachability" an obvious question is whether $R^*(X)$ can be characterized as some limit of the "region of n-step reachability" as n tends to infinity. The latter region is the subset of X consisting of all initial states starting from which there exists a control law that can keep the state of the closed-loop system within X for at least n steps. This subset can be characterized in terms of the function R of (7).

The set $R(X)$ is the set of all states $x \in X$ from which the state can be guaranteed to belong to X at the next time instant by applying suitable control law. By using the same logic as before, we conclude that the set $R[R(X)]$ is the set of all states $x \in R(X)$ from which the state can be guaranteed to belong to $R(X)$ at the next time instant and, therefore, can be guaranteed to belong to the set X for the next two time instants. It can be easily seen that $R[R(X)]$ is also equal to the set of all states $x \in X$ [rather than $x \in R(x)$] from which the state can be guaranteed to belong to $R(X)$ at the next time instant. Similarly, in order that there exist a control law such that starting from the initial state $x \in X$ the state of the closed-loop system is guaranteed to stay in the set X for the next n-time instants, it is necessary and sufficient that

$$x \in R^n(X) \tag{9}$$

where R^n denotes the composition $R\cdot R\cdot \cdots \cdot R$ (n times) and where the function R is defined in (7). It is clear that

$$R^{n+1}(X) \subset R^n(X), \quad \forall n \geq 1. \tag{10}$$

Thus, the set X is reachable for the next n steps if and only if the set $R^n(X)$ is nonempty ($R^n(X) \neq \emptyset$). The elements of $R^n(X)$ are initial states in X, starting from which reachability of the set X is guaranteed for the next n steps by using a suitable control law. It would appear from the above discussion that, based on Definition 1, a set X is infinitely reachable if and only if $\bigcap_{n=1}^{\infty} R^n(X) \neq \emptyset$ and, furthermore, if $\bigcap_{n=1}^{\infty} R^n(X) = R^*(X)$. This conjecture is incorrect in general, as we shall discuss below and demonstrate by example. The reason for this is that although there may exist initial states from which the set X is reachable for any given number of steps, there may not exist a control law that can accomplish reachability (from any of those initial states) for all times. In order to obtain conditions that relate the sets $\bigcap_{n=1}^{\infty} R^n(X)$ and $R^*(X)$, it will be necessary to consider, in addition to the function R of (7) which determines a set of initial states from which reachability is possible, another function C which determines the class of control laws which accomplish reachability. The function C maps subsets of S_x into subsets of $S_x \times S_u$ and is defined by

$$C(Z) = \{(x,u) | x \in Z,\, u \in U(x),\ f[x,u,W(u,x)] \subset Z\}. \tag{11}$$

We shall use the notation

$$C_n(X) = C[R^{n-1}(X)], \qquad n \geq 2 \tag{12}$$

$$C_1(X) = C(X). \tag{13}$$

It can be easily seen that

$$C_{n+1}(X) \subset C_n(X), \qquad n \geq 1. \tag{14}$$

It can also be seen that from (7), (11), (12), and (13) we have

$$P_x[C_n(X)] = R^n(X), \qquad n \geq 1, \tag{15}$$

and hence

$$P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] \subset \bigcap_{n=1}^{\infty} R^n(X) \tag{16}$$

where $P_x[C_n(X)]$ denotes the projection of the set $C_n(X)$ on the space S_x. Notice that by (15) the set $C_n(X)$ completely determines the set $R^n(X)$ of initial states from which the set X is reachable for n steps. However, a closer examination of (11) reveals that the sets $C_n(X)$, $C_{n-1}(X)\cdots C_1(X)$ completely determine the class of all control laws that achieve reachability of X for n steps. We have the following proposition.

Proposition 3: There holds

$$R^*(X) \subset P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] \subset \bigcap_{n=1}^{\infty} R^n(X). \tag{17}$$

Furthermore, the equation

$$P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] = \bigcap_{n=1}^{\infty} R^n(X) \neq \emptyset \tag{18}$$

implies that

$$R^*(X) = P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] = \bigcap_{n=1}^{\infty} R^n(X) \neq \emptyset. \tag{19}$$

When (18) holds, the class of stationary control laws μ which achieves reachability of the set X in the sense of Corollary 1 is the set of all functions $\mu: S_x \to S_u$ for which

$$[x,\mu(x)] \in \bigcap_{n=1}^{\infty} C_n(X), \quad \forall x \in R^*(X).$$

Proof: If $R^*(X) = \emptyset$, i.e., the set X is not infinitely reachable, (17) holds trivially in view of (16). Let $x_0 \in R^*(X)$ and let μ be a stationary control law that achieves strong reachability of $R^*(X)$ in accordance with Corollary 1. Then by the definition of the sets $C_n(X)$ and $R^*(X)$ we have $(x_0,\mu(x_0)) \in C_n(X)$ for all $n \geq 1$. Hence $(x_0,\mu(x_0)) \in \bigcap_{n=1}^{\infty} C_n(X)$ and $x_0 \in P_x[\bigcap_{n=1}^{\infty} C_n(X)]$. Thus we have $R^*(X) \subset P_x[\bigcap_{n=1}^{\infty} C_n(X)]$. In view of (16) the relation (17) is proved. Assume now that $P_x[\bigcap_{n=1}^{\infty} C_n(X)] = \bigcap_{n=1}^{\infty} R^n(X) \neq \emptyset$. Let μ be any control law such that $[x,\mu(x)] \in \bigcap_{n=1}^{\infty} C_n(x)$ for all $x \in \bigcap_{n=1}^{\infty} R^n(X)$. Then by (7) for every $x \in \bigcap_{n=1}^{\infty} R^n(X)$, $f[x,\mu(x),W[x,\mu(x)]] \subset R^n(X)$ for all $n \geq 1$ and therefore

$$f[x,\mu(x)W[x,\mu(x)]] \subset \bigcap_{n=1}^{\infty} R^n(X) \subset X.$$

This implies that the set $\bigcap_{n=1}^{\infty} R^n(X)$ is strongly reachable. In view of Proposition 2 and (7) the desired equation (19) follows. Q.E.D.

The above proposition shows that, at least for the case where (18) holds, the region of infinite-time reachability $R^*(X)$ is the intersection of the regions of n-step reachability $R^n(X)$. In the next section we will demonstrate that (18) holds in some important special cases by introducing a compactness assumption. Furthermore, we will show that the set $R^*(X)$ is obtained as a well-defined limit of the sequence of sets $R^n(X)$.

When (18) fails to hold we have the following possibilities:

$$\emptyset = R^*(X) \neq P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] \neq \bigcap_{n=1}^{\infty} R^n(X) \tag{20}$$

$$\emptyset = R^*(X) = P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] \neq \bigcap_{n=1}^{\infty} R^n(X) \tag{21}$$

$$\emptyset \neq R^*(X) \neq P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] \neq \bigcap_{n=1}^{\infty} R^n(X) \tag{22}$$

$$\emptyset \neq R^*(X) = P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] \neq \bigcap_{n=1}^{\infty} R^n(X). \tag{23}$$

In these cases the region of n-step reachability exhibits instability when passing to the limit. All four cases are possible in general as the following example shows.

Consider the scalar deterministic system

$$x_{k+1} = f(x_k) + g(x_k)u_k + h[f(x_k) + g(x_k)u_k,x_k]$$

where f, g, and h are the functions given by

$$f(x) = \begin{cases} 2x, & \text{if } |x| \neq 1,5,6 \\ x, & \text{if } |x| = 6 \\ 0, & \text{if } |x| = 5 \\ 10x, & \text{if } |x| = 1 \end{cases}$$

$$g(x) = \begin{cases} 1, & \text{if } |x| \neq 5,6 \\ 0, & \text{if } |x| = 5, |x| = 6 \end{cases}$$

and

$$h(z,x) = \begin{cases} 0, & \text{if } |x| = 5, |x| = 6 \\ 1, & \text{if } z \geq 0, |x| \neq 5,6 \\ -1, & \text{if } z < 0, |x| \neq 5,6. \end{cases}$$

Let $U(x) = [-2,2]$ and consider the problem of infinite-time reachability of the set

$$X = [-2,2] \cup \{-6,-5,5,6\}.$$

It can be verified by straightforward calculation that for $n \geq 1$

$$R^n(X) = \{-6,-5,5,6\} \cup [-(1+2^{-n}),-1) \cup (-1,1) \cup (1,(1+2^{-n})]$$

$$R^*(X) = \{-6,6\}, \qquad P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] = \{-6;-5,5,6\}$$

$$\bigcap_{n=1}^{\infty} R^n(X) = \{-6,-5,5,6\} \cup (-1,1),$$

proving that (22) is possible.

If the points $-6,6$ are dropped from X, then

$$R^*(X) = \emptyset$$

$$P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] = \{-5,5\}$$

$$\bigcap_{n=1}^{\infty} R^n(X) = \{-5,5\} \cup (-1,1),$$

proving that (20) is possible.

If the points $-5,5$ are dropped from X, we have

$$R^*(X) = P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] = \{-6,6\}$$

$$\bigcap_{n=1}^{\infty} R^n(X) = \{-6,6\} \cup (-1,1),$$

proving that (23) is possible.

Finally, if the points $-6,-5,5,6$ are dropped from X, we have

$$R^*(X) = P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] = \emptyset$$

$$\bigcap_{n=1}^{\infty} R^n(X) = (-1,1),$$

proving that (21) is possible.

A final comment concerns the interpretation of the sets

$P_x[\bigcap_{n=1}^{\infty} C_n(X)]$ and $\bigcap_{n=1}^{\infty} R^n(X)$ when (18) fails to hold. Assume that we are given an arbitrary but fixed positive integer k. Then there exists a control law $\{\mu_0,\mu_1,\cdots\}$ depending on k such that when the initial state x_0 is any point in $\bigcap_{n=1}^{\infty} R^n(X)$ the states $x_1,\cdots,x_k$ are guaranteed to belong to the set X. Thus, $\bigcap_{n=1}^{\infty} R^n(X)$ can be characterized as the region of reachability for a finite but arbitrary number of steps. The control law that achieves reachability for any fixed number of steps k must satisfy

$$[x,\mu_n(x)] \in C_{k-n}(X), \quad \forall x \in R^{k-n}(X), \qquad n = 0,1,\cdots,k-1$$

and can be selected to be stationary if desired. However, its value $\mu_n(x)$ for $x \notin P_x[\bigcap_{n=1}^{\infty} C_n(X)]$ will depend on the number of steps k. The set $P_x[\bigcap_{n=1}^{\infty} C_n(X)]$ is the set of points on which this control law can be defined independently of k. When (18) holds the control law can be defined on the whole set $\bigcap_{n=1}^{\infty} R^n(X)$ independently of k and thus achieves reachability of X for any number of steps.

IV. Convergence Questions

A question of importance is under what circumstances, given the set X, one can obtain the set $R^*(X)$ as some "limit" of the sequence of sets $\{R^n(X)\}$. An associated question is under what circumstances (18) is satisfied and, therefore, we have $R^*(X) = \bigcap_{n=1}^{\infty} R^n(X)$. The key to these questions is provided, as one would expect, by compactness conditions. Let us assume that the spaces S_x and S_u are equipped with Hausdorff topologies [3]. The product space $S_x \times S_u$ is considered to be a Hausdorff topological space equipped with the product topology. Then the assumption of compactness of the sets $C_n(X)$ in the latter topology has important implications as the following proposition shows.

Proposition 4: Assume that there exists a positive integer n_0 such that the sets $C_n(X)$ are nonempty and compact for all $n \geq n_0$. Then we have the following.

a) The sets $\bigcap_{n=1}^{\infty} C_n(X)$, $R^*(X)$, $P_x[\bigcap_{n=1}^{\infty} C_n(X)]$, $\bigcap_{n=1}^{\infty} R^n(X)$ are nonempty and compact.

b) The set X is infinitely reachable and

$$\emptyset \neq R^*(X) = P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] = \bigcap_{n=1}^{\infty} R^n(X).$$

c) Given any open set $A \subset S_x$ such that $R^*(X) \subset A$, there exists a positive integer Ns uch that $R^*(X) \subset R^n(X) \subset A$ for all $n \geq N$.

Proof: The fact that $\bigcap_{n=1}^{\infty} C_n(X)$ is nonempty and compact follows directly from [3, p. 225, theorem 1.6]. Since the projection P_x: $S_x \times S_u \rightarrow S_x$ is a continuous map and in view of (15), the relation

$$\emptyset \neq P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] = \bigcap_{n=1}^{\infty} P_x[C_n(X)] = \bigcap_{n=1}^{\infty} R^n(X)$$

follows from the conclusion of [3, p. 252, problem 8]. This relation can be proved as follows. We have in general

$$\emptyset \neq P_x\left[\bigcap_{n=1}^{\infty} C_n(X)\right] \subset \bigcap_{n=1}^{\infty} P_x[C_n(X)],$$

and hence it is sufficient to prove the reverse inclusion. If $x \in \bigcap_{n=1}^{\infty} P_x[C_n(X)]$, then there exists a sequence $\{u_n\}$, $n \geq n_0$ such that $(x,u_m) \in C_n(X)$, $m \geq n$. The sequence (x,u_m) by the compactness of $C_n(X)$ has at least one limit point $(x,u) \in C_n(X)$, for all $n \geq n_0$. Hence $(x,u) \in \bigcap_{n=1}^{\infty} C_n(X)$, implying that $x \in P_x[\bigcap_{n=1}^{\infty} C_n(X)]$. Therefore, the reverse inclusion is proved. Now, by using Proposition 4, parts a) and b) are proven. The statement of c) follows again from [3, p. 225, theorem 1.6]. Q.E.D.

Proposition 4c) is particularly interesting since it states that under the assumptions of the proposition the sequence of sets $\{R^n(X)\}$ converges in a well-defined sense to the set $R^*(X)$.

The compactness of the sets $C_n(X)$ can be verified in a number of interesting special cases. An important special case where it can be trivially verified is when the sets X and $U(x)$ are finite (as they will be in any digital computer solution of the problem), and the spaces S_x and S_u are equipped with their discrete topology [3]. In this topology a set is compact if and only if it is finite. Then the sets $C_n(X)$ are finite and therefore compact, and the sequence $\{R^n(X)\}$ will converge to $R^*(X)$ in a finite number of steps. Another special case is when the system equation is of the form

$$x_{k+1} = f(x_k,u_k) + w_k, \qquad k = 0,1,\cdots \tag{24}$$

where S_x, S_u, S_w are Euclidean spaces of appropriate dimension equipped with the usual norm topology. In this topology a set is compact if and only if it is closed and bounded. Consider the case where the sets U and W are independent of x and (x,u), respectively, and assume furthermore that the sets X, U are compact and that the function f is continuous. The set $C(X)$ of (11) can be written for this case as

$$C(X) = \{(x,u) | x \in X, u \in U, f(x,u) \in E\} \tag{25}$$

where the set E is given by

$$E = \{z | z + W \subset X\}. \tag{26}$$

Let us first show that the compactness of X implies the compactness of E. If $z \notin E$, there exists a $w \in W$ such that $x = (z + w) \notin X$. Since X is closed there exists an open ball B_x centered at the origin such that $(x + B_x) \cap X = \emptyset$. This in turn implies that $(z + B_x) \cap E = \emptyset$, i.e., the complement of E is open and therefore E is closed. Since from (26) E is also bounded, the compactness of E is proven. Now from (25) the compactness of E, X, U and the continuity of f imply the compactness of $C(X)$ and therefore, by (15), the compactness of $R(X)$. By using similar reasoning the compactness of $C_n(X)$ can be proven for all n. Thus, whenever these sets are nonempty the conclusions of Proposition 4 hold. When the system of (24) is linear,

$$x_{k+1} = Ax_k + u_k + w_k, \qquad k = 0,1,\cdots,$$

the compactness of the sets $C_n(X)$ can be proven similarly whenever the set X is compact and the set U is closed.

The above special case can be easily generalized to the case where the system equation is given by (24) but the spaces S_x and S_u are instead reflexive Banach spaces equipped with their weak topologies [4]. Then the product space $S_x \times S_u$ with any of the norms

$$||(x,u)|| = \{||x||^p + ||u||^p\}^{1/p}, \qquad 1 \leq p < \infty,$$
$$||(x,u)|| = \max \{||x||,||u||\}$$

is a reflexive Banach space with a weak topology coinciding with the product topology. By assuming that the sets X and U are weakly compact and that the function f is weakly continuous, we can prove again that the assumption of Proposition 4 is satisfied. The proof is entirely similar to the finite-dimensional case and is based on the fact that in a reflexive Banach space a set is compact if and only if it is bounded and weakly closed [4].

V. The Case of a Linear Finite-Dimensional System

In this section we consider a special case which involves the linear discrete-time system

$$x_{k+1} = Ax_k + Bu_k + Gw_k, \qquad k = 0,1,\cdots \tag{27}$$

where $x_k \in E^n$ (n-dimensional Euclidean space), $u_k \in E^m$, $w_k \in E^r$, and the matrices A, B, G have the appropriate dimensions with the matrix A assumed invertible.[1] We assume that the control u_k and the disturbance w_k are restricted to take values in the ellipsoids

$$U = \{u|u'Ru \leq 1\} \tag{28}$$
$$W = \{w|w'Qw \leq 1\} \tag{29}$$

where R and Q are positive-definite symmetric matrices of appropriate dimensions.

Our objective in this section is to show that there exist strongly reachable ellipsoidal sets corresponding to the system (27) and the constraint sets (28) and (29) provided that the system is stabilizable, i.e., there exists a matrix L such that the matrix $(A - BL)$ is stable (has eigenvalues within the unit disk of the complex plane). At the same time we will be able to show that there exists a linear control law that achieves reachability and makes the closed-loop system asymptotically stable; we will give an efficient algorithm for its computation.

In order for an ellipsoid

$$X = \{x|x'Kx \leq 1\}, \tag{30}$$

where K is a positive-definite symmetric matrix, to be a strongly reachable set, we must have, by Proposition 1 and (27),

$$X = \{x| \ni u \in U \text{ s.t. } Ax + Bu + GW \subset X\} \cap X. \tag{31}$$

It can be seen that this condition is equivalent to

$$AX \subset [E + (-B)U] \tag{32}$$

where the set E is defined as

$$E = \{z|z + GW \subset X\}. \tag{33}$$

Conditions (32) and (33) are stated in terms of sets and are, therefore, difficult to verify. A sufficient condition in terms of the matrices K, Q, R and A, B, G in order that the conditions (32) and (33) hold is given by the following proposition. Furthermore, when this sufficient condition holds, the proposition shows that reachability can be achieved by a linear time-invariant control law for which the resulting closed-loop system is asymptotically stable.

Proposition 5: A sufficient condition for the relations (32) and (33) to hold and therefore for the ellipsoid $X = \{x|x'Kx \leq 1\}$ to be strongly reachable is that

$$K = A'(F^{-1} + BR^{-1}B')^{-1}A + \psi \tag{34}$$

for some positive-definite matrix ψ and for some $0 < \beta < 1$ for which the matrix F defined below is positive definite

$$F = \left[(1 - \beta)K^{-1} - \frac{1 - \beta}{\beta} GQ^{-1}G'\right]^{-1} > 0. \tag{35}$$

Under these circumstances a linear time-invariant control law which achieves reachability is given by

$$\mu(x) = -(R + B'FB)^{-1}B'FAx. \tag{36}$$

With this control law the resulting closed-loop system is asymptotically stable.

Proof: Assume that (34) and (35) hold, and consider the ellipsoid

$$\bar{E} = \{z|z'Fz \leq 1\}. \tag{37}$$

We shall show that $\bar{E} \subset E$. By (33) it is sufficient to prove that $[\bar{E} + GW] \subset X$. The support function [6] of the set X is given by

$$\sigma(q|X) = \sup \{\langle q,x\rangle|x \in X\} = (q'K^{-1}q)^{\frac{1}{2}} \tag{38}$$

and the support function of $\bar{E} + GW$ is given by

$$\sigma(q|\bar{E} + GW) = (q'F^{-1}q)^{\frac{1}{2}} + (q'GQ^{-1}G'q)^{\frac{1}{2}}. \tag{39}$$

It can be seen from (35) that the inequality $\sigma(q|\bar{E} + GW) \leq \sigma(q|X)$ holds for all $q \in E^n$ implying $\bar{E} + GW \subset X$, and thus $\bar{E} \subset E$.

In order to prove (32) it is therefore sufficient to prove

$$AX \subset [\bar{E} + (-B)U]. \tag{40}$$

The support function of AX is given by

$$\sigma(q|AX) = (q'AK^{-1}A'q)^{\frac{1}{2}},$$

or, by using (34),

$$\sigma(q|AX) = [q'[(F^{-1} + BR^{-1}B')^{-1} + A'^{-1}\psi A^{-1}]^{-1}q]^{\frac{1}{2}} \leq [q'(F^{-1} + BR^{-1} B')q]^{\frac{1}{2}}. \tag{41}$$

[1] The results of this section can be proven without the assumption of invertibility of A at the expense of somewhat more complicated derivations. The matrix A, however, will be invertible in most cases, including the case where the system (27) results from sampling of a continuous-time system.

The support function of $[\bar{E} + (-B)U]$ is given by

$$\sigma(q|\bar{E} + (-B)U) = (q'F^{-1}q)^{\frac{1}{2}} + (q'BR^{-1}B'q)^{\frac{1}{2}}. \quad (42)$$

By comparing (41) and (42) we obtain

$$\sigma(q|AX) \leq \sigma(q|\bar{E} + (-B)u), \qquad \text{for all } q \in E^n$$

implying (40) and therefore (32) and (33).

In order to show that the control law (36) achieves reachability, it is sufficient to show that

$$x_{k+1} = [A - B(R + B'FB)^{-1}B'FA]x_k \in \bar{E}$$

whenever $x_k \in X$. By denoting

$$L = (R + B'FB)^{-1}B'FA,$$

it is sufficient to show that

$$x_k'(A - BL)'F(A - BL)x_k \leq x_k'Kx_k \leq 1.$$

This last relation is evident from the following identity:

$$K = (A - BL)'F(A - BL) + \psi + L'RL, \quad (43)$$

which is well known in Riccati equation theory.

In order to show that the closed-loop system is asymptotically stable, first notice that from (35)

$$F^{-1} = (1 - \beta)\left(K^{-1} - \frac{1}{\beta}GQ^{-1}G'\right)$$
$$< K^{-1} - \frac{1}{\beta}GQ^{-1}G' \leq K^{-1},$$

implying

$$K < F. \quad (44)$$

Also consider the system

$$x_{k+1} = (A - BL)x_k. \quad (45)$$

Then, by using (43), (44), and (45), we have, for every index $N > 0$, the following:

$$\begin{aligned}
&x_N'Kx_N + \sum_{k=0}^{N-1} x_k'(\psi + L'RL)x_k \\
&< x_N'Fx_N + \sum_{k=0}^{N-1} x_k'(\psi + L'RL)x_k \\
&= x_{N-1}'[(A - BL)'F(A - BL) + \psi \\
&\quad + L'RL]x_{N-1} + \sum_{k=0}^{N-2} x_k'(\psi + L'RL)x_k \\
&= x_{N-1}'Kx_{N-1} + \sum_{k=0}^{N-2} x_k'(\psi + L'RL)x_k \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&< x_1'Kx_1 + x_0'(\psi + L'RL)x_0 \\
&< x_0'[(A - BL)'F(A - BL) + \psi + L'RL]x_0 \\
&= x_0'Kx_0.
\end{aligned}$$

Thus for every $N > 0$ we have

$$x_N'Kx_N + \sum_{k=0}^{N-1} x_k'(\psi + L'RL)x_k < x_0'Kx_0,$$

which implies that the system (45) is asymptotically stable. Q.E.D.

An immediate consequence of the asymptotic stability proved above is that transients due to initial states will vanish eventually during the operation of the closed-loop system. More accurately, for any $\epsilon > 0$ it can be guaranteed that after a sufficient number of steps the state of the system will be confined in the set $X + \epsilon B$ where B is the unit ball in E^n; this will occur for every initial state in E^n.

A question of importance is under what conditions there exist ellipsoids X and corresponding matrices K for which the conditions (34) and (35) in Proposition 5 are satisfied. Furthermore, it is necessary to provide means for the computation of such matrices. This computation is possible by making use of the recursive algorithm

$$K_{i-1} = A(F_i^{-1} + BR^{-1}B')^{-1}A' + \psi \quad (46)$$

$$F_i = \left[(1 - \beta)K_i^{-1} - \frac{1 - \beta}{\beta}GQ^{-1}G'\right]^{-1} \quad (47)$$

$$K_N = \psi \quad (48)$$

where N denotes the time index where the backwards computation starts. This algorithm has been considered in [1], [2] in connection with finite-time reachability problems. The convergence of the algorithm to a steady-state solution, which satisfies the conditions (34) and (35), has been considered in detail in [1]. A closer examination of (46) and (47) shows that the matrices R and ψ must be relatively "small" or else the algorithm will not converge to a positive-definite solution. Now in any practical situation one is given the matrix Q specifying the constraint set for the input disturbance, and there is usually a certain degree of freedom in adjusting the control constraints, i.e., the matrix R, and of course, the matrix ψ, which plays the role of a design parameter. In this sense a possible design procedure is to initially select the matrices R and ψ and, in case the algorithm does not converge to a solution satisfying (34) and (35), to decrease these matrices by multiplication with scalars less than one and repeat the procedure until convergence and satisfaction of the designer. It is important, however, to know under what circumstances there exist matrices R and ψ such that the algorithm converges to a steady state, and furthermore, under what conditions such matrices can be obtained by repeatedly multiplying any initially selected matrices R_1 and ψ_1 by factors of less than one. This is the object of the next proposition which states that the design procedure outlined above is successful provided the system (27) is stabilizable, i.e., if there exists a matrix L such that the matrix $(A - BL)$ is stable. Notice that the system (27) is stabilizable provided that the pair (A,B) is controllable (but not conversely) [11]. This proposition has been proven earlier [1]. Unfortunately, the proof is quite lengthy and, due to space limitation, it will not be reproduced here.

Proposition 6: Assume that the system (27) and the positive-definite symmetric matrix Q are given and that the system (27) is stabilizable. Then given any positive-definite symmetric matrices ψ_1 and R_1 of appropriate dimension, there exists a scalar β_1, $0 < \beta_1 < 1$ such that for every scalar β, $0 < \beta \leq \beta_1$, there exist scalars a_1, b_1 depending on β such that for all matrices $\psi = a\psi_1$, $R = bR_1$ with $0 < a \leq a_1$, $0 < b \leq b_1$, the algorithm of (46)–(48) converges to a positive-definite symmetric matrix K satisfying (34) and (35).

Proposition 6 shows the existence of strongly reachable ellipsoidal sets for the case of the linear system (27) provided that this system is stabilizable and the control constraint set is taken sufficiently large. Furthermore, such ellipsoidal sets can be efficiently computed using the algorithm of (46)–(48). The same algorithm provides the control law (36), which is linear and makes the closed-loop system asymptotically stable.

VI. Conclusions

In this paper we considered the question of whether the state of an uncertain system can be kept within a specified region of the state space for infinite time by using feedback control. This question is basic in problems of feedback control of uncertain systems subject to state constraints over an infinite-time interval since it relates to the behavior of the region of feasibility as the control interval tends to infinity. The notion of strong reachability of a set is of central importance in the problem of the paper since it was proved that in order to achieve infinite-time reachability of the feasible region the state of the system must be confined within some strongly reachable subset of the feasible region. A related question also examined in this paper concerns the limiting behavior of the region of n-step reachability as n tends to infinity, and is important in the analysis of dynamic programmieg algorithms over an infinite-time interval. It has received further attention in [13], where the convergence of a dynamic programming algorithm related to a stochastic optimal control problem similar to the one in [12] is proved.

When the uncertain system is stochastic, the class of admissible control laws may be further restricted by measurability requirements. The question of reachability with a control law within such a restricted class has not been touched upon in this paper. It should be expected, however, that the investigation of this question will benefit from the results presented. As an example, consider the case examined in Section IV involving the system (24) and the question of whether there exists a Borel measurable control law $\mu: X \to U$ that achieves infinite-time reachability of the compact set X. By Proposition 3 this amounts to asking whether there exists a Borel measurable selector for the multivalued mapping

$$M(x) = \left\{u \middle| (x,u) \in \bigcap_{n=1}^{\infty} C_n(X)\right\},$$

which maps the set $R^*(X)$ (assumed nonempty) into the set of all subsets of S_u. Now for every closed subset S of S_u, the set

$$M^{-1}(S) = \{x \in R^*(X) | M(x) \cap S \neq \emptyset\} = P_x\left[\bigcap_{n=1}^{\infty} C_n(X) \cap (R^*(X)xS)\right]$$

is compact by the compactness of $\bigcap_{n=1}^{\infty} C_n(X)$. Hence the mapping M is Borel measurable according to the definition of [7]. By using a theorem of Kuratowski and Rull-Nardzewski it follows that there exists a Borel measurable selector-control law (see [7, corollary 1.1]). Thus for this case, whenever there exists a control law achieving infinite-time reachability of the set X, this control law can be taken to be Borel measurable.

The special case of a linear finite-dimensional system was examined in the latter part of the paper with emphasis on obtaining ellipsoidal sets within which the state can be confined by using feedback control. An efficient algorithm was given for calculating such sets and associated linear control laws. Since the formulation of the regulation problem in terms of hard-state space constraints is attractive in some cases, the design procedure suggested appears to have potential for some practical applications.

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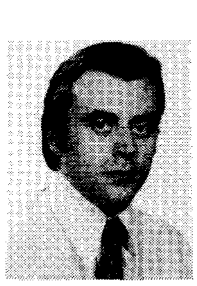

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Extension of Analytical Design Techniques to Multivariable Feedback Control Systems

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Abstract—Analytical design techniques are developed for multivariable feedback control systems. The design includes a saturation constraint and provides for a disturbance at the system output and additive noise at the system input. The system inputs (signal, noise, and disturbance) are assumed to be generated by independent, stationary, stochastic processes that are adequately represented by rational power-spectral-density matrices. System elements are represented by rational transfer function matrices using the bilateral Laplace transform. The design is applicable to linear time-invariant systems. Design formulas are derived for the general case where the transfer function matrix representing the fixed elements of the system may not be square.

The basic design consists of minimizing a weighted sum of the output mean-square errors and the mean-square values of a selected set of saturation signals. A variational technique is used in the optimization, and the technique of spectral factorization is used to obtain a solution. An example is presented to illustrate the design procedure.

Introduction

The analytical design techniques of Newton *et al.* [1] are extended to multivariable feedback control systems in this paper. The system configuration under consideration is shown in Fig. 1. All elements of the system are assumed to be linear and time invariant. The input vectors (signal, noise, and disturbance) are generated by stationary stochastic processes whose power spectral densities are known. The design includes a saturation constraint and provides for a disturbance input when the feedback network can be fixed *a priori*.

The analysis is carried out in the frequency domain using the bilateral Laplace transform. A variational technique employed by Bongiorno [2] is used to minimize a performance index consisting of a weighted sum of the output mean-square errors plus a weighted sum of the mean-square values of a set of saturation signals (i.e., a set of signals that might tend to saturate in amplitude if not constrained).

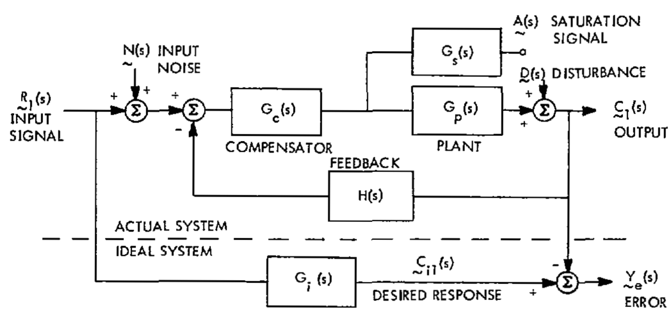

Fig. 1. Multivariable system. Wavy underline is italic boldface in text.

Some aspects of this problem have been solved in recent years; however, most attempts have been hampered by the need to perform matrix spectral factorizations. Youla [3] derived conditions under which a rational matrix may be spectrally factored, and presented an algorithm for accomplishing the factorization. Davis [4] presented an alternate method of performing the factorization, and Tuel [5] developed a numerical solution and a computer program to perform the factorization.

A number of papers have been published that treat related topics. Amara [6] solved the multivariable free-configuration Wiener problem, relying on matrix spectral factorization. Hsieh and Leondes [7] first developed a solution for the semi-free-configuration Wiener problem in the form of a set of simultaneous algebraic equations, thus avoiding the need to perform the matrix spectral factorization. However, Hsieh and Leondes did not prove that a solution to their equations existed and Davis [4] states that their method fails in the case of a predictor. Bongiorno [2] recently solved this same problem using

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