SET INTERSECTION THEOREMS

AND

EXISTENCE OF OPTIMAL SOLUTIONS FOR

CONVEX AND NONCONVEX OPTIMIZATION

Dimitri P. Bertsekas and Paul Tseng

Math. Programming Journal, 2007

NESTED SET SEQUENCE INTERSECTIONS

- **Basic Question:** Given a nested sequence of nonempty closed sets $\{S_k\}$ in $\Re^n$ ($S_{k+1} \subset S_k$ for all k), when is $\cap_{k=0}^{\infty} S_k$ nonempty?

NESTED SET SEQUENCE INTERSECTIONS

- **Basic Question:** Given a nested sequence of nonempty closed sets $\{S_k\}$ in $\Re^n$ ($S_{k+1} \subset S_k$ for all k), when is $\cap_{k=0}^{\infty} S_k$ nonempty?
- Set intersection theorems are significant in at least four major contexts:
  - **Existence of optimal solutions**
  - **Preservation of closedness by linear transformations**
  - **Duality gap issue,** i.e., equality of optimal values of the primal convex problem

    $$\text{minimize}_{x \in X,\, g(x) \leq 0}\, f(x)$$

    and its dual

    $$\text{maximize}_{\mu \geq 0}\, q(\mu) \equiv \inf_{x \in X} \big\{ f(x) + \mu' g(x) \big\}$$

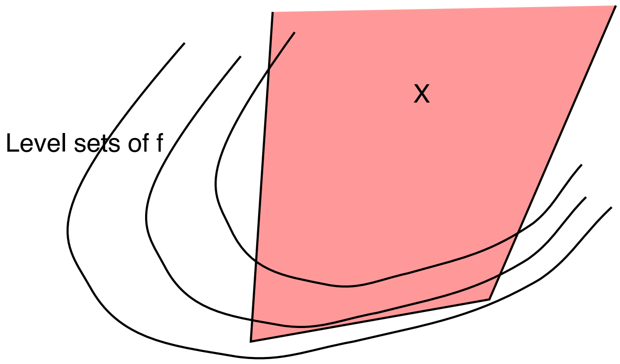
  - **min-max = max-min issue,** i.e., whether

    $$\min_x \max_z \phi(x, z) = \max_z \min_x \phi(x, z),$$

    where ϕ is convex in x and concave in z

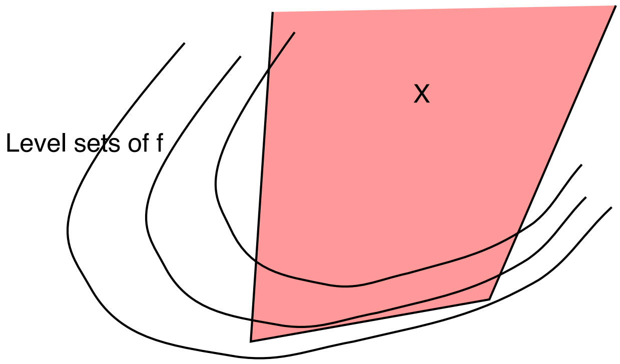
SOME SPECIFIC CONTEXTS I

- **Does a function $f : \Re^n \mapsto (-\infty, \infty]$ attain a minimum over a set X?**
  - This is true iff the intersection of the nonempty sets $\{x \in X \mid f(x) \leq \gamma\}$ is nonempty

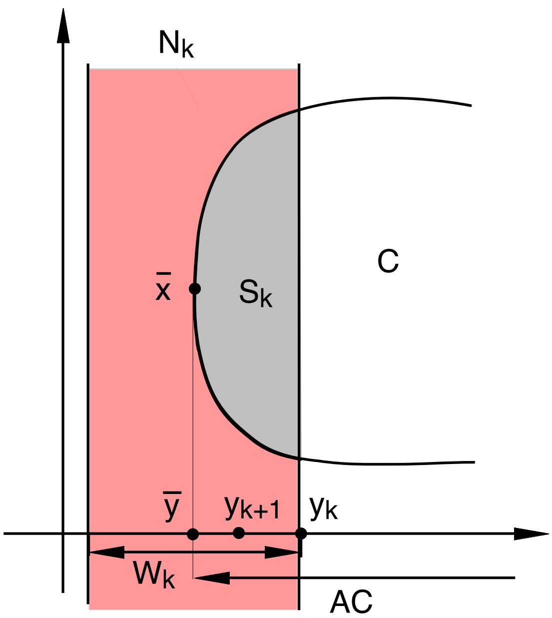


SOME SPECIFIC CONTEXTS I

- **Does a function $f : \Re^n \mapsto (-\infty, \infty]$ attain a minimum over a set X?**
  - This is true iff the intersection of the nonempty sets $\{x \in X \mid f(x) \leq \gamma\}$ is nonempty



- **If C is closed, is AC closed?**



  - Many interesting special cases, e.g., if C_1 and C_2 are closed, is $C_1 + C_2$ closed?

SOME SPECIFIC CONTEXTS II

- **Preservation of closedness by partial minima:** If $F(x,u)$ is closed, is $p(u) = \inf_x F(x,u)$ closed?
  - Critical question in the **duality gap** issue, where

    $$F(x,u) = \begin{cases} f(x) & \text{if } x \in X,\ g(x) \leq u, \\ \infty & \text{otherwise} \end{cases}$$

    and p is the primal function.
  - Critical question regarding **min-max=max-min** where

    $$F(x,u) = \begin{cases} \sup_{z \in Z}\{\phi(x,z) - u'z\} & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

    We have min-max=max-min if

    $$p(u) = \inf_{x \in \Re^n} F(x,u)$$

    is closed.
  - Can be addressed by using the relation

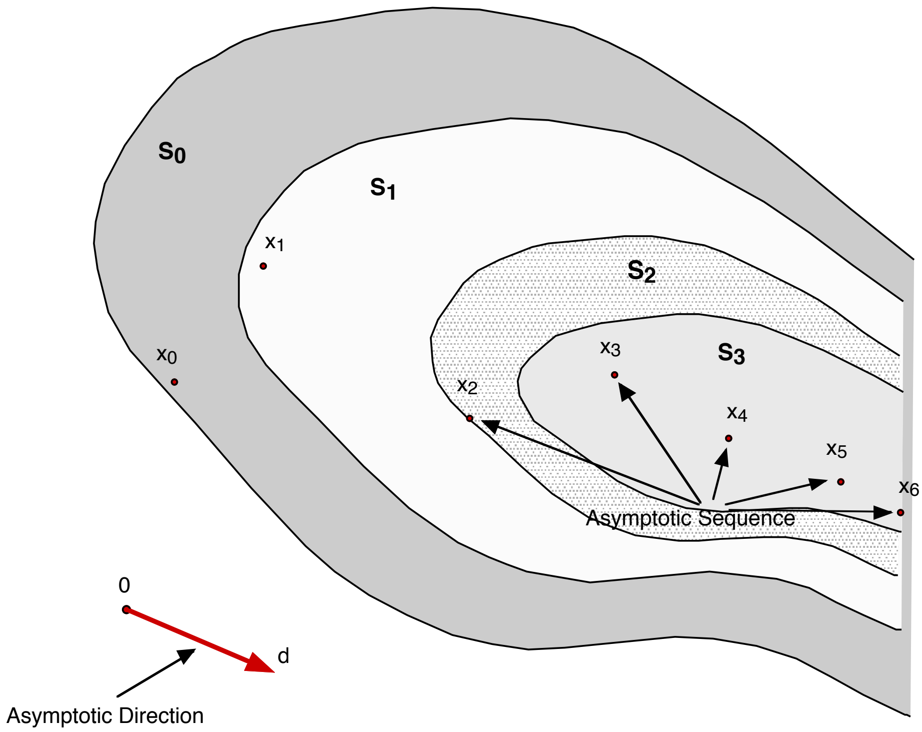
$$\mathrm{Proj}\big(\mathrm{epi}(F)\big) \subset \mathrm{epi}(p) \subset \mathrm{cl}\Big(\mathrm{Proj}\big(\mathrm{epi}(F)\big)\Big)$$

ASYMPTOTIC DIRECTIONS

- Given a sequence of nonempty nested closed sets $\{S_k\}$, we say that a vector $d \neq 0$ is an **asymptotic direction of** $\{S_k\}$ if there exists $\{x_k\}$ s. t.

$$x_k \in S_k, \qquad x_k \neq 0, \qquad k = 0, 1, \ldots$$

$$\|x_k\| \to \infty, \qquad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}.$$

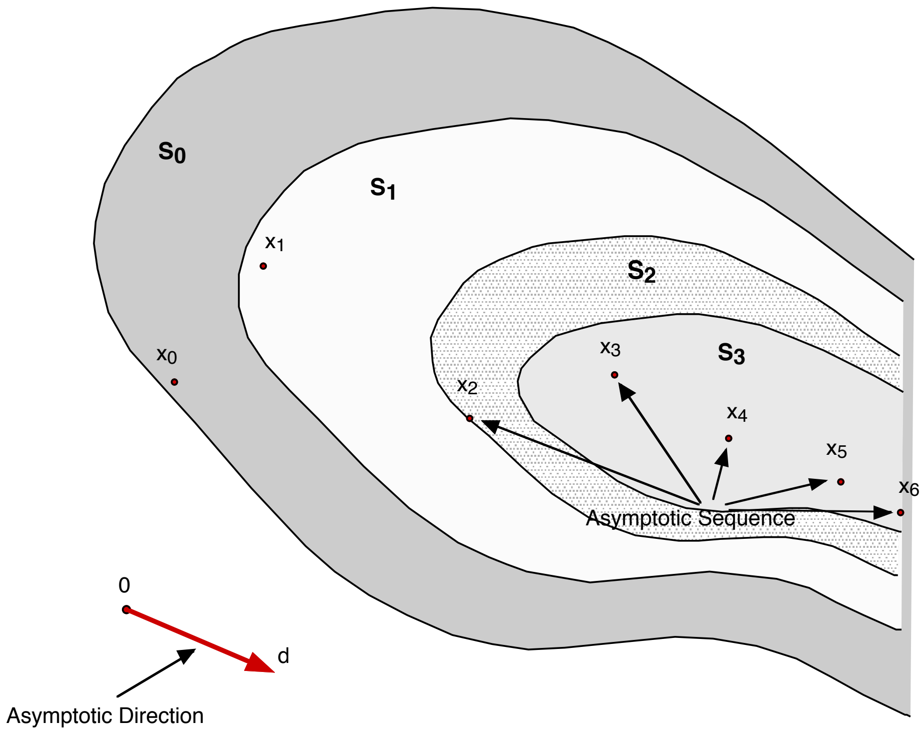


ASYMPTOTIC DIRECTIONS

- Given a sequence of nonempty nested closed sets $\{S_k\}$, we say that a vector $d \neq 0$ is an **asymptotic direction of** $\{S_k\}$ if there exists $\{x_k\}$ s. t.

$$x_k \in S_k, \qquad x_k \neq 0, \qquad k = 0, 1, \ldots$$

$$\|x_k\| \to \infty, \qquad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}.$$



- A sequence $\{x_k\}$ associated with an asymptotic direction d as above is called an **asymptotic sequence** corresponding to d.

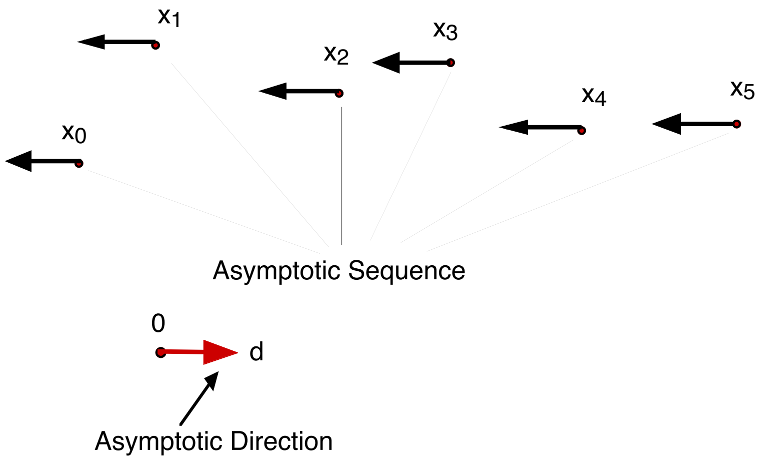
- Generalizes the known notion of asymptotic direction of a set (rather than a nested set sequence).

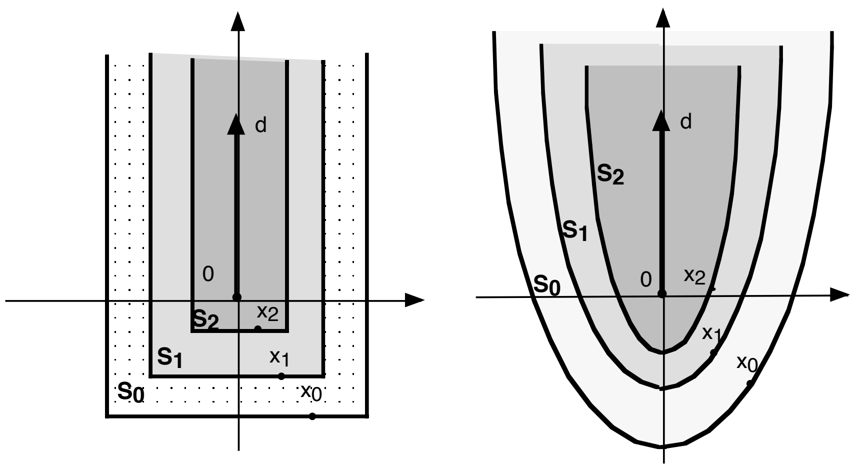
RETRACTIVE ASYMPTOTIC DIRECTIONS

- An asymptotic sequence $\{x_k\}$ and corresponding asymptotic direction are called **retractive** if there exists $\overline{k} \geq 0$ such that

$$x_k - d \in S_k, \qquad \forall\ k \geq \overline{k}.$$

$\{S_k\}$ is called **retractive** if all its asymptotic sequences are retractive.





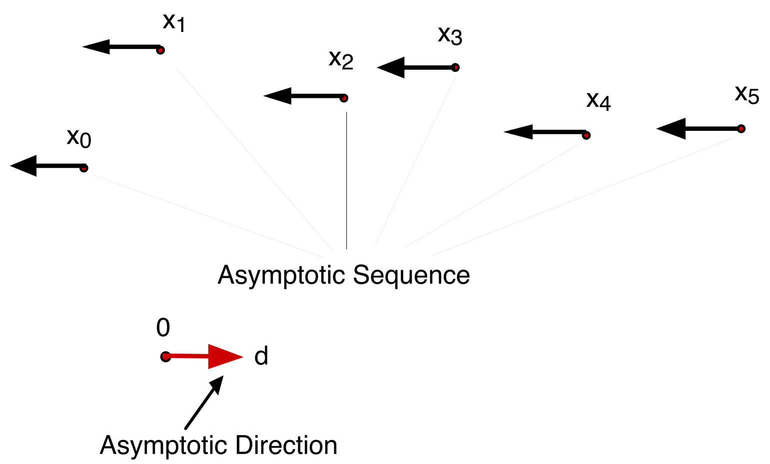
(a) (b)

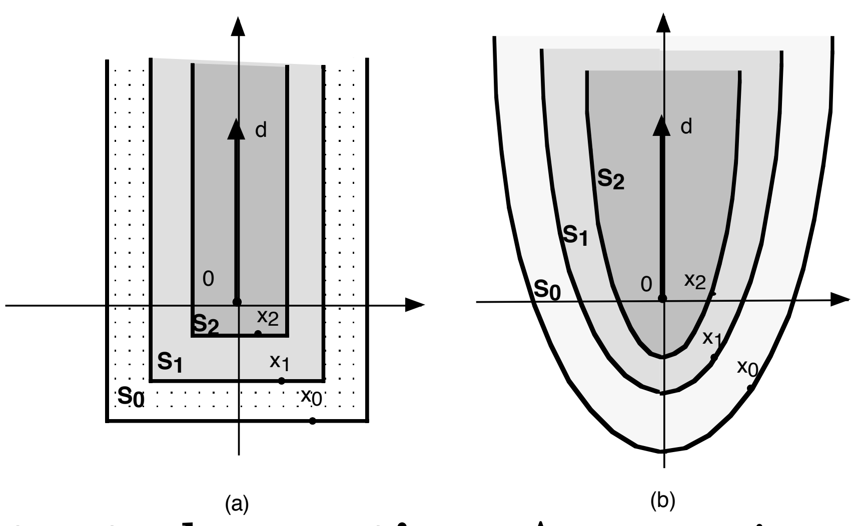
RETRACTIVE ASYMPTOTIC DIRECTIONS

- An asymptotic sequence $\{x_k\}$ and corresponding asymptotic direction are called **retractive** if there exists $\overline{k} \geq 0$ such that

$$x_k - d \in S_k, \qquad \forall\ k \geq \overline{k}.$$

$\{S_k\}$ is called **retractive** if all its asymptotic sequences are retractive.





- **Important observation:** A retractive asymptotic sequence $\{x_k\}$ (for large k) gets closer to 0 when shifted in the opposite direction $-d$.

SET INTERSECTION THEOREM

Proposition: The intersection of a retractive nested sequence of closed sets is nonempty.

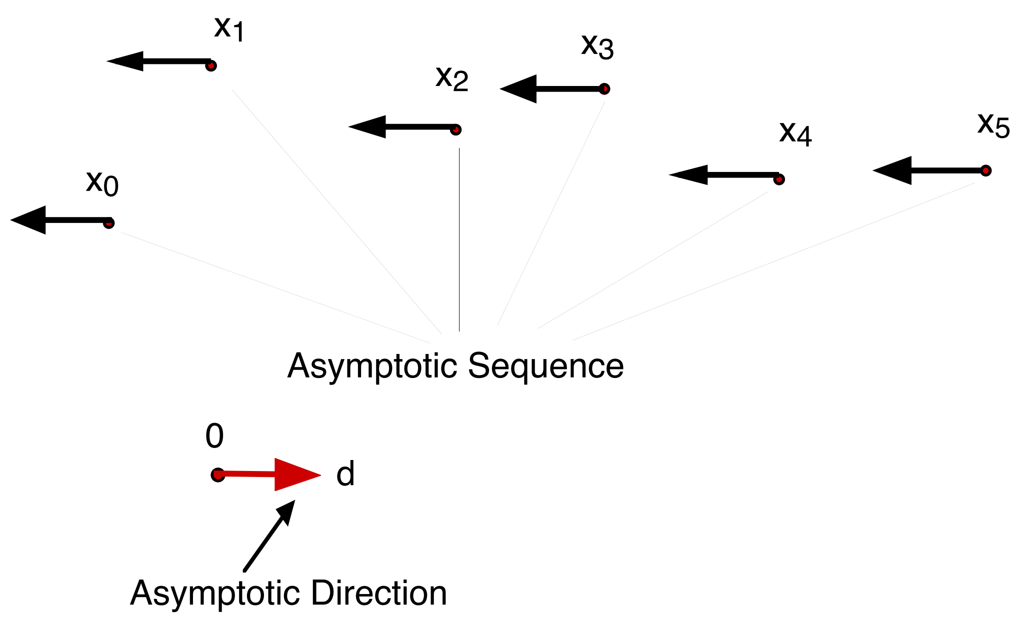
- Key proof ideas:

  (a) Consider x_k a **minimum norm vector** from S_k.

  (b) The intersection $\cap_{k=0}^{\infty} S_k$ is empty iff $\{x_k\}$ is unbounded.

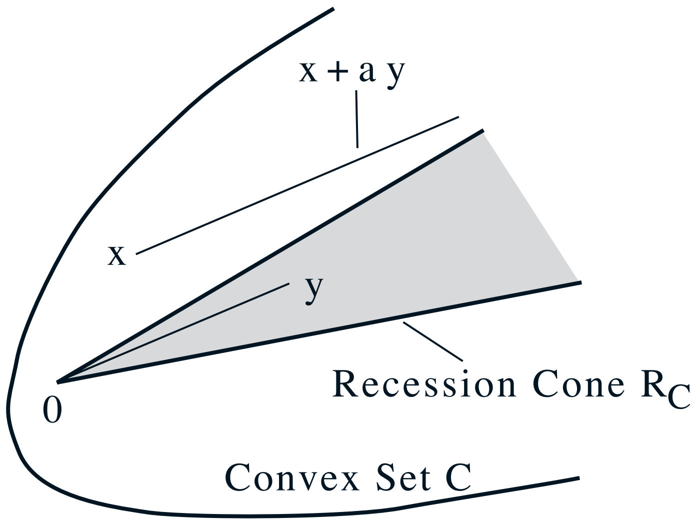
  (c) An asymptotic sequence $\{x_k\}$ consisting of minimum norm vectors from the S_k cannot be retractive, because $\{x_k\}$ **eventually gets closer to 0 when shifted** opposite to the asymptotic direction.

  (d) Hence $\{x_k\}$ is bounded.



CALCULUS OF RETRACTIVE SEQUENCES

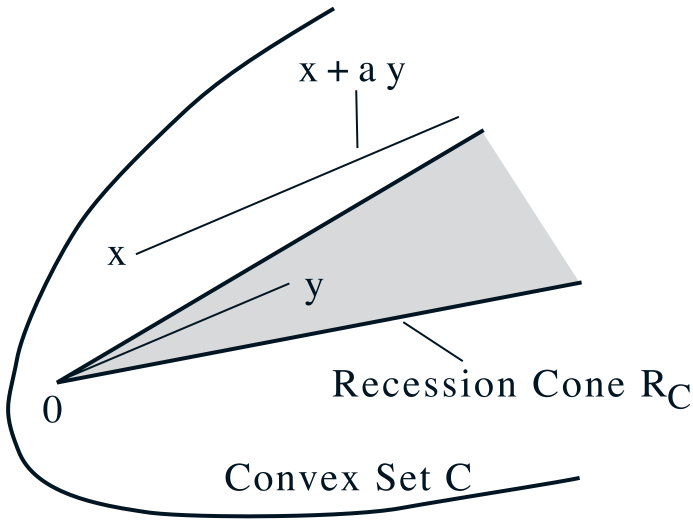
- **Unions and intersections** of retractive set sequences are retractive.
- **Polyhedral sets** are retractive.
- Recall the **recession cone** R_C of a convex set C, and its **lineality space** $L_C = R_C \cap (-R_C)$.



For S_k:**convex**, the set of asymptotic directions of $\{S_k\}$ is the set of nonzero $d \in \cap_k R_{S_k}$.

CALCULUS OF RETRACTIVE SEQUENCES

- **Unions and intersections** of retractive set sequences are retractive.
- **Polyhedral sets** are retractive.
- Recall the **recession cone** R_C of a convex set C, and its **lineality space** $L_C = R_C \cap (-R_C)$.



For S_k:**convex**, the set of asymptotic directions of $\{S_k\}$ is the set of nonzero $d \in \cap_k R_{S_k}$.

- The **vector sum of a compact set and a polyhedral cone** (e.g., a polyhedral set) is retractive.
- The level sets of a continuous **concave** function $\{x \mid f(x) \leq \gamma\}$ are retractive.

EXISTENCE OF SOLUTIONS ISSUES

- Standard results on existence of minima of convex functions generalize with simple proofs using the set intersection theorem.

- Use the set intersection theorem, and

existence of optimal solution

<=> nonemptiness of $\cap$ (nonempty level sets)

- **Example 1:** The set of minima of a closed convex function f over a closed set X is nonempty if there is no asymptotic direction of X that is a direction of recession of f.

- **Example 2:** The set of minima of a closed quasiconvex function f over a retractive closed set X is nonempty if

$$A \cap R \subset L,$$

where A: set of asymptotic directions of X,

$$R = \cap_{k=0}^{\infty} R_{\overline{S}_k}, \qquad L = \cap_{k=0}^{\infty} L_{\overline{S}_k},$$

$$\overline{S}_k = \{x \mid f(x) \le \gamma_k\}$$

and $\gamma_k \downarrow f^*$.

LINEAR AND QUADRATIC PROGRAMMING

- **Frank-Wolfe Th:** Let X be **polyhedral** and

$$f(x) = x'Qx + c'x$$

where Q is symmetric (not necessarily positive semidefinite). If the minimal value of f over X is finite, there exists a minimum of f of over X.

- The proof is straightforward using the set intersection theorem, and

existence of optimal solution

<=> nonemptiness of $\cap$ (nonempty level sets)

LINEAR AND QUADRATIC PROGRAMMING

- **Frank-Wolfe Th:** Let X be **polyhedral** and

$$f(x) = x'Qx + c'x$$

where Q is symmetric (not necessarily positive semidefinite). If the minimal value of f over X is finite, there exists a minimum of f of over X.

- The proof is straightforward using the set intersection theorem, and

existence of optimal solution

<=> nonemptiness of $\cap$ nonempty level sets.

- **Extensions not covered:**
  - X can be the vector sum of a compact set and a polyhedral cone.
  - f can be of the form

$$f(x) = p(x'Qx) + c'x$$

where Q is positive semidefinite and p is a polynomial.

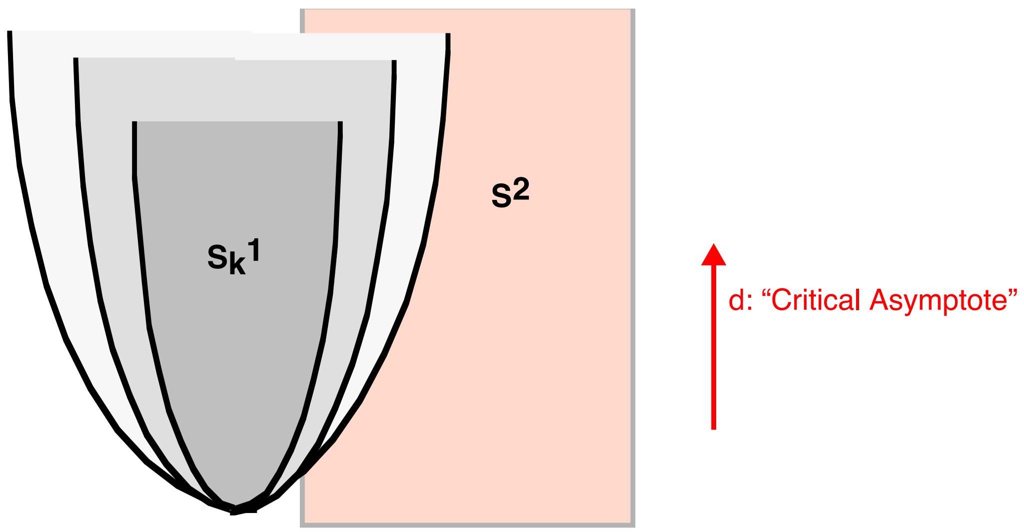
- These extensions need the subsequent theory.
- Reason is that **level sets of quadratic functions (and polynomial) are not retractive**.

MULTIPLE SEQUENCE INTERSECTIONS

- **Key question:** Given $\{S_k^1\}$ and $\{S_k^2\}$, each with nonempty intersection by itself, and with

$$S_k^1 \cap S_k^2 \neq \emptyset,$$

for all k, when does the intersection sequence $\{S_k^1 \cap S_k^2\}$ have an empty intersection?



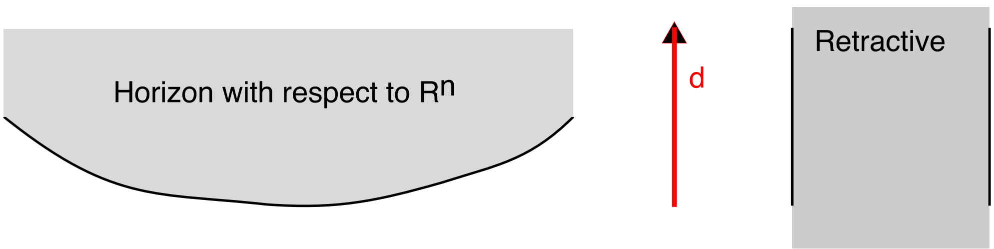
- Examples indicate that the trouble lies with the existence of a **"critical asymptote"**.

- "Critical asymptotes" roughly are: Common asymptotic directions d such that **starting at** $\cap_k S_k^2$ **and looking at the horizon along** d**, we do not meet** $\cap_k S_k^1$ (and similarly with the roles of S_k^1 and S_k^2 reversed).

CRITICAL DIRECTIONS

- We say that an asymptotic direction d of $\{S_k\}$, with $\cap_k S_k \neq \emptyset$ is a **horizon direction with respect to a set** G if for every $x \in G$, we have $x + \alpha d \in \cap_k S_k$ for all α sufficiently large.

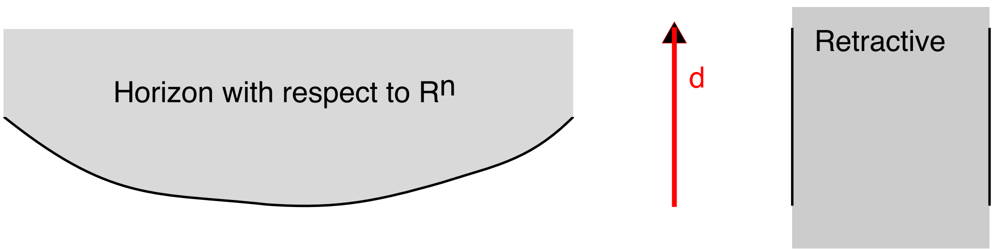
- We say that an asymptotic direction d of $\{S_k\}$ is **noncritical with respect to a set** G if it is either a horizon direction with respect to G or a retractive horizon direction with respect to $\cap_k S_k$. Otherwise, d is **critical with respect to** G.



CRITICAL DIRECTIONS

- We say that an asymptotic direction d of $\{S_k\}$, with $\cap_k S_k \neq \emptyset$ is a **horizon direction with respect to a set** G if for every $x \in G$, we have $x + \alpha d \in \cap_k S_k$ for all α sufficiently large.

- We say that an asymptotic direction d of $\{S_k\}$ is **noncritical with respect to a set** G if it is either a horizon direction with respect to G or a retractive horizon direction with respect to $\cap_k S_k$. Otherwise, d is **critical with respect to** G.



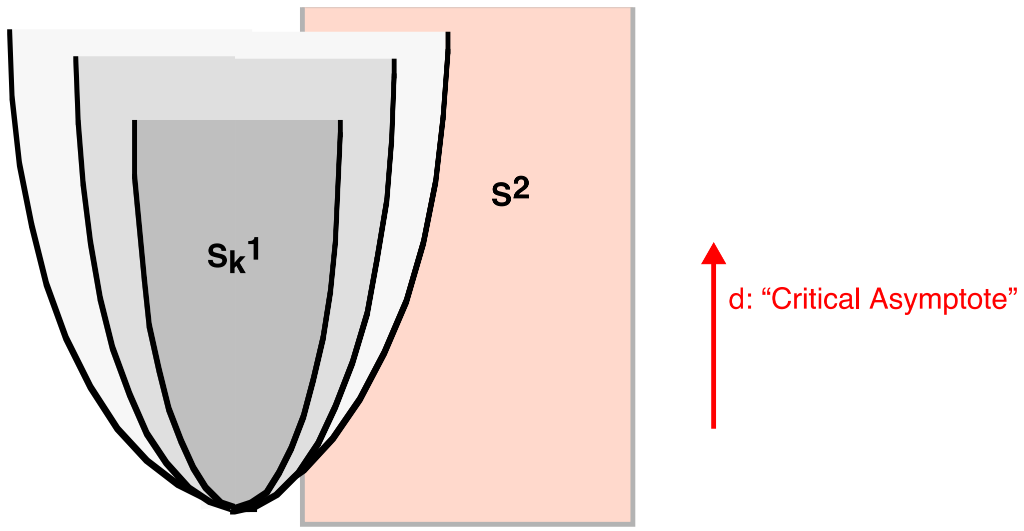
- **Example:** The asymptotic directions of a **level set sequence of a convex quadratic**

$$S_k = \{x \mid x'Qx + c'x + b \leq \gamma_k\}, \qquad \gamma_k \downarrow 0,$$

are noncritical with respect to $\Re^n$. (Extension: Convex polynomials, bidirectionally flat convex fns.)

- **Example:** The as. directions of a **vector sum** S **of a compact and a polyhedral set** are non-critical (are retractive hor. dir. with resp. to S).

EXAMPLE OF CRITICAL DIRECTION



- Two set sequences, all intersections of a finite number of sets are nonempty.
- d shown is the only common asymptotic direction.
- d is noncritical for S^2 with respect to $\cap_k S_k^1$ (because it is retractive).
- d is critical for $\cap_k S_k^1$ with respect to S^2.

CRITICAL DIRECTION THEOREM

- Roughly it says that: **For the intersection of a set sequence $\{S_k^1 \cap S_k^2 \cap \cdots \cap S_k^r\}$ to be empty, some common asymptotic direction must be critical for one of the $\{S_k^j\}$** with respect to all the others.

- **Critical Direction Theorem:** Consider $\{S_k^1\}$ and $\{S_k^2\}$, each with nonempty intersection by itself. If

$$S_k^1 \cap S_k^2 \neq \varnothing \quad \text{for all } k, \text{ and} \quad \cap_{k=0}^{\infty} (S_k^1 \cap S_k^2) = \varnothing,$$

there is a common asymptotic direction that is critical for $\{S_k^1\}$ with respect to $\cap_k S_k^2$ (or for $\{S_k^2\}$ with respect to $\cap_k S_k^1$).

- Extends to any finite number of sequences $\{S_k^j\}$.

CRITICAL DIRECTION THEOREM

- Roughly it says that: **For the intersection of a set sequence $\{S_k^1 \cap S_k^2 \cap \cdots \cap S_k^r\}$ to be empty, some common asymptotic direction must be critical for one of the $\{S_k^j\}$** with respect to all the others.

- **Critical Direction Theorem:** Consider $\{S_k^1\}$ and $\{S_k^2\}$, each with nonempty intersection by itself. If

$$S_k^1 \cap S_k^2 \neq \emptyset \quad \text{for all } k, \text{ and } \quad \cap_{k=0}^{\infty} (S_k^1 \cap S_k^2) = \emptyset,$$

there is a common asymptotic direction that is critical for $\{S_k^1\}$ with respect to $\cap_k S_k^2$ (or for $\{S_k^2\}$ with respect to $\cap_k S_k^1$).

- Extends to any finite number of sequences $\{S_k^j\}$.

- **Special Case:** The intersection of **set sequences defined by convex polynomial functions**

$$S_k^j = \{x \mid p_j(x) \leq \gamma_k^j,\ j = 1, \ldots, r\}, \qquad \gamma_k^j \downarrow 0,$$

is nonempty, if all the $\cap_k S_k^j$ and $S_k^1 \cap \ldots \cap S_k^r$ are nonempty. (For example p_j may be convex quadratic or bidirectionally flat.)

EXISTENCE OF SOLUTIONS THEOREMS

- **Convex Quadratic/Polynomial Problems:**
For $j = 0, 1, \ldots, r$, let $f_j : \Re^n \mapsto \Re$ be polynomial convex functions. Then the problem

$$\begin{aligned} &\text{minimize} \;\; f_0(x) \\ &\text{subject to} \;\; f_j(x) \le 0, \quad j = 1, \ldots, r, \end{aligned}$$

has at least one optimal solution if and only if its optimal value is finite.

EXISTENCE OF SOLUTIONS THEOREMS

- **Convex Quadratic/Polynomial Problems:** For $j = 0, 1, \ldots, r$, let $f_j : \Re^n \mapsto \Re$ be polynomial convex functions. Then the problem

$$\begin{aligned} &\text{minimize} \quad f_0(x) \\ &\text{subject to} \quad f_j(x) \leq 0, \quad j = 1, \ldots, r, \end{aligned}$$

has at least one optimal solution if and only if its optimal value is finite.

- **Extended Frank-Wolfe Theorem:** Let

$$f(x) = x'Qx + c'x$$

where Q is symmetric, and let X be a closed set whose asymptotic directions are retractive horizon directions with respect to X. If the minimal value of f over X is finite, there exists a minimum of f over X.