Quantum expanders from any classical Cayley graph expander

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Aram Harrow (Bristol)

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outline

- Main result.
- Definitions.
- Proof of main result.
- Applying the recipe:
  examples of quantum Cayley graph expanders.
- Related work.
- Coming attractions:
  tensor product expanders and k-designs.

The result

Given:

1. A classical Cayley graph expander on a group G with gap $1-\lambda_2$ and degree d.

2. An irrep $\mu(g)$ of G with dimension N.

3. An efficient method of implementing $\mu(g)$ (such as a QFT on G.)

We have:

An efficient quantum expander with dimension N, degree d and gap $\geq 1-\lambda_2$.

Definition: Cayley graph

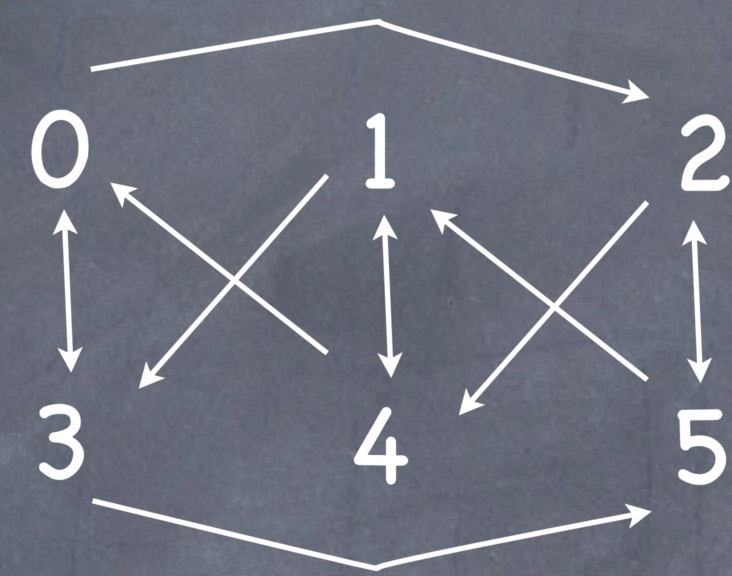
Cayley graph:

1. Given by a group G and a generating set D. $d=|D|$
2. Vertices are elements of G.
3. Neighbours of $g \in G$ are $\{xg : x \in D\}$. Graph is d-regular.

Example: cyclic group

$G = \mathbb{Z}_6$

$D = \{2,3\}$.



$$W = \frac{1}{|D|}\sum_{x\in D} L_x = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Definition: expander

(Classical) expander graph.
Really a family of graphs with $N \to \infty$ vertices and degree $d = O(1)$.

Combinatorial definition: Any not-too-big subset of vertices has lots of neighbours.
Spectral definition: The random walk matrix on the graph has second-largest eigenvalue $\lambda_2 = 1 - \Omega(1)$.

Quantum expander: Spectral definition only.

A family of quantum operations $\mathcal{E}$ acting on an N-dim system.
$d = O(1)$ Kraus operators.
(Typically proportional to unitaries, so $\mathcal{E}(I/N) = I/N$.)

Spectral gap:
As a linear operator on density matrices, $\lambda_2(\mathcal{E}) = 1 - \Omega(1)$.

Representation theory defs

Irrep: μ is a map from G to operators on V_μ such that V_μ has no non-trivial μ-invariant subspace.

Efficiently implementing $\mu(g)$ means taking time poly(log N) to apply $\mu(g)$ to a log N - qubit register. $N := d_\mu = \dim V_\mu$.

Quantum Fourier Transform: U_{QFT}

Implements isomorphism $\mathbb{C}[G] \cong \bigoplus_\mu V_\mu \otimes V_\mu^*$

L_x is the left-multiplication operator: $L_x|g\rangle = |xg\rangle$

Then $U_{\mathrm{QFT}} L_x U_{\mathrm{QFT}}^\dagger = \sum_{\mu \in \hat{G}} |\mu\rangle\langle\mu| \otimes \mu(x) \otimes I_{d_\mu}.$

So, if U_{QFT} and L_x can be implemented efficiently, then so can $\mu(x)$. (Assuming that poly(log |G|) is the same as poly(log d_μ).)

spectra of Cayley graphs

The walk operator is $W = \frac{1}{|D|} \sum_{x \in D} L_x$.

The (normalised) stationary distribution is $\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle$.

In the Fourier basis

The walk operator is $\sum_{\mu} |\mu\rangle\langle\mu| \otimes \frac{1}{|D|} \sum_{x \in D} \mu(x) \otimes I_{d_\mu}$.

The stationary distribution is $|\mu\text{=trivial}\rangle \; |0\rangle \; |0\rangle$.

The second largest eigenvalue is

$$\lambda_2(W) = \max_{\mu \neq \text{trivial}} \left\| \frac{1}{|D|} \sum_{x \in D} \mu(x) \right\|_\infty$$

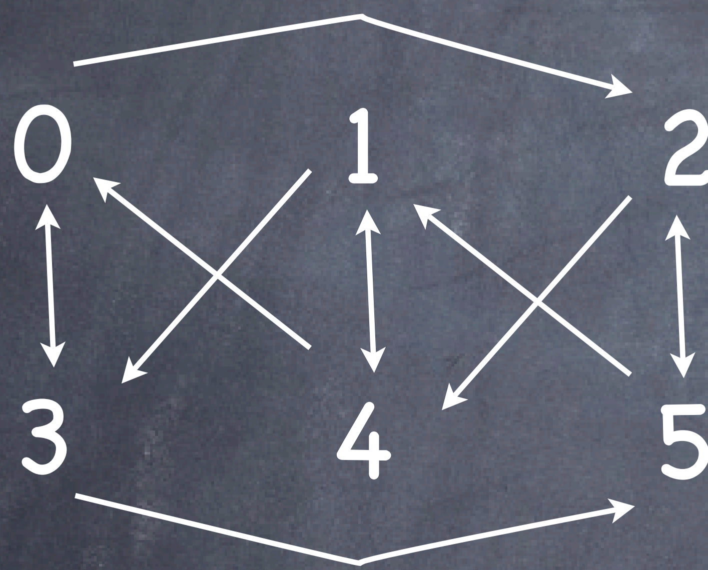
Example: cyclic group

$G = \mathbb{Z}_6$

$D = \{2,3\}$.

Fourier basis: $k \in \{0,1,2,3,4,5\}$

$\mu_k(x) = \omega^{kx}$ $\quad \omega = e^{2\pi i/6}$



$$W = \frac{1}{|D|}\sum_{x\in D} L_x = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

k	$\frac{1}{\lvert D\rvert}\sum_{x\in D}\mu(x)$
0	1
1	$(\omega^2+\omega^3)/2$
2	$(\omega^4+1)/2$
3	$(1+\omega^3)/2$
4	$(\omega^2+1)/2$
5	$(\omega^4+\omega^3)/2$

Warning!
Abelian groups can't have $O(1)$ degree, $\Omega(1)$ gap.

Example: symmetric group

G = S_3
D = {(12), (123)}.

μ	$\frac{1}{\lvert D\rvert}\sum_{x\in D}\mu(x)$
trivial	1
sign	0
2-dim	$\frac{1}{4}\begin{pmatrix} -1+i\sqrt{3} & 2 \\ 2 & -1-i\sqrt{3} \end{pmatrix}$

$$\lambda_2 = \frac{1}{2}$$

The quantum expander construction

Given a classical Cayley graph with generators D⊂G

and given an irrep μ;
the quantum expander is:

$$\mathcal{E}(\rho) = \frac{1}{|D|} \sum_{x \in D} \mu(x)\, \rho\, \mu(x)^\dagger$$

\# Kraus operators = |D| = degree of classical expander
$\mathcal{E}$ is efficient if μ(x) is efficient.
It remains to show that $\lambda_2(\mathcal{E}) \leq \lambda_2(W)$.

Analysis of quantum expander

$$\mathcal{E}(\rho) = \frac{1}{|D|} \sum_{x \in D} \mu(x)\, \rho\, \mu(x)^\dagger$$

As a linear operator (instead of a super-operator), this is:

$$\hat{\mathcal{E}} = \frac{1}{|D|} \sum_{x \in D} \mu(x) \otimes \mu(x)^*$$

We want $\lambda_2(\hat{\mathcal{E}})$.

Now the inevitable representation theory:

$\mu \otimes \mu^*$ is a reducible representation of G, and can be decomposed into irreps. If ν appears with multiplicity m_ν, then

$$\mu(x) \otimes \mu(x)^* \cong \sum_{\nu} |\nu\rangle\langle\nu| \otimes \nu(x) \otimes I_{m_\nu}$$

AND! Schur's Lemma says $m_{\text{trivial}}=1$.

Analysis of quantum expander

$$\hat{\mathcal{E}} = \frac{1}{|D|}\sum_{x\in D}\mu(x)\otimes\mu(x)^*$$

$$\cong \sum_{\nu}|\nu\rangle\langle\nu|\otimes\left(\frac{1}{|D|}\sum_{x\in D}\nu(x)\right)\otimes I_{m_\nu}$$

$m_{trivial}=1$ corresponds to $\lambda_1=1$.

The second largest eigenvalue is

$$\lambda_2(\hat{\mathcal{E}}) = \max_{\substack{\nu\neq\text{trivial}\\ m_\nu>0}}\left\|\frac{1}{|D|}\sum_{x\in D}\nu(x)\right\|_\infty$$

$$\leq \max_{\nu\neq\text{trivial}}\left\|\frac{1}{|D|}\sum_{x\in D}\nu(x)\right\|_\infty = \lambda_2(W)$$

Q.E.D.

Applying the recipe

Recall: We want run-time to be poly(log d_μ), but implementing μ using a QFT usually requires poly(log $|G|$) time. This works when d_μ is sufficiently large.

PSL(2, $\mathbb{F}_q$): The LPS expander. d=6, $\lambda_2 = \sqrt{5} / 3$.
Irreps are large, <u>but</u> no efficient QFT is known.

DOESN'T WORK

SU(2): Another LPS expander. d=6, $\lambda_2 = \sqrt{5} / 3$.
Irreps are large, <u>but</u> no efficient QFT is known.
quant-ph/0407140 claims to implement $\mu(x)$ in time poly(log d_μ), but the algorithm is incomplete.

DOESN'T WORK

Applying the recipe

S_n: The Kassabov expander. d=O(1), $\lambda_2 = 1-\Omega(1)$ WORKS

Irreps are large: $\log d_\mu \approx (\log |S_n|) / 2$

QFT runs in poly($\log |S_n|$) = poly(n).

S_{N+1}: The Kassabov expander. d=O(1), $\lambda_2 = 1-\Omega(1)$ WORKS (for any N)

There is an N-dimensional irrep that can be directly implemented in time poly(log N).

$H \wr H \wr \ldots$: zig-zag product [Rozenman-Shalev-Wigderson] WORKS

$|H| = O(1)$. $H=[H,H]$.

Has large irreps and efficient QFT.

Aff(2, $\mathbb{F}_q$): Margulis expander. d=8, $\lambda_2 \le 5\sqrt{2} / 8$. WORKS (for any N)

No efficient QFT but one irrep can be efficiently constructed. [Eisert-Gross; 0710.0651]

Related work

quantum zig-zag product:

[Ben-Aroya, Schwartz, Ta-Shma; 0709.0911]

Not the same as applying my construction to the classical zig-zag product.

Another QFT-based construction:

[Ben-Aroya, Ta-Shma; 0702129]

Not yet known to be efficient.

Quantum Margulis expanders

[Eisert-Gross; 0710.0651]

Also yields efficient constant-gap, constant-degree expanders for any dimension.

Coming attractions!

expander

approx. 1-design
$\{p_i, U_i\}$ s.t.

$$\sum_i p_i U_i \rho U_i^\dagger \approx \int dU\, U \rho U^\dagger$$

approx. k-design
$\{p_i, U_i\}$ s.t.

$$\sum_i p_i U_i^{\otimes k} \rho (U_i^\dagger)^{\otimes k} \approx \int dU\, U^{\otimes k} \rho (U^\dagger)^{\otimes k}$$

k-tensor product expander

with M. Hastings: random unitaries are tensor product expanders.

with R. Low: random circuits are tensor product expanders. (we think)


THANKS