A Series of Lectures on
Approximate Dynamic Programming
Lecture 4

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APPROXIMATE DYNAMIC PROGRAMMING III
1. Approximation in Policy Space

2. Tail Problem Approximation
Approximation in Policy Space

Using a Parametric Approximation Architecture for Policies

- Parametrize policies with a parameter vector \( r = (r_0, \ldots, r_{N-1}) \):
  \[
  \pi(r) = \{ \tilde{\mu}_0(x_0, r_0), \ldots, \tilde{\mu}_{N-1}(x_{N-1}, r_{N-1}) \}
  \]

- Compute/train off-line the parameters using some optimization

- Great advantage: After off-line training, the on-line calculation of the controls is very fast

An important use: Implement policies obtained by approximation in value space

- Train off-line a cost function approximation and compute many state-control pairs \((x^s_k, u^s_k), s = 1, \ldots, q\)

- Train a policy approximation architecture on these pairs. For example by solving for each \( k \) the least squares problem
  \[
  \min \sum_{s=1}^{q} \| u^s_k - \tilde{\mu}_k(x^s_k, r_k) \|^2 + (\text{Regularization term})
  \]

- This idea applies more generally. Generate many “good” state-control pairs \((x^s_k, u^s_k)\), using a software or human “expert” and train in policy space

Bertsekas (M.I.T.)
Approximate Dynamic Programming
Cost Optimization Approach for Training a Policy Architecture

- Minimize the cost $J_{\pi(r)}(x_0)$ over $r$
- Aim directly for an optimal policy within the parametric class
- Gradient-based optimization may be a possibility
- Random search in the space of $r$ is another possibility (e.g., cross entropy method)

An important special case: Policy parametrization through cost function parametrization

- For a given value space parametrization $r = (r_0, \ldots, r_{N-1})$, we define

$$\tilde{\mu}_k(x_k, r_k) = \arg \min_{u_k \in U_k(x_k)} E\left\{ g_k(x_k, u_k, w_k) + \tilde{J}_{k+1}(f_k(x_k, u_k, w_k), r_k) \right\}$$

- Has achieved success in a number of test problems (e.g., tetris)
An Example: Tetris (Often Used as Testbed in Competitions)

- Number of states $> 2^{200}$ (for $10 \times 20$ board)
- $J^*(x)$: optimal score starting from board position $x$
- Common choice: 22 features, readily recognized by tetris players as capturing important aspects of the board position (heights of columns, etc)
- Long history of successes and failures
Lookahead Minimization

Cost-to-go Approximation

First $\ell$ Steps

Tail problem approximation

\[
\min_{u_k, \mu_{k+1}, \ldots, \mu_{k+\ell-1}} E \left\{ g_k(x_k, u_k, w_k) + \sum_{m=k+1}^{k+\ell-1} g_k(x_m, \mu_m(x_m), w_m) + \tilde{J}_{k+\ell}(x_{k+\ell}) \right\}
\]
Tail Problem Approximation Ideas

Obtain $\tilde{J}_{k+\ell}$ as the cost-to-go of a simplified problem which is solved exactly or approximately.

Enforced decomposition of interconnected subsystems

Applies to problems involving a collection $I$ of interconnected subsystems, with each subsystem $i \in I$ applying control $u^i_k$ at time $k$:

- One-at-a-time optimization: Obtain $\tilde{J}_{k+\ell}$ by optimizing one subsystem at a time, with controls of other subsystems fixed at nominal values.
- Constraint relaxation: Artificially decouple subsystems by modifying the constraint set.
- Lagrangean relaxation: Artificially decouple subsystems by using Lagrange multipliers (we will not cover).

Probabilistic approximation

Simplify the probabilistic structure (e.g., replace random variables with deterministic).

Aggregation

Reduce the size of the problem; e.g., by “combining” states into aggregate states.
Let $u_k = (u_k^1, \ldots, u_k^n)$, with $u_k^i$ corresponding to the $i$th subsystem.

To compute cost-to-go approximation $\tilde{J}_k(x_k)$:

- Start with subsystem 1, optimize over $(u_k^1, \ldots, u_{N-1}^1)$, with all future controls of other subsystems $i \neq 1$ held at nominal values ($\tilde{u}_k^i, \ldots, \tilde{u}_{N-1}^i$).
- Fix the nominal values of subsystem 1 to the optimal sequence thus obtained.
- Repeat for all subsystems $i = 2, \ldots, n$ (with intermediate adjustment of the nominal control values).
Example: Optimize the Routes of $n$ Vehicles Through a Road Network

- **Aim:** Execute a number of tasks with given values
- The value of a task is collected only once; a finite horizon is assumed
- This is a very complex combinatorial problem
- The single vehicle problem is typically much simpler (e.g., can be solved exactly or with a high-quality heuristic)
- **Solve (suboptimally) the tail subproblem one-vehicle-at-a-time.** The nominal decisions of the other vehicles can be determined using some heuristic
Enforced Decomposition: Constraint Decoupling by Relaxation

- Let \( x_k = (x^1_k, \ldots, x^n_k), u_k = (u^1_k, \ldots, u^n_k), w_k = (w^1_k, \ldots, w^n_k) \), with \( (x^i_k, u^i_k, w^i_k) \) corresponding to the \( i \)th subsystem.

- Assume that the only coupling between subsystems is the control constraint

\[
(u^1_k, \ldots, u^n_k) \in U, \quad \text{e.g., } u^i_k \in U^i, \ u^1_k + \cdots + u^n_k \leq b_k
\]

- Approximate \( U \) with a decomposed constraint \( U^1 \times \cdots \times U^n \).

- The problem decomposes into \( n \) decoupled subproblems. Let \( \tilde{J}^i_k \) be the optimal cost to go functions for the \( i \)th decoupled subproblem (obtained by DP off-line).

- Use one-step lookahead with cost-to-go approximation

\[
\tilde{J}_{k+1}(x_{k+1}) = \tilde{J}^1_{k+1}(x^1_{k+1}) + \cdots + \tilde{J}^n_{k+1}(x^n_{k+1})
\]
Example: Production Planning

Constraint Relaxation

A work center producing $n$ product types

- $x_k^i, u_k^i, w_k^i$: the amounts stored, produced, and demanded of product $i$ at time $k$
- State is the stock vector $x_k = (x_k^1, \ldots, x_k^n)$, where $x_{k+1}^i = x_k^i + u_k^i - w_k^i$
- $U$ represents the (shared) production capacity of the work center
- In a more complex version (involving equipment failures), $U$ depends on a random parameter $\alpha_k$ that changes according to a Markov chain
Modify the probability distributions \( P(w_k \mid x_k, w_k) \) to simplify the calculation of \( \tilde{J}_{k+\ell} \) and/or the lookahead minimization.

Certainty equivalent control (inspired by linear-quadratic control problems):

- Replace uncertain quantities with deterministic nominal values.
- The lookahead and tail problems are deterministic so they can be solved by DP or by special deterministic methods.
- Use expected values or forecasts to determine nominal values; update policy when forecasts change (on-line replanning).
- A variant: Partial certainty equivalence. Fix only some uncertain quantities to nominal values.
- A generalization: Approximate \( E\{\cdot\} \) by limited simulation.
Construct a “smaller” aggregate tail problem by introducing aggregate states
Use the exact costs-to-go of the aggregate tail problem as approximate costs-to-go for the original

Aggregation examples:
- State discretization-interpolation schemes
- Grouping of states into subsets, which serve as aggregate states
- Feature-based discretization; aggregate problem operates in the space of features
## Concluding Remarks

### What we covered
- Approximate DP for finite horizon problems with perfect state information
- Approximation in value space
- Approximation in policy space; possibly in combination with approximation in value space

### What we did not cover
- **Approximate DP for infinite horizon problems**
  - Lookahead and rollout ideas apply with essentially no change
  - Special training methods for approximation in value space
  - Temporal difference methods [e.g., TD(\(\lambda\)) and others]; TD(\(\lambda\)) is closely related with the proximal algorithm, but implemented by simulation (see internet videolecture)
- **Imperfect state information problems** can be converted to (much more complex) perfect state information problems. Approximate DP is essential for any kind of solution
- A variety of important lookahead/approximation in value space schemes: Model predictive control, open-loop feedback control, and others
- **Alternative cost criteria:** minimax/games, multiplicative/exponential cost, etc
- **Approximation error bound analysis**
Thank you!