Incremental Gradient, Subgradient, and Proximal Methods for Convex Optimization

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Problems with Many Additive Cost and Constraint Components

\[
\text{minimize } \sum_{i=1}^{m} f_i(x) \quad \text{subject to } \quad x \in X = \bigcap_{\ell=1}^{q} X_\ell,
\]

where \( f_i : \mathbb{R}^n \mapsto \mathbb{R} \) are convex, and the sets \( X_\ell \) are closed and convex.

Incremental algorithm: Typical iteration

- Choose indexes \( i_k \in \{1, \ldots, m\} \) and \( \ell_k \in \{1, \ldots, q\} \).
- Perform a subgradient iteration or a proximal iteration

\[
x_{k+1} = P_{X_{\ell_k}} \left( x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k) \right) \quad \text{or} \quad x_{k+1} = \arg \min_{x \in X_{\ell_k}} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k} \| x - x_k \|_2^2 \right\}
\]

where \( \alpha_k \) is a positive stepsize and \( \tilde{\nabla} \) denotes (any) subgradient.

Motivation

- Avoid processing all the cost components at each iteration
- Use a simpler constraint to simplify the projection or the proximal minimization
Joint and individual works with A. Nedic and M. Wang.
Focus on convergence, rate of convergence, component formation, and component selection.

Work on **incremental gradient methods and extended Kalman filter** for least squares, 1994-1997 (DPB).
Work on **incremental subgradient methods** with A. Nedic, 2000-2010.
Work on **incremental proximal methods**, 2010-2012 (DPB).
Work on **incremental constraint projection methods** with M. Wang 2012-2014 (following work by A. Nedic in 2011).

See our websites.
Outline

1. Incremental Algorithms

2. Two Methods for Incremental Treatment of Constraints

3. Convergence Analysis
**Problem:** \( \min_{x \in X} \sum_{i=1}^{m} f_i(x) \), where \( f_i \) and \( X \) are convex.

**Long history:** LMS (Widrow-Hoff, 1960, for linear least squares w/out projection), former Soviet Union literature 1960s, stochastic approximation literature 1960s, neural network literature literature 1970s

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**Basic incremental subgradient method**

\[ x_{k+1} = P_X \left( x_k - \alpha_k \nabla f_{i_k}(x_k) \right) \]

- **Stepsizes selection possibilities:**
  - \( \sum_{k=0}^{\infty} \alpha_k = \infty \) and \( \sum_{k=0}^{\infty} \alpha_k^2 < \infty \)
  - \( \alpha_k \): Constant
  - Dynamically chosen (based on estimate of optimal cost)

- **Index \( i_k \) selection possibilities:**
  - Cyclically
  - Fully randomized/equal probability \( 1/m \)
  - Reshuffling/randomization within a cycle (frequent practical choice)
Convergence Mechanism

Quadratic One-Dimensional Example: \( \min_{x \in \mathbb{R}} \sum_{i=1}^{m} (c_i x - b_i)^2 \)

Conceptually, the idea generalizes to higher dimensions, but is hard to treat/quantify analytically.

Adapting the stepsize \( \alpha_k \) to the farout and confusion regions is an important issue.

Shaping the confusion region is an important issue.
Variants of Incremental Subgradient Methods

Method with momentum/extrapolation/heavy ball: $\beta_k \in [0, 1)$

\[
x_{k+1} = P_X \left( x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_k) + \beta_k (x_k - x_{k-1}) \right)
\]

Accelerates in the farout region, decelerates in the confusion region.

Aggregated incremental gradient method

\[
x_{k+1} = P_X \left( x_k - \alpha_k \sum_{j=0}^{m-1} \tilde{\nabla} f_{i_{k-j}}(x_{k-j}) \right)
\]

- Proposed for differentiable $f_i$, no constraints, cyclic index selection, and constant stepsize, by Blatt, Hero, and Gauchman (2008).
- Recent work by Schmidt, Le Roux, and Bach (2013), randomized index selection, and constant stepsize.
- A fundamentally different convergence mechanism (relies on differentiability and aims at cost function descent). Works even with a constant stepsize (no region of confusion).
Select index $i_k$ and set

$$x_{k+1} = \arg\min_{x \in \mathcal{X}} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}$$

Many similarities with incremental subgradient

- Similar stepsize choices
- Similar index selection schemes
- Can be written as

$$x_{k+1} = P_{\mathcal{X}}(x_k - \alpha_k \tilde{\nabla} f_{i_k}(x_{k+1}))$$

where $\tilde{\nabla} f_{i_k}(x_{k+1})$ is a special subgradient at $x_{k+1}$ (index advanced by 1)

Compared to incremental subgradient

- Likely more stable
- May be harder to implement
Select index $i_k$ and set

$$x_{k+1} = \arg \min_{x \in X} \left\{ f_{i_k}(x) + \sum_{j=1}^{m-1} \tilde{\nabla} f_{i_k-j}(x_{k-j+1})'(x - x_k) + \frac{1}{2\alpha_k} \| x - x_k \|^2 \right\}$$

where $\tilde{\nabla} f_{i_k-j}(x_{k-j+1})$ is special subgradient at $x_{k-m+1}$ (index advanced by 1)

- Can be written as

$$x_{k+1} = P_x \left( x_k - \alpha_k \sum_{j=0}^{m-1} \tilde{\nabla} f_{i_k-j}(x_{k-j+1}) \right)$$

- More stable (?) than incremental subgradient or proximal
- May be harder to implement
- Convergence can be shown if $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$
Typical iteration

Choose $i_k \in \{1, \ldots, m\}$ and do a subgradient or a proximal iteration

$$
x_{k+1} = P_X(x_k - \alpha_k \nabla f_{i_k}(x_k)) \quad \text{or} \quad x_{k+1} = \arg \min_{x \in X} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k} \|x - x_k\|^2 \right\}
$$

where $\alpha_k$ is a positive stepsize and $\nabla$ denotes (any) subgradient.

- Idea: Use proximal when easy to implement; use subgradient otherwise
- A very flexible implementation
- The proximal iterations still require diminishing $\alpha_k$ for convergence
Convergence Analysis

Under Lipschitz continuity-type assumptions:

- Convergence to the optimum for diminishing stepsize.
- Convergence to a neighborhood of the optimum for constant stepsize.
- Faster convergence for randomized index selection (relative to a worst-case cyclic choice).

Notes:

- Fundamentally different from the proximal gradient method, which applies when $m = 2$, 
  \[
  \min_{x \in \mathcal{X}} \left\{ f_1(x) + f_2(x) \right\},
  \]
  
  and $f_1$ is differentiable. This is a cost descent method and can use a constant stepsize.
- Aggregated version possible
Problem

\[
\begin{align*}
\text{minimize} \quad & \sum_{i=1}^{m} f_i(x) \\
\text{subject to} \quad & x \in \cap_{\ell=1}^{q} X_{\ell},
\end{align*}
\]

where \( f_i : \mathbb{R}^n \mapsto \mathbb{R} \) are convex, and the sets \( X_{\ell} \) are closed and convex.

Equivalent Problem (Assuming \( f_i \) are Lipschitz Continuous)

\[
\begin{align*}
\text{minimize} \quad & \sum_{i=1}^{m} f_i(x) + \gamma \sum_{\ell=1}^{q} \text{dist}(x, X_{\ell}) \\
\text{subject to} \quad & x \in \mathbb{R}^n,
\end{align*}
\]

where \( \gamma \) is sufficiently large (the two problems have the same set of minima).

Proximal iteration on the \( \text{dist}(x, X_{\ell}) \) function is easy

Project on \( X_{\ell} \) and interpolate:

\[
x_{k+1} = (1 - \beta_k)x_k + \beta_k P_{x_i k}(x_k), \quad \beta_k = \min \left\{ 1, \frac{\alpha_k \gamma}{\text{dist}(x_k; X_{i k})} \right\}
\]

(since \( \gamma \) is large, usually \( \beta_k = 1 \)).
minimize \( \sum_{i=1}^{m} f_i(x) \) \( \text{subject to} \quad x \in \bigcap_{\ell=1}^{q} X_{\ell}, \)

where \( f_i : \mathbb{R}^n \rightarrow \mathbb{R} \) are convex, and the sets \( X_{\ell} \) are closed and convex.

**Incremental constraint projection algorithm**

- Choose indexes \( i_k \in \{1, \ldots, m\} \) and \( \ell_k \in \{1, \ldots, q\} \).
- Perform a subgradient iteration or a proximal iteration

\[
x_{k+1} = P_{X_{\ell_k}}(x_k - \alpha_k \nabla f_{i_k}(x_k)) \quad \text{or} \quad x_{k+1} = \arg \min_{x \in X_{\ell_k}} \left\{ f_{i_k}(x) + \frac{1}{2\alpha_k} \| x - x_k \|^2 \right\}
\]

where \( \alpha_k \) is a positive stepsize and \( \nabla \) denotes (any) subgradient.

First proposal and analysis of the case where \( m = 1 \) and some of the constraints are explicit

\[
X_{\ell} = \{ x \mid g_{\ell}(x) \leq 0 \}
\]

was by A. Nedic (2011). Connection to feasibility/alternating projection methods.
Comparison of the two methods

Second method does not require a penalty parameter $\gamma$, but needs a linear regularity assumption: For some $\eta > 0$,

$$\|x - P \cap_{\ell=1}^q X_\ell(x)\| \leq \eta \max_{\ell=1,\ldots,q} \|x - P_{X_\ell}(x)\|, \quad \forall x \in \mathbb{R}^n$$

Both methods require diminishing stepsize $\alpha_k$. Unclear how to construct an aggregated version, or any version that is convergent with a constant stepsize.

The second method involves an interesting two-time scale convergence analysis (the subject of the remainder of this talk).
Problem

\[
\text{minimize } \sum_{i=1}^{m} f_i(x) \quad \text{subject to } \quad x \in X = \cap_{\ell=1}^{q} X_{\ell},
\]

Typical iteration

- Choose randomly indexes \( i_k \in \{1, \ldots, m\} \) and \( \ell_k \in \{1, \ldots, q\} \).
- Set
  \[
  x_{k+1} = P_{X_{\ell_k}} (x_k - \alpha_k \tilde{\nabla} f_{i_k}(\bar{x}_k))
  \]
- \( \bar{x}_k = x_k \) or \( \bar{x} = x_{k+1} \).
- \( \sum_{k=0}^{\infty} \alpha_k = \infty \) and \( \sum_{k=0}^{\infty} \alpha_k^2 < \infty \) (diminishing stepsize is essential).

Two-way progress

- Progress to feasibility: The projection \( P_{X_{\ell_k}} (\cdot) \).
- Progress to optimality: The “subgradient” iteration \( x_k - \alpha_k \tilde{\nabla} f_{i_k}(\bar{x}_k) \).
Progress to feasibility should be faster than progress to optimality. Gradient stepsizes $\alpha_k$ should be $<<$ than the feasibility stepsize of 1.
Nearly independent sampling

\[ \inf_{k \geq 0} \text{Prob}(\ell_k = X_\ell \mid \mathcal{F}_k) > 0, \quad \ell = 1, \ldots, q, \]

where \( \mathcal{F}_k \) is the history of the algorithm up to time \( k \).

Cyclic sampling

Deterministic or random reshuffling every \( q \) iterations.

Most distant constraint sampling

\[ \ell_k = \arg \max_{\ell=1,\ldots,q} \| x_k - P_{X_\ell}(x_k) \| \]

Markov sampling

Generate \( \ell_k \) as the state of an ergodic Markov chain with states 1, \ldots, q.
Random independent uniform sampling

Each index $i \in \{1, \ldots, m\}$ is chosen with equal probability $1/m$, independently of earlier choices.

Cyclic sampling

Deterministic or random reshuffling every $m$ iterations.

Markov sampling

Generate $i_k$ as the state of a Markov chain with states $1, \ldots, m$, and steady state distribution $\{1/m, \ldots, 1/m\}$. 
Assuming Lipschitz continuity of the cost, linear regularity of the constraint, and nonemptiness of the optimal solution set, \( \{x_k\} \) converges to some optimal solution \( x^* \) w.p. 1, under any combination of the preceding sampling schemes.

Idea of the convergence proof

There are two convergence processes taking place:

- **Progress towards feasibility**, which is fast (geometric thanks to the linear regularity assumption).
- **Progress towards optimality**, which is slower (because of the diminishing stepsize \( \alpha_k \)).

This two-time scale convergence analysis idea is encoded in a **coupled supermartingale convergence theorem**, which governs the evolution of two measures of progress

\[
E[\text{dist}^2(x_k, X)] : \text{Distance to the constraint set, which is fast}
\]

\[
E[\text{dist}^2(x_k, X^*)] : \text{Distance to the optimal solution set, which is slow}
\]
Concluding Remarks

- Incremental methods exhibit interesting and complicated convergence behavior
- Proximal variants enhance reliability
- Constraint projection variants provide flexibility and enlarge the range of potential applications
- Issues not discussed:
  - Distributed asynchronous implementation
  - Incremental Gauss-Newton methods (Extended Kalman Filter)
Thank you!