Massachusetts Institute of Technology
Department of Physics

8.276 Spring 2007 **April 5, 2007**
Solution to Problem set #6

1. (10 points)

$$\begin{array}{ccc} \nu_\ell & \longleftarrow \; \pi \; \longrightarrow & \ell \\ p & E & p \end{array}$$

a) With $c = 1$, $E = m_\pi = p + \sqrt{p^2 + m_\ell^2}$.

Solve for p: $(m_\pi - p)^2 = p^2 + m_\ell^2 \rightarrow p = \dfrac{m_\pi^2 - m_\ell^2}{2m_\pi}$

$$E_\ell = m_\pi - p = \frac{m_\pi^2 + m_\ell^2}{2m_\pi}$$

$$1 - \frac{v_\ell}{c} = 1 - \frac{p}{E_\ell} = \frac{2m_\ell^2}{m_\pi^2 + m_\ell^2} = \begin{cases} 0.73 & \text{for } \mu \\ 0.27 \times 10^{-4} & \text{for e} \end{cases}$$

b)

$$\frac{|\mathcal{M}_{\pi \mathrm{e}}|^2}{|\mathcal{M}_{\pi\mu}|^2} = \frac{1 - v_\mathrm{e}/c}{1 - v_\mu/c} = 0.37 \times 10^{-4}$$

c)

$$\frac{\mathrm{d}p}{\mathrm{d}E} = \frac{\mathrm{d}p}{\mathrm{d}m_\pi} = \frac{m_\pi^2 + m_\ell^2}{2m_\pi^2}$$

$$p^2 \frac{\mathrm{d}p}{\mathrm{d}E} = \frac{(m_\pi^2 - m_\ell^2)^2}{4m_\pi^2} \cdot \frac{m_\pi^2 + m_\ell^2}{2m_\pi^2}$$

$$\frac{\varrho_\mathrm{e}}{\varrho_\mu} = \frac{(m_\pi^2 - m_\mathrm{e}^2)^2}{(m_\pi^2 - m_\mu^2)^2} \cdot \frac{m_\pi^2 + m_\mathrm{e}^2}{m_\pi^2 + m_\mu^2} = 3.49$$

(note: formula in textbook answer is incorrect)

d)

$$\Gamma(\pi^+ \rightarrow \ell^+ \nu_\ell) \propto \frac{(m_\pi^2 - m_\ell^2)^2 (m_\pi^2 + m_\ell^2)(2m_\ell^2)}{8m_\pi^4 (m_\pi^2 + m_\ell^2)}$$

$$\propto \frac{m_\ell^2}{m_\pi^4} (m_\pi^2 - m_\ell^2)^2$$

$$\Longrightarrow \quad \frac{\Gamma(\pi^+ \rightarrow \mathrm{e}^+ \nu_\mathrm{e})}{\Gamma(\pi^+ \rightarrow \mu^+ \nu_\mu)} = \frac{m_\mathrm{e}^2 (m_\pi^2 - m_\mathrm{e}^2)^2}{m_\mu^2 (m_\pi^2 - m_\mu^2)^2} = 1.28 \times 10^{-4}$$

in agreement with experiment

2. (15 points)

As discussed in class, the way you do DIS on a nucleon (neutron and proton) is to use an isoscalar target, such as the deuteron.

For neutrino scattering (interactions only with negatively charged quarks), we can write

$$\begin{aligned} d^{\mathrm{p}}(x) + d^{\mathrm{n}}(x) &= d(x) + u(x) \equiv Q(x) \\ \bar{u}^{\mathrm{p}}(x) + \bar{u}^{\mathrm{n}}(x) &= \bar{u}(x) + \bar{d}(x) \equiv \bar{Q}(x). \end{aligned}$$

Then,

$$\frac{\mathrm{d}^2\sigma(\nu_\mu \mathrm{N} \to \mu^- \mathrm{X})}{\mathrm{d}x\mathrm{d}y} = C \cdot x \cdot \left[Q(x) + (1-y)^2 \bar{Q}(x)\right].$$

Similarly,

$$\frac{\mathrm{d}^2\sigma(\bar{\nu}_\mu \mathrm{N} \to \mu^+ \mathrm{X})}{\mathrm{d}x\mathrm{d}y} = C \cdot x \cdot \left[\bar{Q}(x) + (1-y)^2 Q(x)\right]$$

The interactions with s or $\bar{s}$ are suppressed by the Cabibbo angle factor. We can ignore heavy quarks in the sea such as $c\bar{c}$ or $b\bar{b}$.

C represents all the constants and factors common to ν and $\bar{\nu}$ scattering: Eq(10.30)

a) Let $P = \displaystyle\int_0^1 xQ(x)\mathrm{d}x$ = the quark momentum content.

$\bar{P} = \displaystyle\int_0^1 x\bar{Q}(x)\mathrm{d}x$ = the antiquark momentum content.

$$\int_0^1 (1-y)^2 \mathrm{d}y = \frac{1}{3}$$

Integrating over y (from 0 to 1):

$$\begin{aligned} \sigma^{\nu N} &= C\left[P + \frac{1}{3}\bar{P}\right] \\ \sigma^{\bar{\nu} N} &= C\left[\bar{P} + \frac{1}{3}P\right] \\ R = \frac{\sigma^{\bar{\nu} N}}{\sigma^{\nu N}} &= \frac{3\bar{P} + P}{3P + \bar{P}} = 0.5 \end{aligned}$$

Solve for $\dfrac{\bar{P}}{P}$:

$$\begin{aligned} 3\bar{P} + P &= \frac{3}{2} + \frac{1}{2}\bar{P} \\ \frac{5}{2}\bar{P} &= \frac{1}{2}P \qquad \longrightarrow \qquad \frac{\bar{P}}{P} = \frac{1}{5} = 0.2 \end{aligned}$$

$$\text{(in reality, } \frac{\bar{P}}{P} \approx 0.05\text{)}$$

b)

$$\text{use } \langle y \rangle = \frac{\int_0^1 y \cdot \frac{\mathrm{d}\sigma}{\mathrm{d}y} \mathrm{d}y}{\int_0^1 \frac{\mathrm{d}\sigma}{\mathrm{d}y} \mathrm{d}y}$$

For neutrinos,

$$\begin{aligned} \int_0^1 \frac{\mathrm{d}\sigma^{\nu N}}{\mathrm{d}y} \mathrm{d}y &\propto \int_0^1 \left[1 + (1-y)^2 \frac{\bar{P}}{P} \right] \mathrm{d}y \\ &\propto \left(1 + \frac{1}{3} \frac{\bar{P}}{P} \right). \end{aligned}$$

For antineutrinos,

$$\begin{aligned} \int_0^1 \frac{\mathrm{d}\sigma^{\bar{\nu} N}}{\mathrm{d}y} \mathrm{d}y &\propto \int_0^1 \left[\frac{\bar{P}}{P} + (1-y)^2 \right] \mathrm{d}y \\ &\propto \left(\frac{\bar{P}}{P} + \frac{1}{3} \right). \end{aligned}$$

For neutrinos,

$$\begin{aligned} \int_0^1 y \cdot \frac{\mathrm{d}\sigma^{\nu N}}{\mathrm{d}y} \mathrm{d}y &\propto \int_0^1 \left[y + y(1-y)^2 \frac{\bar{P}}{P} \right] \mathrm{d}y \\ &= \frac{1}{2} + \frac{1}{12} \frac{\bar{P}}{P}. \end{aligned}$$

For antineutrinos,

$$\begin{aligned} \int_0^1 y \cdot \frac{\mathrm{d}\sigma^{\bar{\nu} N}}{\mathrm{d}y} \mathrm{d}y &\propto \int_0^1 \left[y \frac{\bar{P}}{P} + y(1-y)^2 \right] \mathrm{d}y \\ &= \frac{1}{12} + \frac{1}{2} \frac{\bar{P}}{P}. \end{aligned}$$

So, with $\bar{P}/P = 0.2$,

$$\begin{aligned} \langle y \rangle^{\nu} &= \frac{6 + \bar{P}/P}{12 + 4\bar{P}/P} = 0.48 \\ \langle y \rangle^{\bar{\nu}} &= \frac{1 + 6\bar{P}/P}{4 + 12\bar{P}/P} = 0.34. \end{aligned}$$

3. (10 points) Neglecting scattering from antiquarks,

$$\begin{aligned}
\frac{\mathrm{d}\sigma^{\mathrm{CC}}(\nu)}{\mathrm{d}y} &= \frac{G^2 s}{2\pi}\left[Q + (1-y)^2\bar{Q}\right] = \frac{G^2 sQ}{2\pi} \\
\frac{\mathrm{d}\sigma^{\mathrm{CC}}(\bar{\nu})}{\mathrm{d}y} &= \frac{G^2 s}{2\pi}\left[\bar{Q} + (1-y)^2 Q\right] = \frac{G^2 s}{2\pi}(1-y)^2 Q \\
\frac{\mathrm{d}\sigma^{\mathrm{NC}}(\nu)}{\mathrm{d}y} &= \frac{G^2 s}{2\pi}\left\{g_{\mathrm{L}}^2\left[Q + (1-y)^2\bar{Q}\right] + g_{\mathrm{R}}^2\left[\bar{Q} + (1-y)^2 Q\right]\right\} \\
&= \frac{G^2 s}{2\pi}\left[g_{\mathrm{L}}^2 + g_{\mathrm{R}}^2(1-y)^2\right] Q \\
\frac{\mathrm{d}\sigma^{\mathrm{NC}}(\bar{\nu})}{\mathrm{d}y} &= \frac{G^2 s}{2\pi}\left\{g_{\mathrm{L}}^2\left[\bar{Q} + (1-y)^2 Q\right] + g_{\mathrm{R}}^2\left[Q + (1-y)^2\bar{Q}\right]\right\} \\
&= \frac{G^2 s}{2\pi}\left[g_{\mathrm{L}}^2(1-y)^2 + g_{\mathrm{R}}^2\right] Q.
\end{aligned}$$

These are the cross sections <u>per nucleon</u> for isoscalar targets:

$$Q \equiv \int x Q(x)\mathrm{d}x = \int x\left[u(x) + d(x)\right]\mathrm{d}x$$

$$\text{Note:}\quad \int_0^1 (1-y)^2 \mathrm{d}y = \frac{1}{3}$$

Coupling of Z^0 to fermion $f \propto (T_3 - z_f \sin^2\theta_{\mathrm{W}})$, so

$$\begin{aligned}
g_{\mathrm{L}}(u) &= \frac{1}{2} - \frac{2}{3}\sin^2\theta_{\mathrm{W}} \\
g_{\mathrm{R}}(u) &= -\frac{2}{3}\sin^2\theta_{\mathrm{W}} \\
g_{\mathrm{L}}(d) &= -\frac{1}{2} + \frac{1}{3}\sin^2\theta_{\mathrm{W}} \\
g_{\mathrm{R}}(d) &= \frac{1}{3}\sin^2\theta_{\mathrm{W}}
\end{aligned}$$

$$\begin{aligned}
\frac{\sigma^{\mathrm{NC}}(\nu)}{\sigma^{\mathrm{CC}}(\nu)} = \frac{\int_0^1 \mathrm{d}y\,\sigma^{\mathrm{NC}}(\nu)}{\int_0^1 \mathrm{d}y\,\sigma^{\mathrm{CC}}(\nu)} &= g_{\mathrm{L}}(u)^2 + g_{\mathrm{L}}(d)^2 + \frac{1}{3}\left(g_{\mathrm{R}}(u)^2 + g_{\mathrm{R}}(d)^2\right) \\
&= \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_{\mathrm{W}}\right)^2 + \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_{\mathrm{W}}\right)^2 + \frac{1}{3}\cdot\frac{5}{9}\sin^4\theta_{\mathrm{W}} \\
&= \boxed{\frac{1}{2} - \sin^2\theta_{\mathrm{W}} + \frac{20}{27}\sin^4\theta_{\mathrm{W}}}
\end{aligned}$$

$$\begin{aligned}
\frac{\sigma^{\mathrm{NC}}(\bar{\nu})}{\sigma^{\mathrm{CC}}(\bar{\nu})} &= 3\cdot\left[\frac{1}{3}\left(g_{\mathrm{L}}(u)^2 + g_{\mathrm{L}}(d)^2\right) + g_{\mathrm{R}}(u)^2 + g_{\mathrm{R}}(d)^2\right] \\
&= \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_{\mathrm{W}}\right)^2 + \left(-\frac{1}{2} + \frac{1}{3}\sin^2\theta_{\mathrm{W}}\right)^2 + 3\cdot\frac{5}{9}\sin^4\theta_{\mathrm{W}} \\
&= \boxed{\frac{1}{2} - \sin^2\theta_{\mathrm{W}} + \frac{20}{9}\sin^4\theta_{\mathrm{W}}}
\end{aligned}$$

4. (10 points)

a) Neutral-current cross section for neutron-electron scattering:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y} = \frac{2G^2 mE}{\pi}\left[g_{\mathrm{L}}^2 + g_{\mathrm{R}}^2(1-y)^2\right]$$

To obtain total cross section for neutrino energy E, integrate over y from 0 to 1.

$$\begin{aligned}\sigma &= \frac{2G^2 mE}{\pi}\int_0^1 \left(g_{\mathrm{L}}^2 + g_{\mathrm{R}}^2(1-y)^2\right)\mathrm{d}y \\ &= \frac{2G^2 mE}{\pi}\left(g_{\mathrm{L}}^2 + \frac{1}{3}g_{\mathrm{R}}^2\right)\end{aligned}$$

i) $\nu_\mu \mathrm{e} \to \nu_\mu \mathrm{e}: \quad g_{\mathrm{L}} = -\frac{1}{2} + \sin^2\theta_{\mathrm{W}} \quad g_{\mathrm{R}} = \sin^2\theta_{\mathrm{W}}$

$$\begin{aligned}\frac{\sigma}{E} &= \frac{2G^2 m}{\pi}\left\{\frac{1}{4} - \sin^2\theta_{\mathrm{W}} + (\sin^2\theta_{\mathrm{W}})^2 + \frac{1}{3}(\sin^2\theta_{\mathrm{W}})^2\right\} \\ &= \frac{2G^2 m}{\pi}\left\{\frac{4}{3}(\sin^2\theta_{\mathrm{W}})^2 - \sin^2\theta_{\mathrm{W}} + \frac{1}{4}\right\}\end{aligned}$$

ii) $\bar{\nu}_\mu \mathrm{e} \to \bar{\nu}_\mu \mathrm{e}: \quad g_{\mathrm{L}} = \sin^2\theta_{\mathrm{W}} \quad g_{\mathrm{R}} = -\frac{1}{2} + \sin^2\theta_{\mathrm{W}}$

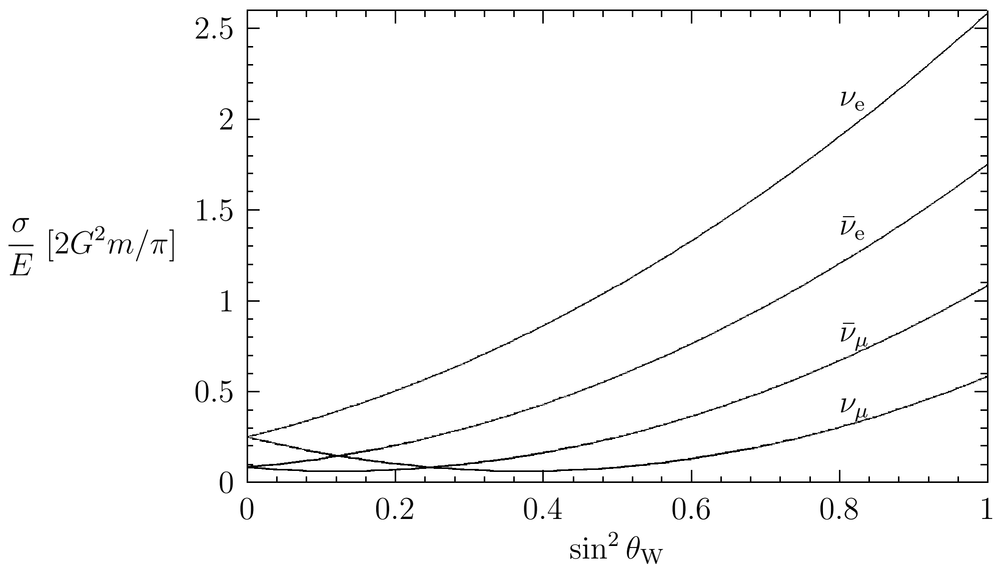
$$\begin{aligned}\frac{\sigma}{E} &= \frac{2G^2 m}{\pi}\left\{(\sin^2\theta_{\mathrm{W}})^2 + \frac{1}{3}\left(\frac{1}{4} - \sin^2\theta_{\mathrm{W}} + (\sin^2\theta_{\mathrm{W}})^2\right)\right\} \\ &= \frac{2G^2 m}{\pi}\left\{\frac{4}{3}(\sin^2\theta_{\mathrm{W}})^2 - \frac{1}{3}\sin^2\theta_{\mathrm{W}} + \frac{1}{12}\right\}\end{aligned}$$

iii) $\nu_{\mathrm{e}} \mathrm{e} \to \nu_{\mathrm{e}} \mathrm{e}: \quad g_{\mathrm{L}} = \frac{1}{2} + \sin^2\theta_{\mathrm{W}} \quad g_{\mathrm{R}} = \sin^2\theta_{\mathrm{W}}$

$$\begin{aligned}\frac{\sigma}{E} &= \frac{2G^2 m}{\pi}\left\{\frac{1}{4} + \sin^2\theta_{\mathrm{W}} + (\sin^2\theta_{\mathrm{W}})^2 + \frac{1}{3}(\sin^2\theta_{\mathrm{W}})^2\right\} \\ &= \frac{2G^2 m}{\pi}\left\{\frac{4}{3}(\sin^2\theta_{\mathrm{W}})^2 + \sin^2\theta_{\mathrm{W}} + \frac{1}{4}\right\}\end{aligned}$$

iv) $\bar{\nu}_{\mathrm{e}} \mathrm{e} \to \bar{\nu}_{\mathrm{e}} \mathrm{e}: \quad g_{\mathrm{L}} = \sin^2\theta_{\mathrm{W}} \quad g_{\mathrm{R}} = \frac{1}{2} + \sin^2\theta_{\mathrm{W}}$

$$\begin{aligned}\frac{\sigma}{E} &= \frac{2G^2 m}{\pi}\left\{(\sin^2\theta_{\mathrm{W}})^2 + \frac{1}{3}\left(\frac{1}{4} + \sin^2\theta_{\mathrm{W}} + (\sin^2\theta_{\mathrm{W}})^2\right)\right\} \\ &= \frac{2G^2 m}{\pi}\left\{\frac{4}{3}(\sin^2\theta_{\mathrm{W}})^2 + \frac{1}{3}\sin^2\theta_{\mathrm{W}} + \frac{1}{12}\right\}\end{aligned}$$



b)

$$\frac{\mathrm{d}\sigma(\nu_\mu)}{\mathrm{d}x} = 0 \quad \rightarrow \quad \frac{8}{3}\sin^2\theta_{\mathrm{W}} - 1 = 0 \quad \rightarrow \quad \sin^2\theta_{\mathrm{W}} = \frac{3}{8}$$

$$\frac{\mathrm{d}\sigma(\bar{\nu}_\mu)}{\mathrm{d}x} = 0 \quad \rightarrow \quad \frac{8}{3}\sin^2\theta_{\mathrm{W}} - \frac{1}{3} = 0 \quad \rightarrow \quad \sin^2\theta_{\mathrm{W}} = \frac{1}{8}$$

c)

$c_{\mathrm{V}} = g_{\mathrm{L}} + g_{\mathrm{R}} = 0$ for purely axial-vector coupling.

For $\bar{\nu}_{\mathrm{e}}$e scattering, $(\sin^2\theta_{\mathrm{W}})^2 - \frac{1}{2} + (\sin^2\theta_{\mathrm{W}})^2 = 0 \quad \rightarrow \quad (\sin^2\theta_{\mathrm{W}})^2 = \frac{1}{4} = 0.250$

5. (15 points)

a) Assume neutrino mass eigenstates ν_1 and ν_2 which are linear combinations of the weak-interaction eigenstates ν_e and ν_μ:

$$\begin{aligned} \nu_1 &= \quad \nu_\mathrm{e} \cos\theta + \nu_\mu \sin\theta \\ \nu_2 &= -\nu_\mathrm{e} \sin\theta + \nu_\mu \cos\theta \end{aligned}$$

The mass eigenstates will propagate according to:

$$\begin{aligned} \nu_1(t) &= \nu_1(0) e^{-iE_1 t} \\ \nu_2(t) &= \nu_2(0) e^{-iE_2 t} \qquad \text{setting } \hbar = 1 \end{aligned}$$

At time $t = 0$, assume $\nu_\mathrm{e}(0) = 1$ and $\nu_\mu(0) = 0$.
So $\nu_1(0) = \cos\theta$ and $\nu_2(0) = -\sin\theta$.
Writing $\nu_\mu(t)$ in terms of $\nu_1(t)$ and $\nu_2(t)$,

$$\begin{aligned} \nu_\mu(t) &= \nu_1(t)\sin\theta + \nu_2(t)\cos\theta \\ &= \cos\theta\sin\theta\left(e^{-iE_1 t} - e^{-iE_2 t}\right) \end{aligned}$$

$$\begin{aligned} \text{Intensity} = |\nu_\mu(t)|^2 &= \cos^2\theta\sin^2\theta\left\{2 - \left[e^{i(E_2-E_1)t} - e^{-i(E_2-E_1)t}\right]\right\} \\ &= 2\cos^2\theta\sin^2\theta\left(1 - \cos\left(E_2 - E_1\right)t\right) \\ &= 4\cos^2\theta\sin^2\theta\sin^2\frac{(E_2-E_1)t}{2} \\ &= \sin^2 2\theta \sin^2\frac{(E_2-E_1)t}{2} \end{aligned}$$

Now

$$E_i = pc + \frac{m_i^2 c^3}{2p} \qquad i = 1,2$$

(momentum conserved: p constant)

So

$$E_2 - E_1 = \frac{(m_2^2 - m_1^2)c^3}{2p} \equiv \frac{\triangle m^2 c^3}{2p}$$

And since $t = \dfrac{\ell}{c}$,

$$\underset{\substack{\uparrow \\ \text{put back in!}}}{\frac{(E_2-E_1)t}{2\hbar}} = \frac{\triangle m^2 \cdot \ell \cdot c^2}{4p\hbar} = \frac{(\triangle m^2 c^4)\ell}{4(pc)(\hbar c)} = \frac{1.27\triangle m^2 \ell}{pc}$$

with $\triangle m^2 c^4$ in eV^2
ℓ in meters
(pc) in MeV

So

$$P \equiv |\nu_\mu(t)|^2 = \sin^2 2\theta \sin^2\left(\frac{1.27\triangle m^2\ell}{pc}\right).$$

b) Oscillation length L is defined by setting phase in

$$\cos\left(\frac{E_2 - E_1}{\hbar}\right) t = 2\pi$$

$$\begin{aligned}
\frac{\triangle m^2 c^3}{2p\hbar} \cdot \frac{L}{c} &= 2\pi \\
\Longrightarrow L &= 4\pi \frac{(pc)(\hbar c)}{\triangle m^2 c^4} \qquad \text{(note that Eq. (10.26) is incorrect)} \\
&= \pi \frac{(pc)}{1.27 \triangle m^2}
\end{aligned}$$

For $\triangle m^2 = 1.5 \times 10^{-4} (\mathrm{eV}/c^2)^2$

$$L = 1.65 \times 10^4 \; (pc)$$

So

$$\begin{aligned}
\text{for } pc = 0.4\text{MeV}, &\quad L = \;\; 6.6 \times 10^3 \text{m} = 6.6\text{km} \\
pc = 0.9\text{MeV}, &\quad L = 1.49 \times 10^4 \text{m} \approx 15\text{km} \\
pc = \;\, 14\text{MeV}, &\quad L = \;\; 2.3 \times 10^5 \text{m} \approx \frac{1}{10} \text{ radius of earth}
\end{aligned}$$