COMMENTARIES ON METHODS TO ESTIMATE GROUND VIBRATIONS ELICITED BY FAST MOVING LOADS

EDUARDO KAUSEL
Professor of Civil and Environmental Engineering
Massachusetts Institute of Technology
Cambridge, Massachusetts – USA

ABSTRACT
This paper presents a brief survey and some general comments on the numerical methods available for the estimation of ground borne vibrations elicited by fast moving loads.

1. INTRODUCTION
Among the early pioneers who exerted a profound influence on the theory of moving dynamic sources are Kelvin, Stokes, and Lamb, who in the course of the 19th and early part of the 20th century provided the theoretical framework and fundamental solutions, or Green’s functions, for full spaces and half-spaces that now lie at the heart of the solutions for travelling loads.

The classical problem of a point load moving along a straight line within a homogeneous, infinite full space was considered, among others, by Eason et al (1956) [1] and by Payton (1964) [2], and their solutions can also be found in the treatise on elastodynamics by Eringen and Suhubi (1975) [3]. Extensions to the elastic half-space (Cole and Huth, 1958 [4]; Eason, 1965 [5]; Gakenheimer and Miklowitz, 1969 [6]) and to the layered half-space followed soon thereafter (De Barros and Luco, 1994 [7]), so the problem is now well understood, at least conceptually. Still, some important computational issues remain as yet unresolved, even in the case of linear problems.

By and large, there exist three classes of methods to address the problem of moving loads, which are: Analytical methods; Semi-numerical (semi-analytical) methods; and Numerical or purely discrete methods. We shall consider these briefly in the following.
2. ANALYTICAL METHODS

Strictly speaking, analytical methods can be applied solely to cases of very simple geometric and material conditions, such as homogeneous full spaces and half-spaces subjected to line loads and point loads of constant amplitude and everlasting duration, which move at constant speed along a straight line on the surface of the medium. While the mathematical formalism based on the superposition of Green’s functions used in these cases can also be extended to more complex conditions, such as layered soils, the resulting integral expressions are invariably of such complexity that they can be evaluated only by numerical means, and the required computational effort is usually formidable.

For this method to be useful, it is necessary that the response functions anywhere in the medium caused by a stationary source of either impulsive or harmonic variation in time be readily available. In other words, one must know how to obtain either the impulse response functions for a non-moving, impulsive point source (or line source) anywhere in the layered medium, or alternatively, the frequency-response functions for a harmonic load. Having these functions, it is then possible to construct the solution by superposition of elementary impulses throughout the line of action of the moving load.

Omitting for simplicity both the direction of the source as well as the direction of the response at the receiver, let \( g(x,y,z,z',t) \) be a generic impulse response function of the medium, observed at any arbitrary elevation \( z \), for an impulsive, stationary point load \( \delta(x)\delta(y)\delta(z-z')\delta(t) \) acting at elevation \( z' \). For further notational simplicity, we assume also that the receiver is in the vertical plane of the load, and we also choose to omit the vertical elevations \( z' \) of both the source and of the receiver in the arguments of \( g \), and write simply \( g(x,t) \).

Consider a rather general type of time-varying source \( p(x,t) \) that acts along a horizontal line of action \( y = 0, z = z' \), usually at the free surface, i.e. \( z' = 0 \). Applying the principle of superposition, the generic response at the receiver can be expressed formally as

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\xi,\tau) g(x-\xi, t-\tau) d\xi d\tau
\]

Sources that are also distributed in the third dimension \( y \) can be obtained by an appropriate additional superposition along that coordinate direction. Carrying out Hankel and/or Fourier transforms in space-time as may be appropriate, one can also express the displacements in the frequency-wavenumber domain, and obtain the response in space time from inverse transforms that must be evaluated numerically.

Of the many possible types of moving loads, which may be local in both space and time, in most typical cases it is assumed that the point load (or line load) exists forever and begins in the remote past at an infinite distance to the left, passes through the origin at \( t = 0 \), and then marches on to the right with constant amplitude and velocity. Such a load is of the form \( p(\xi,\tau) = \delta(\xi-V\tau) \), in which case the above double integral simplifies to the single integral
\begin{equation}
    u(x,t) = \int_{-\infty}^{\infty} g(x-V\tau, t-\tau) d\xi \ d\tau
\end{equation}

After evaluation, it will be found that the motion pattern very much depends on the speed of propagation, and that it changes characteristics once the source speed \( V \) exceeds the material wave velocities in the medium. At subsonic speeds, the waves outrun the load, while at supersonic speeds, a shock wave trailing behind the load with its characteristic Mach cone can be observed. It is unlikely, however, that supersonic waves should be elicited by fast moving trains because even at 360 km/hr, a train moves at 100 m/s, which is less than the typical material wave velocities in the ground.

From the preceding we see that the generic response is obtained in terms of the generic Green’s function, which must thus be “known”. It can also readily be shown that the response is quasi-static in the sense that the displacement pattern appears to be invariant to an observer that moves with the same speed and in the same direction as the load. Now, even in the case of a homogeneous elastic half-space subjected to an impulsive source at the surface, these functions are available only in a few cases, see Eringen and Suhubi [3] or Kausel [8]:

- **Lamb-Pekeris-Mooney Problem**: Vertical point load at the surface, vertical displacements in space-time observed at the surface for any Poisson’s ratio, but horizontal (radial) displacements only for Poisson’s ratio \( \nu \leq 0.2631 \) (this is the value at which the false roots of the Rayleigh function turn complex). Solutions for harmonic loads exist only in terms of integral transforms that must be evaluated numerically.
- **Chao’s Problem**: Horizontal point load at the surface, solution for horizontal displacements at the surface available only for \( \nu = 0.25 \), and vertical displacements for \( \nu \leq 0.2631 \) (obtained from the vertical load case via the reciprocity principle). Again, solutions for harmonic loads exist only in terms of integral transforms.

Closed-form solutions also exist for line loads (either of the in-plane or anti-plane variety), but the solutions to moving line loads break down for sub-sonic speeds of the loads because the 2-D half-space has no static stiffness, in which case the displacements associated with an everlasting line load are infinitely large.

Other than those listed above, no other closed-form solutions exist at present, although in some cases solutions may be obtained by numerical means as discussed in the next section. Still, the idealized cases listed above could be used as yardsticks against which one can judge the correctness and quality of semi-numerical and/or discrete implementations. This is so, because the canonical solutions have the virtue of being exact, so they are valid even at high frequencies.

3. SEMI-ANALYTICAL (SEMI-NUMERICAL) METHODS

A first strategy to deal with more general types of sources is to rely on the boundary element method (BEM), which requires the surface to be discretized, inasmuch as this method uses the Green’s functions for a full space. Still, the BEM is highly effective because the fundamental
solutions have a very simple structure and are known in both the time and frequency domains. However, it generally breaks down when dealing with layered soils or inhomogeneous media, because modeling the transitions in material properties demands a substantial increase in boundary nodes.

The BEM can also be generalized to cases where the geometry is two-dimensional and the load propagates in the third dimension along which material and geometric properties remain constant. This allows, for example, modeling a free surface that contains two rails at a finite distance apart. The method is based on the Green’s functions for the so-called 2 ½ D problem, which constructs the full solution by numerical integration over axial wavenumbers (Tadeu and Kausel, 2000 [9]). Because of the need to evaluate numerically this integral, this option requires more effort than a conventional BEM, but it can still be quite effective as long as one deals with homogeneous half-spaces. Still, for loads that move at constant speed, this integral can be dispensed with.

On the other hand, the formalism of the previous section can readily be extended to layered media by means of integral transforms into the frequency-wavenumber domain. In virtually every case, however, the requisite inverse transforms into space-time can only be accomplished numerically, and this is an operation that is fraught with difficulties. This is so, because in the presence of layering, the kernels of the transforms are highly wavy, and especially so when considering the high-frequency problems that are typical of fast moving loads.

A highly effective way to sidestep the difficulties in obtaining the Green’s functions for sources anywhere in layered media consists in using the Thin Layer Method (TLM), which combines analytical solutions in the horizontal direction with a finite element type of solution in the direction of layering e.g. Kausel [10 → 16]. Because the TLM is based on the principle of superposition, it is restricted to linear materials, although the damping (attenuation) could exhibit any variation with frequency. For layered soils of finite depth, the TLM is available in both space-frequency [10,11] and space-time [14]. Straightforward extensions to layered half-spaces that rely on approximations known as paraxial boundaries exist as well, but only for a formulation in the frequency-domain (Hull and Kausel, 1984 [17]). For time domain formulations, layered half-spaces can be dealt with by using a more elaborate yet rigorous algorithm, see Park and Kausel (2006 [18,19,20]). Being a semi-discrete method, the TLM requires a sufficiently fine discretization (i.e. layering) to be able to transmit waves vertically, and especially so at high frequencies and/or near the location of the source. Nonetheless, the computational effort is typically substantially smaller than that required by a full-fledged solution with conventional, three-dimensional finite elements, and it avoids also the artificial reflections at the lateral boundaries of the finite element model.

4. NUMERICAL SOLUTIONS

A fairly common strategy these days is to formulate the problem of moving loads by means of solid, finite elements in three dimensions, ideally using also absorbing boundaries of some kind, such as the paraxial boundaries referred to earlier, or the rather effective transition elements obtained using the Perfectly Matched Layer technique (PML). Finite elements offer
the added advantage that one can also model non-linear effects near the load, such as those within the ballast that forms the rail bed. If so, what could be at fault with this approach?

In this time and age of fast personal computers, it would seem that we could solve without much ado and in exquisite detail even enormous models of soil-structure systems (Bielak et al, 2003 [21]). This is largely true as long as we deal with two-dimensional or axisymmetric systems subjected to linear soil behavior. However, in the case of large three-dimensional geometries, and even if we assume the models to remain fully elastic, we quickly encounter significant practical difficulties and barriers. Conceptually, it is not that we do not know how to solve these problems—it would seem that we just need to add enough finite elements—but that the associated numerical models exhibit polynomial growth, and thus remain computationally infeasible, unless we take substantial short-cuts, such as allowing coarse elements, restricting our models to low frequencies, tolerating boundary artifacts, not to mention neglecting the inherent non-linearity of soils near the loads. We refer to this problem as the intrinsic numerical complexity. For example, fast trains surely elicit vibrations of significant amplitude well beyond 100 Hz. In a soil whose shear wave velocity near the surface is 150 m/s, this frequency would imply the transmission of waves with wavelengths on the order of 1.5m, and with six to ten finite elements required per wavelength, this would demand solid elements no larger than some 20cm on each side. To be useful, the complete model would have to be several tens of meters long, wide and deep, which would entail millions of degrees of freedom. In most situations, such large models are simply untenable.

A second problem, which we can refer to as intrinsic material complexity, has to do with the fact that, no matter how much effort and expense we should devote to that effect, we can never really know in glorious detail the mechanical characteristics of the soil at small and large strains, in loading and unloading, either over a wide region or at some specific fixed location near the rail bed—not to mention that the rail extends for hundreds of kilometers. For example, the in-situ non-linear characteristics of soils in both loading and unloading cannot be measured unambiguously from either lab or field experiments—or even with both methods combined—simply because it is impossible for us to extract soil samples without seriously disrupting the soils’ fabric and thus alter their constitutive properties. Indeed, the very concept of “undisturbed soil sample” is an oxymoron. While freezing and other modern soil extraction strategies may be helpful in obtaining samples from which orders of magnitude of the material parameters at large strains can be inferred, the results thus obtained have much uncertainty, especially from the point of view of wave propagation. Then again, the very act of excavating the soil and placing a structure at that location elicits additional disruptions of the soil, and seasonal variations caused by either ground freezing or depth of the phreatic surface (water table) could have substantial effects on guided waves and on wave amplification near the surface. For that matter, we never really know the state of stresses—especially lateral stresses—that exists in-situ either before or after the structure is in place, all of which places intrinsic limitations on how accurately we can model inelastic behavior with advanced numerical models. It is also clear that we can never test the soil throughout the entire vicinity of a structure, so our models can never fully account for spatial inhomogeneities such as lenses of sand or clay, boulders, undulating or dipping layers, and the like. Thus most practical problems exhibit irreducible material complexity for which an “exact” answer does not exist. Of course, this does not prevent us from actually estimating ground vibrations using models based on simplifying assumptions, including linear models with imperfectly small models that
are made with coarse elements. The point here is that one cannot construct a three-dimensional
finite element representation of this problem and then delude oneself into believing that the
solution thus obtained is rigorous.

5. CONCLUSIONS

In the past four decades or so, we have gained enormously in our ability to model analytically
and numerically the complex vibration problems that relate to the interaction of soils and
structures, and to predict and measure train-induced vibrations in the field. Despite—or
perhaps because—of these successes, we must learn to temper the temptation to use models
whose complexity exceeds the knowable and the measurable. Inappropriate sophisticated tools
will always give very precise answers, but to the wrong problem.

REFERENCES

Mech., 25, 433-436
force”, Int. J. of Engineering Science, 2, 581-609
by a point load traveling on the surface”, J. Appl. Mech., 36, 505-515
a moving point load”, Wave Motion, 19, 189-210
Cambridge University Press, Cambridge, UK
R74-11, Soils Publication No. 336, Department of Civil Engineering.
An Explicit Solution”, Bulletin of the Seismological Society of America, 72, (5), 1459-
12481. See also Errata in BSSA 79 (4), p. 1508.
Research Report R82-50, Department of Civil Engineering.
Journal for Numerical Methods in Engineering, 37, 927-941
Chapter V in Wave Motion in Earthquake Engineering, WIT Press.


