Distributed Asynchronous Computation of Fixed Points and Applications in Dynamic Programming

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Joint Work with Huizhen (Janey) Yu

Summary

Background Review

- Asynchronous iterative fixed point methods
- Convergence issues
- Dynamic programming (DP) applications

New Research on Asynchronous DP Algorithms

- Asynchronous policy iteration (asynchronous value and policy updates by multiple processors)
- Failure of the “natural" algorithm (Williams-Baird counterexample, 1993)
- A radical modification of policy iteration/evaluation: Aim to solve an optimal stopping problem instead of solving a linear system
- Convergence properties are restored/enhanced
- Generalizations and abstractions

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Current work

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- H. Yu and D. P. Bertsekas, "Q-Learning and Policy Iteration Algorithms for Stochastic Shortest Path Problems," to appear in Annals of OR
- D. P. Bertsekas and H. Yu, “Distributed Asynchronous Policy Iteration," Proc. Allerton Conference, Sept. 2010
- More works in progress

Connection to early work on theory of totally asynchronous algorithms

- D. P. Bertsekas, "Distributed Dynamic Programming," IEEE Transactions on Aut. Control, Vol. AC-27, 1982
- D. P. Bertsekas, "Distributed Asynchronous Computation of Fixed Points," Mathematical Programming, Vol. 27, 1983
- D. P. Bertsekas and J. N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods, Prentice-Hall, 1989

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Outline

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Some History: The First “Real" Implementation of a Distributed Asynchronous iterative Algorithm

Routing algorithm of the ARPANET (1969)

- Based on the Bellman-Ford shortest path algorithm

$$J(i) = \min_{j=0,1,\dots,n} \{a_{ij} + J(j)\}, \qquad i = 1, \dots, n,$$

where:

- $J(i)$: Estimated distance of node i to destination node 0 ($J(0) = 0$)
- a_{ij}: Length of the link connecting i to j

- Original ambitious implementation included congestion dependent a_{ij}
- Failed miserably because of violent oscillations
- The algorithm was “stabilized" by making a_{ij} essentially constant (1974)
- Showing convergence of this algorithm and its DP extensions was my entry point into the field (1979)

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- Initial proposal by Rosenfeld (1967) involved experimentation but no convergence analysis
- Sup-norm linear contractive iterations by Chazan and Miranker (1969)
- Sup-norm nonlinear contractive iterations by Miellou (1975), Robert (1976), Baudet (1978)
- DP algorithms (Bertsekas 1982, 1983) - both sup norm contractive and also noncontractive/monotone iterations (e.g., shortest path)
- Many subsequent works (nonexpansive, network, and other iterations)

Types of Asynchronism

- All asynchronous iterative algorithms proposed up to the early 80s involved "total asynchronism" and either sup-norm contractive or monotone mappings
- Subsequent work also involved "partial asynchronism," which could be applied to additional types of iterations (e.g., gradient methods for optimization)

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Distributed "Totally" Asynchronous Framework for Fixed Point Computation

- Consider solution of general fixed point problem $J = TJ$, or

$$J(i) = T_i\big(J(1), \ldots, J(n)\big), \qquad i = 1, \ldots, n$$

- Network of processors, with a separate processor i for each component $J(i)$, $i = 1, \ldots, n$

Asynchronous Fixed Point Algorithm

- Processor i updates $J(i)$ at a subset of times $\mathscr{T}_i \subset \{0, 1, \ldots\}$
- Processor i receives (possibly outdated values) $J(j)$ from other processors $j \neq i$
- Update of processor i [with “delays" $t - \tau_{ij}(t)$]

$$J^{t+1}(i) = \begin{cases} T_i\big(J^{\tau_{i1}(t)}(1), \ldots, J^{\tau_{in}(t)}(n)\big) & \text{if } t \in \mathscr{T}_i, \\ J^t(i) & \text{if } t \notin \mathscr{T}_i. \end{cases}$$

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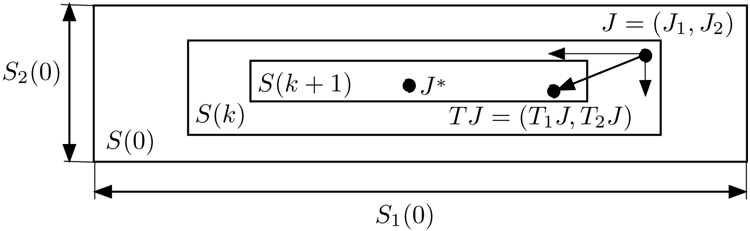
Distributed Convergence of Fixed Point Iterations

A general theorem for “totally asynchronous" iterations, i.e., $\mathscr{T}_i$ are infinite sets and $\tau_{ij}(t) \to \infty$ as $t \to \infty$ (Bertsekas, 1983)



- Assume there is a nested sequence of sets $S(k+1) \subset S(k)$ such that
  - (Synchronous Convergence Condition) We have

    $$TJ \in S(k+1), \qquad \forall\, J \in S(k),$$

    and the limit points of all sequences $\{J^k\}$ with $J^k \in S(k)$, for all k, are fixed points of T.
  - (Box Condition) $S(k)$ is a Cartesian product:

    $$S(k) = S_1(k) \times \cdots \times S_n(k)$$

Then, if $J^0 \in S(0)$, every limit point of $\{J^t\}$ is a fixed point of T.

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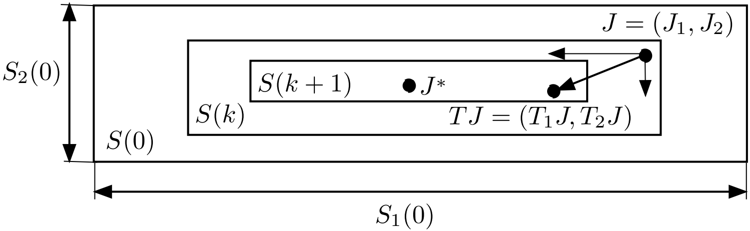

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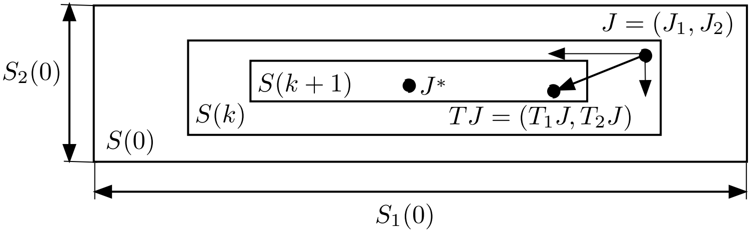

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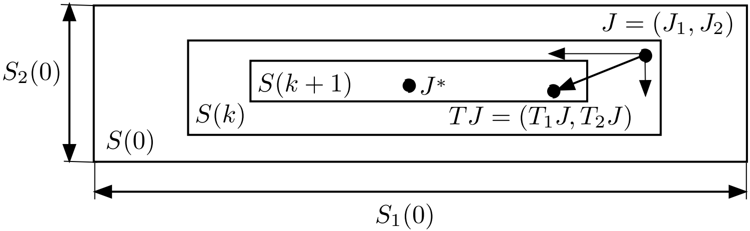

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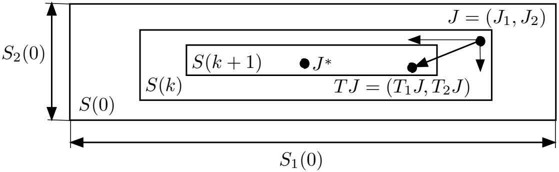
Applications of the Theorem



Major contexts where the theorem applies:

- T is a sup-norm contraction with fixed point J^* and modulus α:
$$S(k) = \{J \mid \|J - J^*\|_\infty \le \alpha^k B\}, \qquad \text{for some scalar } B$$
- T is monotone ($TJ \le TJ'$ for $J \le J'$) with fixed point J^*. Moreover,
$$S(k) = \{J \mid T^k \underline{J} \le J \le T^k \bar{J}\}$$
for some $\underline{J}$ and $\bar{J}$ with
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Both of these apply to various DP problems:

- 1st context applies to discounted problems
- 2nd context applies to undiscounted problems (e.g., shortest paths).

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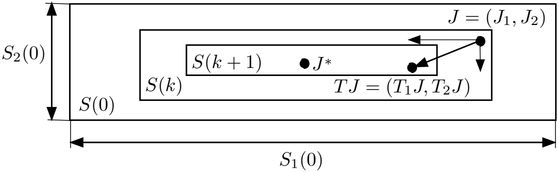

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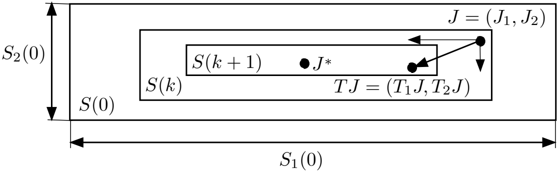

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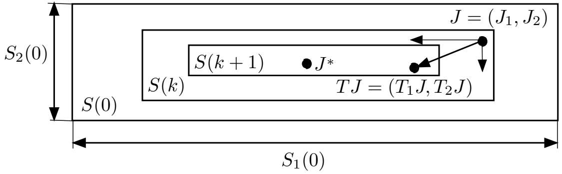

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  for some $\underline{J}$ and $\bar{J}$ with

$$\underline{J} \le T\underline{J} \le T\bar{J} \le \bar{J},$$

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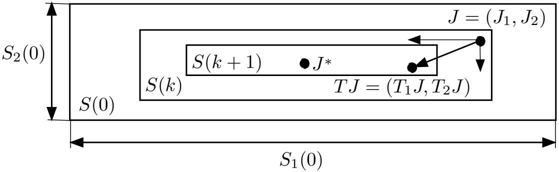
Both of these apply to various DP problems:

- 1st context applies to discounted problems
- 2nd context applies to undiscounted problems (e.g., shortest paths)

Applications of the Theorem



Major contexts where the theorem applies:

- T is a sup-norm contraction with fixed point J^* and modulus α:

$$S(k) = \{J \mid \|J - J^*\|_\infty \leq \alpha^k B\}, \qquad \text{for some scalar } B$$

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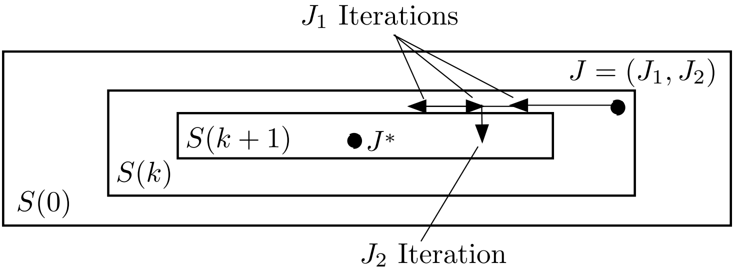
Asynchronous Convergence Mechanism



- Once we are in $S(k)$, iteration of any one component (say J_1) keeps us in the current box
- Once all components have been iterated once, we pass into the next box $S(k+1)$ (permanently)

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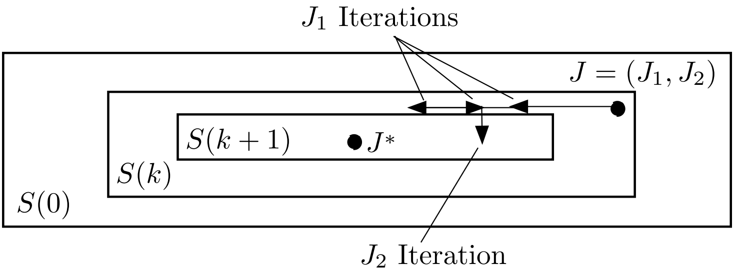

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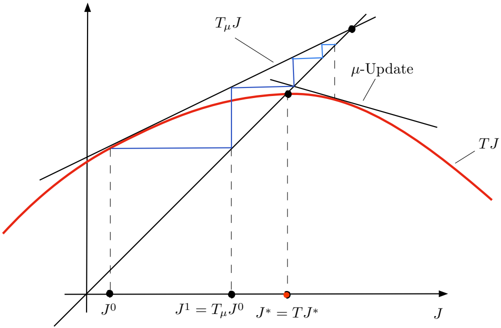
Finding Fixed Points of Parametric Mappings

- Problem: Find a fixed point J^* of a mapping $T : \Re^n \mapsto \Re^n$ of the form

$$(TJ)(i) = \min_{\mu \in \mathcal{M}_i} (T_\mu J)(i), \qquad i = 1, \ldots, n,$$

  where μ is a parameter from some set $\mathcal{M}_i$.

- Common idea (central in DP policy iteration): Instead of T, we iterate with some T_μ, then change μ once in a while



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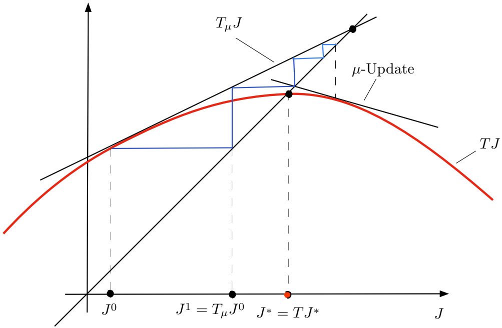

Distributed Asynchronous Parametric Fixed Point Iterations

Partition J and μ into components, $J(1), \dots, J(n)$ and $\mu(1), \dots, \mu(n)$

Asynchronous Algorithm

- Processor i, at each time does one of three things:
  - Does nothing or
  - Updates $J(i)$ by setting it to

$$J(i) := (T_{\mu(i)} J)(i)$$

    or
  - Updates $J(i)$ and $\mu(i)$ by setting

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A very chaotic environment!

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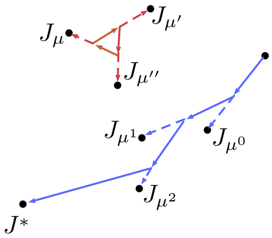
Difficulty with Parametric Fixed Point Algorithm

J_μ : Fixed Point of T_μ

$J*$: Fixed Point of T



- T_μ has different fixed point than T ... so the target of the iterations keeps changing
- A (restrictive) convergence result: If each T_μ is both a sup-norm contraction and is monotone, convergence can be shown assuming that $T_{\mu^0}J^0 \leq J^0$
- Without the condition $T_{\mu^0}J^0 \leq J^0$, the synchronous algorithm converges but the asynchronous algorithm may oscillate

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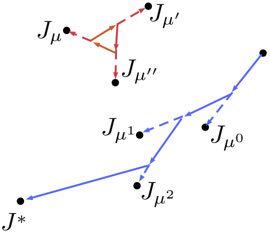

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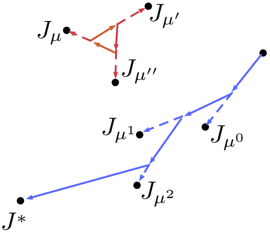

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Addressing the Difficulty: A High-Level View

Assume that each T_μ is a sup-norm contraction

- Our approach: Embed both T and T_μ within another (uniform) contraction mapping G_μ that has the same fixed point for all μ
- The uniform contraction mapping G_μ operates on a larger space
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Outline

Dynamic Programming - Markovian Decision Problems (MDP)

- **System:** Controlled Markov chain w/ transition probabilities $p_{ij}(u)$
- **States:** $i = 1, \ldots, n$
- **Controls:** $u \in U(i)$
- **Cost per stage:** $g(i, u, j)$
- **Stationary policy:** State to control mapping μ; apply $\mu(i)$ when at state i
- **Discounted MDP:** Find policy μ that minimizes the expected value of the infinite horizon cost:

$$\sum_{k=0}^{\infty} \alpha^k g\big(i_k, \mu(i_k), i_{k+1}\big)$$

where

$$i_k = \text{ state at time } k, \qquad i_{k+1} = \text{ state at time } k+1,$$

$$\alpha : \text{ discount factor } \quad 0 < \alpha < 1$$

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Major DP Results

- $J^*(i)$ = Optimal cost starting from state i
- $J_\mu(i)$ = Optimal cost starting from state i using policy μ
- Bellman's equation:

$$J^*(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J^*(j)\big), \qquad i = 1,\ldots,n$$

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Shorthand Notation - Fixed Point View

- Denote by T and T_μ the mappings that map $J \in \Re^n$ to the vectors TJ and $T_\mu J$ with components

$$(TJ)(i) \stackrel{\text{def}}{=} \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J(j)\big), \qquad i = 1, \ldots, n,$$

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- Bellman's equations are written as

$$J^* = TJ^* = \min_\mu T_\mu J^*, \qquad J_\mu = T_\mu J_\mu$$

- Key structure: T and T_μ are sup-norm contractions with modulus α,

$$\|TJ - TJ'\|_\infty = \max_{i=1,\ldots,n} \big|(TJ)(i) - (TJ')(i)\big| \le \alpha \max_{i=1,\ldots,n} \big|J(i) - J'(i)\big| = \alpha \|J - J'\|_\infty$$

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Major (Synchronous) Methods for Finding Fixed Point of T

- Value iteration (generic fixed point method): Start with any J^0, iterate by

$$J^{t+1} = TJ^t$$

- Policy iteration (special method for T of the form $T = \min_\mu T_\mu$): Start with any J^0 and μ^0. Given J^t and μ^t, iterate by:
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- Both methods converge to J^*:
    - Value iteration, thanks to contraction of T
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- Typically, policy iteration (with a reasonable choice of m) is more efficient because application of T_μ is cheaper than application of T

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Major (Synchronous) Methods for Finding Fixed Point of T

- Value iteration (generic fixed point method): Start with any J^0, iterate by
$$J^{t+1} = TJ^t$$
- Policy iteration (special method for T of the form $T = \min_\mu T_\mu$): Start with any J^0 and μ^0. Given J^t and μ^t, iterate by:
    - Policy evaluation: $J^{t+1} = (T_{\mu^t})^m J^t$ (m applications of T_{μ^t} on J^t; $m = \infty$ is possible)
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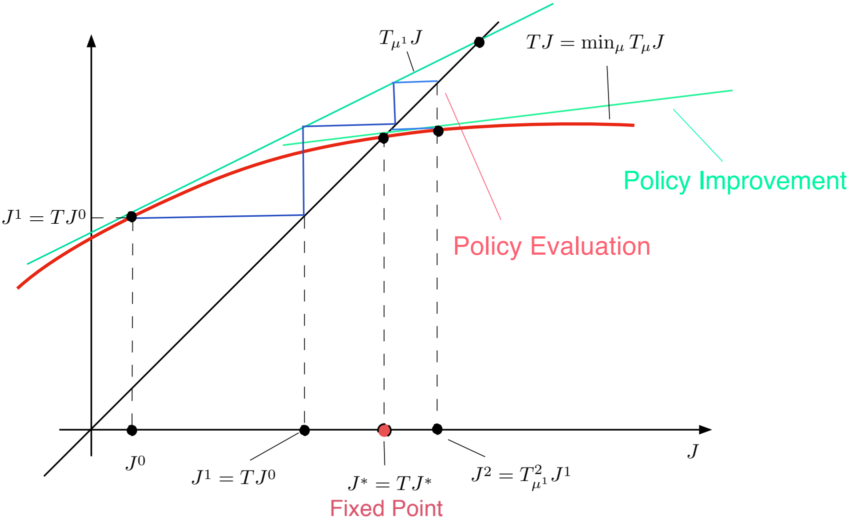
Graphical Interpretation of Policy Iteration



Failure of Asynchronous Policy Iteration: W-B Example

Counterexample by Williams and Baird (1993)

- Deterministic discounted MDP with 6 states arranged in a circle
- 2 controls available in half the states, 1 control available in the other half
- Policy evaluations and improvements are one state at a time, no “delays"
- A cycle of 15 iterations is constructed that repeats the initial conditions

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Rectifying the Difficulty - Q-Factors

- Q-factors, $Q(i,u)$ are functions of state-control pairs (i,u)
- The optimal Q-factors are given by

$$Q^*(i,u) = \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha J^*(j)\big)$$

$Q^*(i,u)$: Cost of starting at i, using u first, then use optimal policy.

- Bellman's equation $J^* = TJ^*$ is written as

$$J^*(i) = \min_{u \in U(i)} Q^*(i,u)$$

- These two equations constitute a fixed point equation in (Q,J)
- For a fixed policy μ, the corresponding fixed point equation in (Q_μ, J_μ) is

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Sup-Norm Uniform Contraction Property

- Consider Q-factors $Q(i,u)$ and costs $J(i)$. For any μ, define mapping

$$(Q,J) \quad \mapsto \quad \big(F_\mu(Q,J), M_\mu(Q,J)\big)$$

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- Key fact: A sup-norm contraction w/ common fixed point (Q^*, J^*) for all μ

$$\max\big\{\|F_\mu(Q,J)-Q^*\|_\infty, \|M_\mu(Q,J)-J^*\|_\infty\big\} \le \alpha \max\big\{\|Q-Q^*\|_\infty, \|J-J^*\|_\infty\big\}$$

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$$Q := F_\mu(Q,J), \qquad J := M_\mu(Q,J), \qquad \mu := \text{arbitrary}$$

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Asynchronous Policy Iteration: A More Detailed View

Asynchronous distributed policy iteration algorithm: Maintains J^t, μ^t, and V^t, where

$$V^t(i) = Q^t\big(i, \mu^t(i)\big) \qquad \approx \text{Q-factors of current policy}$$

- Processor i at time t does one of three things:
  - Does nothing
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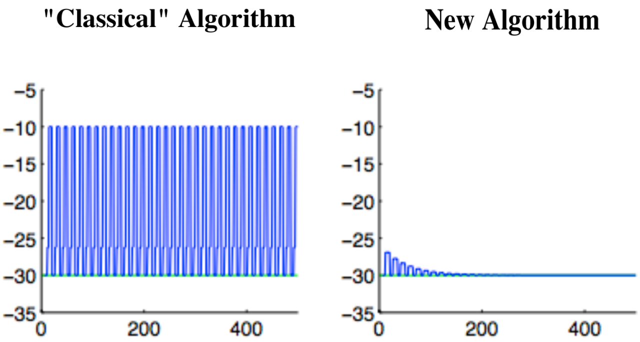
Williams and Baird Counterexample



Outline

Generalized Mappings T and T_μ

- The preceding analysis uses only the contraction property of the discounted MDP (not monotonicity or the probabilistic structure)
- Abstract mappings T and T_μ:
  - Introduce a mapping $H(i,u,J)$ and denote
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  sets $\mu^{t+1}(i)$ to a u that attains the minimum.

Generalized Mappings T and T_μ

- The preceding analysis uses only the contraction property of the discounted MDP (not monotonicity or the probabilistic structure)
- Abstract mappings T and T_μ:
  - Introduce a mapping $H(i, u, J)$ and denote

    $$(TJ)(i) = \min_{u \in U(i)} H(i, u, J), \qquad (T\mu J)(i) = H\big(i, \mu(i), J\big)$$

    i.e., $TJ = \min_\mu T_\mu J$, where the min is taken separately for each component
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    $$\big|H(i, u, J) - H(i, u, J')\big| \leq \alpha \|J - J'\|_\infty$$

- Asynchronous algorithm: At time t, processor i does one of three things:
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DP Applications with Generalized T and T_μ

- DP models beyond discounted with standard policy evaluation
  - Optimistic/modified policy iteration for semi-Markov and minimax discounted problems
  - Stochastic shortest path problems
  - Semi-Markov decision problems
  - Stochastic zero sum games
  - Sequential minimax problems
- Multi-agent aggregation
  - Each agent updates costs at all states within a subset
  - Each agent uses detailed costs for the local states and aggregate costs for other states, as communicated by other agents

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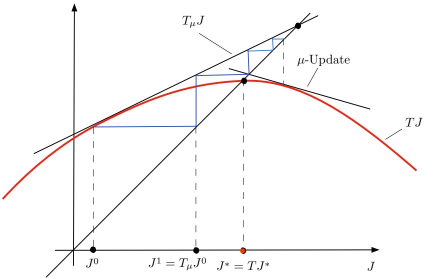
Fixed Points of Concave Sup-Norm Contractions



- Find a fixed point of a mapping $T : \Re^n \mapsto \Re^n$, where the components $T_i(\cdot) : \Re^n \mapsto \Re$ are concave sup-norm contractions
- T is the minimum of a collection of linear mappings T_μ involving the concave conjugate functions of the concave functions T_i
- An asynchronous parametric fixed point algorithm can be used:
  - Linearize at the current J
  - Iterate using the linearized mapping
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- All of the above are done asynchronously

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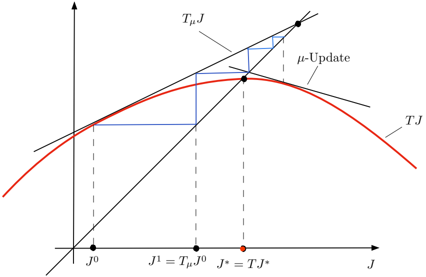

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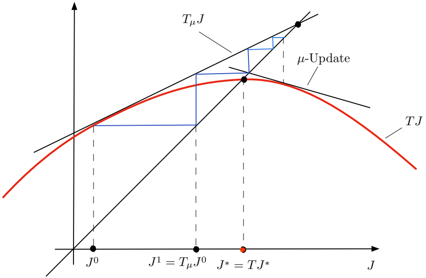

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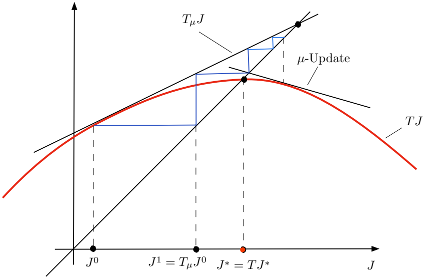

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Concluding Remarks

- Asynchronous policy iteration has fragile convergence properties
- We have provided a new approach to correct the difficulties
- Key idea: Embed the problem into one that involves a uniform sup-norm contraction
- Extensions to generalized DP models involving:
  - Other DP models (e.g., stochastic shortest path, games, etc)
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Thank You!