2

Particle Dynamics

Understanding and utilizing the response of charged particles to electromagnetic forces is the basis of particle optics and accelerator theory. The goal is to find the time-dependent position and velocity of particles, given specified electric and magnetic fields. Quantities calculated from position and velocity, such as total energy, kinetic energy, and momentum, are also of interest. The nature of electromagnetic forces is discussed in Chapter 3. In this chapter, the response of particles to general forces will be reviewed. These are summarized in laws of motion. The Newtonian laws, treated in the first sections, apply at low particle energy. At high energy, particle trajectories must be described by relativistic equations. Although Newton's laws and their implications can be understood intuitively, the laws of relativity cannot since they apply to regimes beyond ordinary experience. Nonetheless, they must be accepted to predict particle behavior in high-energy accelerators. In fact, accelerators have provided some of the most direct verifications of relativity.

This chapter reviews particle mechanics. Section 2.1 summarizes the properties of electrons and ions. Sections 2.2-2.4 are devoted to the equations of Newtonian mechanics. These are applicable to electrons from electrostatic accelerators of in the energy range below 20 kV. This range includes many useful devices such as cathode ray tubes, electron beam welders, and microwave tubes. Newtonian mechanics also describes ions in medium energy accelerators used for nuclear physics. The Newtonian equations are usually simpler to solve than relativistic formulations. Sometimes it is possible to describe transverse motions of relativistic particles using Newtonian equations with a relativistically corrected mass. This approximation is treated in Section 2.10. In the second part of the chapter, some of the principles of special relativity are derived from two basic postulates, leading to a number of useful formulas summarized in Section 2.9.

2.1 CHARGED PARTICLE PROPERTIES

In the theory of charged particle acceleration and transport, it is sufficient to treat particles as dimensionless points with no internal structure. Only the influence of the electromagnetic force, one of the four fundamental forces of nature, need be considered. Quantum theory is unnecessary except to describe the emission of radiation at high energy.

This book will deal only with ions and electrons. They are simple, stable particles. Their response to the fields applied in accelerators is characterized completely by two quantities: mass and charge. Nonetheless, it is possible to apply much of the material presented to other particles. For example, the motion of macroparticles with an electrostatic charge can be treated by the methods developed in Chapters 6-9. Applications include the suspension of small objects in oscillating electric quadrupole fields and the acceleration and guidance of inertial fusion targets. At the other extreme are unstable elementary particles produced by the interaction of high-energy ions or electrons with targets. Beamlines, acceleration gaps, and lenses are similar to those used for stable particles with adjustments for different mass. The limited lifetime may influence hardware design by setting a maximum length for a beamline or confinement time in a storage ring.

An electron is an elementary particle with relatively low mass and negative charge. An ion is an assemblage of protons, neutrons, and electrons. It is an atom with one or more electrons removed. Atoms of the isotopes of hydrogen have only one electron. Therefore, the associated ions (the proton, deuteron, and triton) have no electrons. These ions are bare nucleii consisting of a proton with 0, 1, or 2 neutrons. Generally, the symbol Z denotes the atomic number of an ion or the number of electrons in the neutral atom. The symbol Z* is often used to represent the number of electrons removed from an atom to create an ion. Values of Z* greater than 30 may occur when heavy ions traverse extremely hot material. If Z* = Z, the atom is fully stripped. The atomic mass number A is the number of nucleons (protons or neutrons) in the nucleus. The mass of the atom is concentrated in the nucleus and is given approximately as Am_p, where mp is the proton mass.

Properties of some common charged particles are summarized in Table 2.1. The meaning of the rest energy in Table 2.1 will become clear after reviewing the theory of relativity. It is listed in energy units of million electron volts (MeV). An electron volt is defined as the energy gained by a particle having one fundamental unit of charge ($q = \pm e = \pm 1.6 \times 10^{-19}$ coulombs) passing

TABLE 2.1 Charged Particle Properties

Particle	Charge (coulomb)	Mass (kg)	Rest Energy (MeV)	A	Z	Z^*
Electron (β particle)	-1.60×10^{-19}	9.11×10^{-31}	0.511	—	—	—
Proton	$+1.60 \times 10^{-19}$	1.67×10^{-27}	938	1	1	1
Deuteron	$+1.60 \times 10^{-19}$	3.34×10^{-27}	1875	2	1	1
Triton	$+1.60 \times 10^{-19}$	5.00×10^{-27}	2809	3	1	1
He^+	$+1.60 \times 10^{-19}$	6.64×10^{-27}	3728	4	2	1
He^{++} (α particle)	$+3.20 \times 10^{-19}$	6.64×10^{-27}	3728	4	2	2
C^+	$+1.6 \times 10^{-19}$	1.99×10^{-26}	1.12×10^4	12	6	1
U^+	$+1.6 \times 10^{-19}$	3.95×10^{-25}	2.22×10^5	238	92	1

through a potential difference of one volt. In MKS units, the electron volt is

$$I\ eV = (1.6 \times 10^{-19}\ C)\ (1\ V) = 1.6 \times 10^{-19}\ J.$$

Other commonly used metric units are keV (10^3 eV) and GeV (10^9 eV). Relativistic mechanics must be used when the particle kinetic energy is comparable to or larger than the rest energy. There is a factor of 1843 difference between the mass of the electron and the proton. Although methods for transporting and accelerating various ions are similar, techniques for electrons are quite different. Electrons are relativistic even at low energies. As a consequence, synchronization of electron motion in linear accelerators is not difficult. Electrons are strongly influenced by magnetic fields; thus they can be accelerated effectively in a circular induction accelerator (the betatron). High-current electron beams (~10 kA) can be focused effectively by magnetic fields. In contrast, magnetic fields are ineffective for high-current ion beams. On the other hand, it is possible to neutralize the charge and current of a high-current ion beam easily with light electrons, while the inverse is usually impossible.

2.2 NEWTON'S LAWS OF MOTION

The charge of a particle determines the strength of its interaction with the electromagnetic force. The mass indicates the resistance to a change in velocity. In Newtonian mechanics, mass is constant, independent of particle motion.

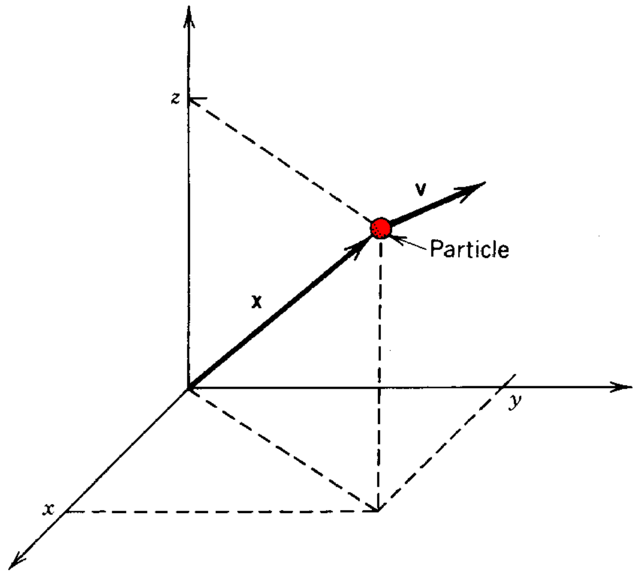

Figure 2.1. Position and velocity vectors of a particle in Cartesian coordinates.

The Newtonian mass (or *rest mass*) is denoted by a subscript: m_e for electrons, m_p for protons, and m_o for a general particle. A particle's behavior is described completely by its position in three-dimensional space and its velocity as a function of time. Three quantities are necessary to specify position; the position **x** is a vector. In the Cartesian coordinates (Figure 2.1), **x** can be written

$$\boldsymbol{x} = (x,y,z). \tag{2.1}$$

The particle velocity is

$$\boldsymbol{v} = (v_x,v_y,v_z) = (dx/dt,dy/dt,dz/dt) = d\boldsymbol{x}/dt, \tag{2.2}$$

Newton's first law states that a moving particle follows a straight-line path unless acted upon by a force. The tendency to resist changes in straight-line motion is called the momentum, **p**. Momentum is the product of a particle's mass and velocity,

$$\boldsymbol{p} = m_o\boldsymbol{v} = (p_x,p_y,p_z). \tag{2.3}$$

Newton's second law defines force F through the equation

$$d\boldsymbol{p}/dt = \boldsymbol{F}. \tag{2.4}$$

In Cartesian coordinates, Eq. (2.4) can be written

$$dp_x/dt = F_x, \quad dp_y/dt = F_y, \quad dp_z/dt = F_z. \tag{2.5}$$

Motions in the three directions are decoupled in Eq. (2.5). With specified force components, velocity components in the x, y, and z directions are determined by separate equations. It is important to note that this decoupling occurs only when the equations of motion are written in terms of Cartesian coordinates. The significance of straight-line motion is apparent in Newton's first law, and the laws of motion have the simplest form in coordinate systems based on straight lines. Caution must be exercised using coordinate systems based on curved lines. The analog of Eq. (2.5) for cylindrical coordinates (r, 0, z) will be derived in Chapter 3. In curvilinear coordinates, momentum components may change even with no force components along the coordinate axes.

2.3 KINETIC ENERGY

Kinetic energy is the energy associated with a particle's motion. The purpose of particle accelerators is to impart high kinetic energy. The kinetic energy of a particle, T, is changed by applying a force. Force applied to a static particle cannot modify T; the particle must be moved. The change in T (work) is related to the force by

$$\Delta T = \int \mathbf{F}\cdot\mathbf{dx}. \tag{2.6}$$

The integrated quantity is the vector dot product; **dx** is an incremental change in particle position.
In accelerators, applied force is predominantly in one direction. This corresponds to the symmetry axis of a linear accelerator or the main circular orbit in a betatron. With acceleration along the z axis, Eq. (2.6) can be rewritten

$$\Delta T = \int F_z dz = \int F_z \; (dz/dt) \; dt. \tag{2.7}$$

The chain rule of derivatives has been used in the last expression. The formula for T in Newtonian mechanics can be derived by (1) rewriting F, using Eq. (2.4), (2) taking T = 0 when v, = 0, and (3) assuming that the particle mass is not a function of velocity:

$$T = \int m_o v_z \; (dv_z/dt) \; dt = m_o v_z^2/2. \tag{2.8}$$

The differential relationship $d(m_o v_z^2/2)/dt = m_o v_z\, dv_z/dt$ leads to the last expression. The differences of relativistic mechanics arise from the fact that assumption 3 is not true at high energy.

When static forces act on a particle, the potential energy U can be defined. In this circumstance, the sum of kinetic and potential energies, T + U, is a constant called the total energy. If the force is axial, kinetic and potential energy are interchanged as the particle moves along the z axis, so that U = U(z). Setting the total time derivative of T + U equal to 0 and assuming $\partial U/\partial t = 0$ gives

$$m_o v_z (dv_z/dt) = -(\partial U/\partial z)(dz/dt). \tag{2.9}$$

The expression on the left-hand side equals $F_z v_z$. The static force and potential energy are related by

$$\boldsymbol{F_z} = -\partial U/\partial z, \quad \boldsymbol{F} = -\nabla U. \tag{2.10}$$

where the last expression is the general three-dimensional form written in terms of the vector gradient operator,

$$\nabla = \boldsymbol{u_x}\partial/\partial x + \boldsymbol{u_y}\partial/\partial y + \boldsymbol{u_z}\partial/\partial z. \tag{2.11}$$

The quantities $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$ are unit vectors along the Cartesian axes.

Potential energy is useful for treating electrostatic accelerators. Stationary particles at the source can be considered to have high U (potential for gaining energy). This is converted to kinetic energy as particles move through the acceleration column. If the potential function, U(x, y, z), is known, focusing and accelerating forces acting on particles can be calculated.

2.4 GALILEAN TRANSFORMATIONS

In describing physical processes, it is often useful to change the viewpoint to a frame of reference that moves with respect to an original frame. Two common frames of reference in accelerator theory are the stationary frame and the rest frame. The stationary frame is identified with the laboratory or accelerating structure. An observer in the rest frame moves at the average velocity of the beam particles; hence, the beam appears to be at rest. A coordinate transformation converts quantities measured in one frame to those that would be measured in another moving with velocity u. The transformation of the properties of a particle can be written symbolically as

$$(\boldsymbol{x},\boldsymbol{v},m,\boldsymbol{p},T) \Rightarrow (\boldsymbol{x}',\boldsymbol{v}',m',\boldsymbol{p}',T')$$

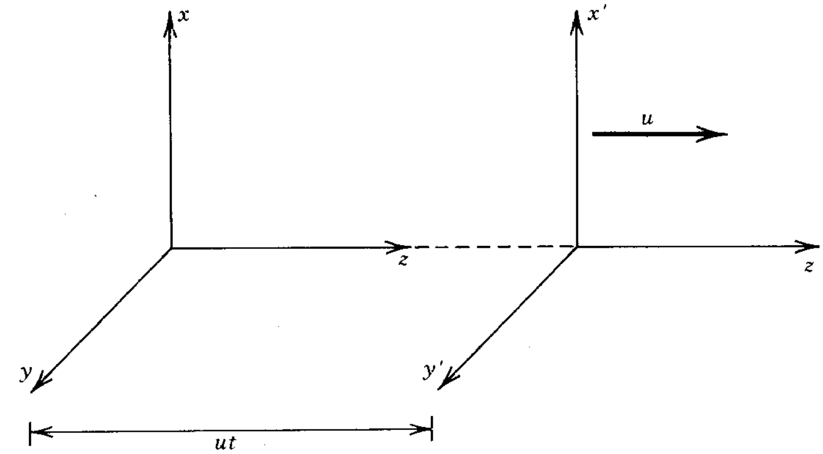

Figure 2.2. Galilean transformation between coordinate systems

where primed quantities are those measured in the moving frame. The operation that transforms quantities depends on **u**. If the transformation is from the stationary to the rest frame, **u** is the particle velocity **v**.

The transformations of Newtonian mechanics (Galilean transformations) are easily understood by inspecting Figure 2.2. Cartesian coordinate systems are defined so that the z axes are colinear with u and the coordinates are aligned at t = 0. This is consistent with the usual convention of taking the average beam velocity along the z axis. The position of a particle transforms as

$$x' = x, \quad y' = y, \quad z' = z - ut. \tag{2.12}$$

Newtonian mechanics assumes inherently that measurements of particle mass and time intervals in frames with constant relative motion are equal: m' = m and dt' = dt. This is not true in a relativistic description. Equations (2.12) combined with the assumption of invariant time intervals imply that *dx' = dx* and *dx'/dt' = dx/dt*. The velocity transformations are

$$v_x' = v_x, \quad v_y' = v_y, \quad v_z' = v_x, - u. \tag{2.13}$$

Since m' = m, momenta obey similar equations. The last expression shows that velocities are additive. The axial velocity in a third frame moving at velocity w with respect to the x' frame is related to the original quantity by $v_z'' = v_z - u - w$.

Equations (2.13) can be used to determine the transformation for kinetic energy,

$$T' = T + \tfrac{1}{2}m_o(-2uv_z + u^2). \tag{2.14}$$

Measured kinetic energy depends on the frame of reference. It can be either larger or smaller in a moving frame, depending on the orientation of the velocities. This dependence is an important factor in beam instabilities such as the two-stream instability.

2.5 POSTULATES OF RELATIVITV

The principles of special relativity proceed from two postulates:

1.The laws of mechanics and electromagnetism are identical in all inertial frames of reference.

2.Measurements of the velocity of light give the same value in all inertial frames.

Only the theory of special relativity need be used for the material of this book. General relativity incorporates the gravitational force, which is negligible in accelerator applications. The first postulate is straightforward; it states that observers in any *inertial frame* would derive the same laws of physics. An inertial frame is one that moves with constant velocity. A corollary is that it is impossible to determine an absolute velocity. Relative velocities can be measured, but there is no preferred frame of reference. The second postulate follows from the first. If the velocity of light were referenced to a universal stationary frame, tests could be devised to measure absolute velocity. Furthermore, since photons are the entities that carry the electromagnetic force, the laws of electromagnetism would depend on the absolute velocity of the frame in which they were derived. This means that the forms of the Maxwell equations and the results of electrodynamic experiments would differ in frames in relative motion. Relativistic mechanics, through postulate 2, leaves Maxwell's equations invariant under a coordinate transformation. Note that invariance does not mean that measurements of electric and magnetic fields will be the same in all frames. Rather, such measurements will always lead to the same governing equations.

The validity of the relativistic postulates is determined by their agreement with experimental measurements. A major implication is that no object can be induced to gain a measured velocity faster than that of light,

$$c = 2.998\times 10^{8} \ m/s. \qquad (2.15)$$

This result is verified by observations in electron accelerators. After electrons gain a kinetic energy above a few million electron volts, subsequent acceleration causes no increase in electron velocity, even into the multi-GeV range. The constant velocity of relativistic particles is important in synchronous accelerators, where an accelerating electromagnetic wave must be

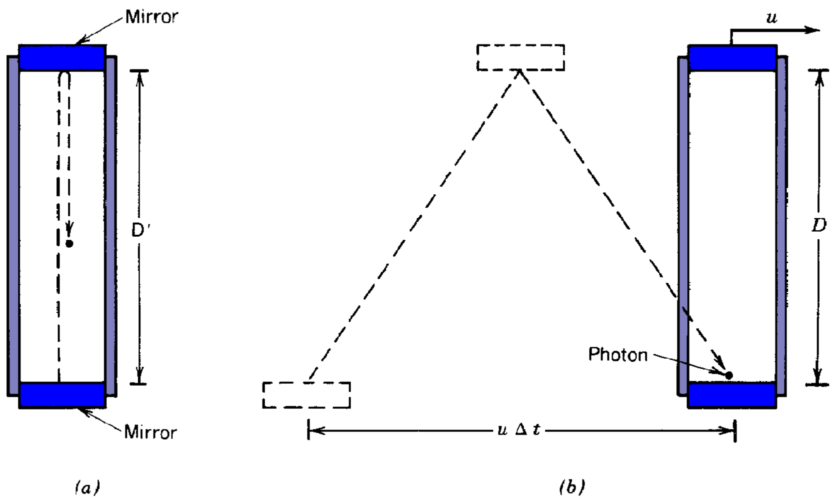

Figure 2.3 Effect of time dilation on the observed rates of a photon clock. (a) Clock rest frame. (b) Stationary frame.

matched to the motion of the particle.

2.6 TIME DILATION

In Newtonian mechanics, observers in relative motion measure the same time interval for an event (such as the decay of an unstable particle or the period of an atomic oscillation). This is not consistent with the relativistic postulates. The variation of observed time intervals (depending on the relative velocity) is called *time dilation*. The term *dilation* implies extending or spreading out.

The relationship between time intervals can be demonstrated by the clock shown in Figure 2.3, where double transits (back and forth) of a photon between mirrors with known spacing are measured. This test could actually be performed using a photon pulse in a mode-locked laser. In the rest frame (denoted by primed quantities), mirrors are separated by a distance D', and the photon has no motion along the z axis. The time interval in the clock rest frame is

$$\Delta t' = 2D'/c. \qquad (2.16)$$

If the same event is viewed by an observer moving past the clock at a velocity - u, the photon appears to follow the triangular path shown in Figure 2.3b. According to postulate 2, the photon still travels with velocity c but follows a longer path in a double transit. The distance traveled in the laboratory frame is

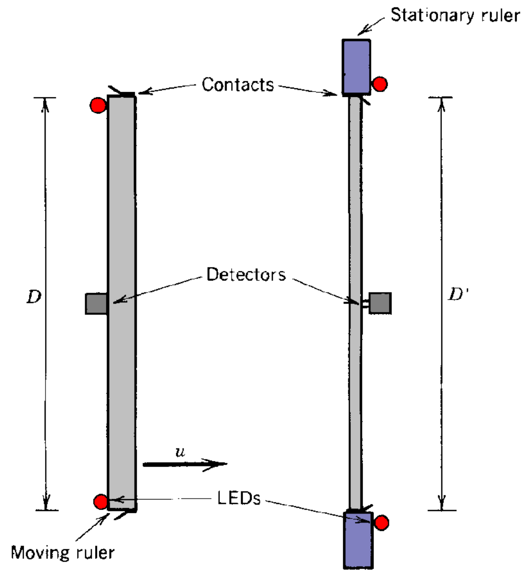

Figure 2.4. Experiment to demonstrate invariance of transverse lengths between frames in relative motion

$$c\Delta t = 2\left[D^2 + (u\Delta t/2)^2\right]^{1/2},$$

or

$$\Delta t = \frac{2D/c}{(1 - u^2/c^2)^{1/2}} . \qquad (2.17)$$

In order to compare time intervals, the relationship between mirror spacing in the stationary and rest frames (D and D') must be known. A test to demonstrate that these are equal is illustrated in Figure 2.4. Two scales have identical length when at rest. Electrical contacts at the ends allow comparisons of length when the scales have relative motion. Observers are stationed at thecenters of the scales. Since the transit times of electrical signals from the ends to the middle are equal in all frames, the observers agree that the ends are aligned simultaneously. Measured length may depend on the magnitude of the relative velocity, but it cannot depend on the direction since there is no preferred frame or orientation in space. Let one of the scales move; the observer in the scale rest frame sees no change of length. Assume, for the sake of argument, that the stationary observer measures that the moving scale has shortened in the transverse direction, D < D'. The situation is symmetric, so that the roles of stationary and rest frames can be interchanged. This leads to conflicting conclusions. Both observers feel that their clock is the same length but the other is shorter. The only way to resolve the conflict is to take D = D'. The key to the argument is that the observers agree on simultaneity of the comparison events (alignment of the ends). This is not true in tests to compare axial length, as discussed in the next section. Taking D = D', the relationship between time intervals is

$$\Delta t = \frac{\Delta t'}{(1-u^2/c^2)^{1/2}} . \tag{2.18}$$

Two dimensionless parameters are associated with objects moving with a velocity u in a stationary frame:

$$\beta = u/c, \quad \gamma = (1-u^2/c^2)^{-1/2}. \tag{2.19}$$

These parameters are related by

$$\gamma = (1-\beta^2)^{-1/2}, \tag{2.20}$$

$$\beta = (1-1/\gamma^2)^{1/2}. \tag{2.21}$$

A time interval Δt measured in a frame moving at velocity u with respect to an object is related to an interval measured 'in the rest frame of the object, Δt', by

$$\Delta t = \gamma \Delta t'. \tag{2.22}$$

For example, consider an energetic π^+ pion (rest energy 140 MeV) produced by the interaction of a high-energy proton beam from an accelerator with a target. If the pion moves at velocity 2.968×10^8 m/s in the stationary frame, it has a β value of 0.990 and a corresponding γ value of 8.9. The pion is unstable, with a decay time of 2.5×10^{-8} s at rest. Time dilation affects the decay time measured when the particle is in motion. Newtonian mechanics predicts that the average distance traveled from the target is only 7.5 in, while relativistic mechanics (in agreement with observation) predicts a decay length of 61 in for the high-energy particles.

2.7 LORENTZ CONTRACTION

Another familiar result from relativistic mechanics is that a measurement of the length of a moving object along the direction of its motion depends on its velocity. This phenomenon is

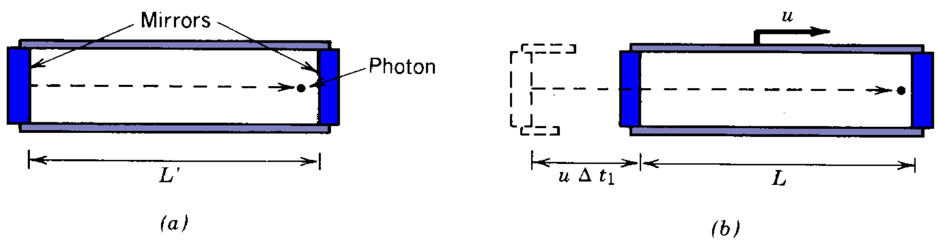

Figure 2.5 Lorentz contraction of a photon clock. (a) Clock rest frame. (bl) Stationary frame

known as Lorentz contraction. The effect can be demonstrated by considering the clock of Section 2.6 oriented as shown in Figure 2.5.

The detector on the clock measures the double transit time of light between the mirrors. Pulses are generated when a photon leaves and returns to the left-hand mirror. Measurement of the single transit time would require communicating the arrival time of the photon at the right-hand mirror to the timer at the left-hand mirror. Since the maximum speed with which this information can be conveyed is the speed of light, this is equivalent to a measurement of the double transit time. In the clock rest frame, the time interval is $\Delta t' = 2L'/c$.

To a stationary observer, the clock moves at velocity u. During the transit in which the photon leaves the timer, the right-hand mirror moves away. The photon travels a longer distance in the stationary frame before being reflected. Let Δt_1, be the time for the photon to travel from the left to right mirrors. During this time, the right-hand mirror moves a distance u At,. Thus,

$$c\Delta t_1 = (L + u\Delta t_1),$$

where L is the distance between mirrors measured in the stationary frame. Similarly, on the reverse transit, the left-hand mirror moves toward the photon. The time to complete this leg is

$$\Delta t_2 = (L - u\Delta t_2)/c.$$

The total time for the event in the stationary frame is

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{L}{c-u} + \frac{L}{c+u} ,$$

or

$$\Delta t = \frac{2L/c}{1-u^2/c^2} .$$

Time intervals cannot depend on the orientation of the clock, so that Eq. (2.22) holds. The above equations imply that

$$L = L'/\gamma. \tag{2.23}$$

Thus, a moving object appears to have a shorter length than the same object at rest.

The acceleration of electrons to multi-GeV energies in a linear accelerator provides an interesting example of a Lorentz contraction effect. Linear accelerators can maintain longitudinal accelerating gradients of, at most, a few megavolts per meter. Lengths on the kilometer scale are required to produce high-energy electrons. To a relativistic electron, the accelerator appears to be rushing by close to the speed of light. The accelerator therefore has a contracted apparent length of a few meters. The short length means that focusing lenses are often unnecessary in electron linear accelerators with low-current beams.

2.8 LORENTZ TRANSFORMATIONS

Charged particle orbits are characterized by position and velocity at a certain time, (**x**, **v**, t). In Newtonian mechanics, these quantities differ if measured in a frame moving with a relative velocity with respect to the frame of the first measurement. The relationship between quantities was summarized in the Galilean transformations.

The Lorentz transformations are the relativistic equivalents of the Galilean transformations. In the same manner as Section 2.4, the relative velocity of frames is taken in the z direction and the z and z' axes are colinear. Time is measured from the instant that the two coordinate systems are aligned (z = z' = 0 at t = t' = 0). The equations relating position and time measured in one frame (unprimed quantities) to those measured in another frame moving with velocity u (primed quantities) are

$$x' = x, \tag{2.24}$$

$$y' = y, \tag{2.25}$$

$$z' = \frac{z-ut}{(1-u^2/c^2)^{1/2}} = \gamma(z-ut), \tag{2.26}$$

$$t' = \frac{t-uz/c^2}{(1-u^2/c^2)^{1/2}} = \gamma\left(t-\frac{uz}{c^2}\right). \tag{2.27}$$

Particle Dynamics

The primed frame is not necessarily the rest frame of a particle. One major difference between the Galilean and Lorentz transformations is the presence of the γ factor. Furthermore, measurements of time intervals are different in frames in relative motion. Observers in both frames may agree to set their clocks at t = t' = 0 (when z = z' = 0), but they will disagree on the subsequent passage of time [Eq. (2.27)]. This also implies that events at different locations in z that appear to be simultaneous in one frame of reference may not be simultaneous in another.

Equations (2.24)-(2.27) may be used to derive transformation laws for particle velocities. The differentials of primed quantities are

$$dx' = dx, \quad dy' = dy, \quad dz' = \gamma(dz - u dt),$$

$$dt' = \gamma dt\ (1 - u v_z / c^2).$$

In the special case where a particle has only a longitudinal velocity equal to u, the particle is at rest in the primed frame. For this condition, time dilation and Lorentz contraction proceed directly from the above equations.

Velocity in the primed frame is dx'/dt'. Substituting from above,

$$v_x' = \frac{v_x}{\gamma\ (1 - u v_z / c^2)} \ . \tag{2.28}$$

When a particle has no longitudinal motion in the primed frame (i.e., the primed frame is the rest frame and $v_z = u$), the transformation of transverse velocity is

$$v_x' = \gamma\ v_x. \tag{2.29}$$

This result follows directly from time dilation. Transverse distances are the same in both frames, but time intervals are longer in the stationary frame.

The transformation of axial particle velocities can be found by substitution for dz' and dt',

$$\frac{dz'}{dt'} = \frac{\gamma dt\ (dz/dt - u)}{\gamma dt\ (1 - u v_z/c^2)} \ ,$$

or

$$v_z' = \frac{v_z - u}{1 - u v_z/c^2} \ . \tag{2.30}$$

This can be inverted to give

$$v_z = \frac{v_z^{/} + u}{1 + u v_z^{/}/c^2} \; . \tag{2.31}$$

Equation (2.31) is the relativistic velocity addition law. If a particle has a velocity v_z' in the primed frame, then Eq. (2.31) gives observed particle velocity in a frame moving at *-u*. For v_z' approaching c, inspection of Eq. (2.31) shows that v_z also approaches c. The implication is that there is no frame of reference in which a particle is observed to move faster than the velocity of light. A corollary is that no matter how much a particle's kinetic energy is increased, it will never be observed to move faster than c. This property has important implications in particle acceleration. For example, departures from the Newtoniain velocity addition law set a limit on the maximum energy available from cyclotrons. In high-power, multi-MeV electron extractors, saturation of electron velocity is an important factor in determining current propagation limits.

2.9 RELATIVISTIC FORMULAS

The motion of high-energy particles must be described by relativistic laws of motion. Force is related to momentum by the same equation used in Newtonian mechanics

$$d\mathbf{p}/dt = \mathbf{F}. \tag{2.32}$$

This equation is consistent with the Lorentz transformations if the momentum is defined as

$$\mathbf{p} = \gamma m_o \mathbf{v}. \tag{2.33}$$

The difference from the Newtonian expression is the γ factor. It is determined by the total particle velocity v observed in the stationary frame, $\gamma = (1-v^2/c^2)^{-½}$. One interpretation of Eq. (2.33) is that a particle's effective mass increases as it approaches the speed of light. The relativistic mass is related to the rest mass by

$$m = \gamma m_o. \tag{2.34}$$

The relativistic mass grows without limit as v_z approaches c. Thus, the momentum increases although there is a negligible increase in velocity.

In order to maintain Eq. (2.6), relating changes of energy to movement under the influence of a force, particle energy must be defined as

$$E = \gamma m_o c^2. \tag{2.35}$$

The energy is not zero for a stationary particle, but approaches $m_o c^2$, which is called the rest energy. The kinetic energy (the portion of energy associated with motion) is given by

$$T = E - m_o c^2 = m_o c^2(\gamma - 1). \tag{2.36}$$

Two useful relationships proceed directly from Eqs. (2.20). (2.33), and (2.35):

$$E = \sqrt{c^2 p^2 + m_o^2 c^4}, \tag{2.37}$$

where $p^2 = \mathbf{p} \cdot \mathbf{p}$, and

$$\mathbf{v} = c^2 \mathbf{p}/E. \tag{2.38}$$

The significance of the rest energy and the region of validity of Newtonian mechanics is clarified by expanding Eq. (2.35) in limit that $v/c \ll 1$.

$$E = \frac{m_o c^2}{\sqrt{1 - v^2/c^2}} = m_o c^2 \; (1 + v^2/c^2 + ...). \tag{2.39}$$

The Newtonian expression for T [Eq. (2.8)] is recovered in the second term. The first term is a constant component of the total energy, which does not affect Newtonian dynamics. Relativistic expressions must be used when $T \geq m_o$c2. The rest energy plays an important role in relativistic mechanics.

Rest energy is usually given in units of electron volts. Electrons are relativistic when T is in the MeV range, while ions (with a much larger mass) have rest energies in the GeV range. Figure 2.6 plots β for particles of interest for accelerator applications as a function of kinetic energy. The Newtonian result is also shown. The graph shows saturation of velocity at high energy and the energy range where departures from Newtonian results are significant.

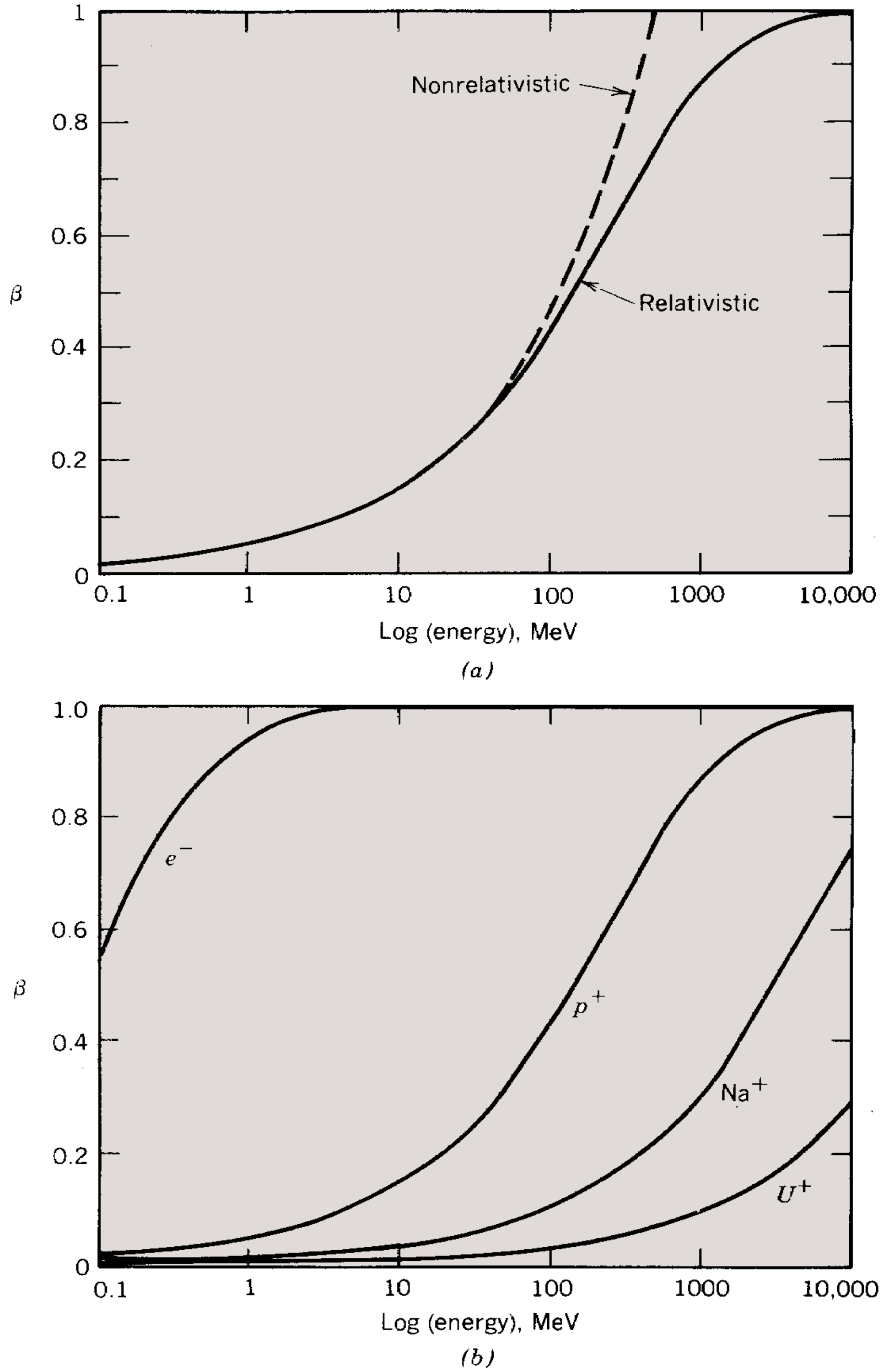

Figure 2.6. Particle velocity normalized to the speed of light as a function of kinetic energy. (a) Protons: solid line, relativistic predicted, dashed line, Newtonian predicition. (b) Relativistic predictions for various particles.

2.10 NONRELATIVISTIC APPROXIMATION FOR TRANSVERSE MOTION

A relativistically correct description of particle motion is usually more difficult to formulate and solve than one involving Newtonian equations. In the study of the transverse motions of charged particle beams, it is often possible to express the problem in the form of Newtonian equations with the rest mass replaced by the relativistic mass. This approximation is valid when the beam is well directed so that transverse velocity components are small compared to the axial velocity of beam particles. Consider the effect of focusing forces applied in the x direction to confine particles along the z axis. Particles make small angles with this axis, so that v_x is always small compared to v_z. With $\mathbf{F} = \mathbf{u}_x F_x$, Eq. (2.32) can be written in the form

$$\gamma m_o v_x \left(\frac{d\gamma/dt}{\gamma} + \frac{dv_x/dt}{v_x} \right) = F_x. \tag{2.40}$$

Equation (2.39) can be rewritten as

$$E = m_o c^2 \; [1 + (v_z^2 + v_x^2)/2c^2 + 3(v_z^2 + v_x^2)^2/8c^4 + \ldots].$$

When $v_x \ll v_z$, relative changes in γ resulting from the transverse motion are small. In Eq. (2.40), the first term in parenthesis is much less than the second, so that the equation of motion is approximately

$$\gamma m_o \frac{dv_x}{dt} = F_x. \tag{2.41}$$

This has the form of a Newtonian expression with m_o replaced by γm_o.