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THEORETICAL CONSIDERATIONS ON THE SMITH-CUNDALL-ET AL BOUNDARY

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Introduction

In 1979, Smith\(^4\) proposed a simple method to eliminate -at least partially - the elastic wave reflections (or echoes) that develop at the boundaries of finite element representations of infinite media, which is based on the principle of virtual images. The procedure entails computing the dynamic solution with at least two different boundary conditions: first, with normal stresses and tangential displacements at the boundaries equal to zero, then with the tangential stresses and normal displacements at the boundaries equal to zero. These two solutions are then averaged. In the case of a homogeneous solid space with a single plane boundary, the procedure would completely eliminate these reflections, as can be demonstrated by simple symmetry-antisymmetry considerations. In the presence of more than one boundary interface, however, this procedure generally fails because the waves are reflected more than once. In the virtual image analogy, this difficulty is manifested by the appearance of many, or indeed infinitely many virtual images. In Smith's paper\(^4\), as well as those that make reference to it, the statement is made that an exact solution for this case would require \(2^n\) independent solutions to the problem, where \(n\) is the number of reflections (virtual images!) that can occur. Since in most practical cases this number is infinitely large (as for example in the case of two parallel boundaries or "mirrors"), the procedure would fail and lose its computational appeal.

More recently, Cundall, Kunar, Carpenter and Martí\(^1\), and Kunar and Martí\(^3\) proposed a simple and ingenious solution to overcome the problem of multiple reflections: two overlapping boundary regions with complementary boundary conditions are coupled to the finite element regions, and the reflected waves are "cancelled as they occur", averaging the solutions at every four time-steps. Heuristic concepts were advanced by these authors to justify this scheme, but no rigorous proof

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was presented, either to demonstrate the adequacy of the method, to rationalize its practical implementation ("constant velocity" and "constant stress" at the boundary during four time steps, etc.), or to explain the origin of the "numerical shock" that may develop under certain circumstances. Such a theoretical analysis was taken up by Durón (2), who investigated the mathematical basis of the method. A brief description of his findings is presented in the following, while a detailed report will be presented elsewhere in an expanded version of this paper.

Generalized Smith-Cundall et al. Boundary

Due to space considerations, only a brief description will be presented of a generalization of this boundary for the one-dimensional case. Nevertheless, it is straightforward to extend the formulation to two- and three-dimensional cases. (Durón, 1982). With reference to Figure 1, a semi-infinite rod is subjected to a pulse \( f(t - x/c) \) which travels with speed \( c \) towards the boundary, placed at the origin of coordinates. This pulse splits into two identical pulses traveling along each boundary branch (no reflections occur at \( C \), because each of the two branches have \( 1 \) of the stiffness and mass of the rod). The boundary conditions at branch points A and B are:

a) Displacements are prescribed at point A (Dirichlet boundary)
b) Stresses are prescribed at point B (Neumann boundary)

Numerous options are available for the two conditions (a) and (b). For example, if the displacements and stresses vanish at the boundary, then we have the classical fixed and free boundaries. In Cundall et al's option, the displacements change linearly with time over 4 intervals (constant velocity), while the stresses are held constant during this time. More generally, the prescribed values can be any arbitrary functions of time, which should be optimized in some sense.

Let \( u(x,t) \) and \( v(x,t) \) represent the displacements of branches A and B for times prior to the arrival of higher order reflections (which occur at \( C \) when primary reflections with opposite phase meet there). These displacements are then

\[
\begin{align*}
u(t,x) &= f(t - \frac{x}{c}) - f(t + \frac{x}{c}) + h(t + \frac{x}{c}) \\
v(t,x) &= f(t - \frac{x}{c}) + f(t + \frac{x}{c}) - g(t + \frac{x}{c})
\end{align*}
\]

The first term represents the incident wave, the second is the primary reflection, and \( h \) and \( g \) are the waves generated by the enforcement of non-zero displacements \( u(t) \) and stresses \( p(t) \) at the boundary points A and B, respectively. At the boundary \( (x = 0) \), we would have

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\[ u = h(t), \ v = 2f - g, \ (p \sim \gamma) \]

It is easy to show that if one imposes at A the displacement \( v(t) \)
observed at B, and the reaction \( r \) at A is imposed as external force \( p \)
on B, then \( h = g = f \), and the reflection cancels completely. However,
such a scheme is not practical since it would require the instantaneous
knowledge of information not readily available at the time of the com-
putation. All one can do is approximate these quantities (to some
arbitrary degree of accuracy) by the use of forward differences, (not
all of them leading to stable schemes). For example, one could use
(with \( j \) referring to the discrete time steps)

\[ u_j = (\alpha_j u_{j-1} + \alpha_{j-2} u_{j-2} + \ldots) + (\beta_j - j v_{j-1} + \gamma_j 2v_{j-2} + \ldots) \]

\[ p_j = (\alpha_j p_{j-1} + \alpha_{j-2} p_{j-2} + \ldots) + (\beta_j - j r_{j-1} + \gamma_j 2r_{j-2} + \ldots) \]

By an appropriate choice of these coefficients, as for example

\[ u_j = \frac{1}{2} (2u_{j-1} - u_{j-2}) + \frac{1}{2} (2v_{j-1} - v_{j-2}) \]

\[ p_j = \frac{1}{2} (2p_{j-1} - p_{j-2}) + \frac{1}{2} (2r_{j-1} - r_{j-2}) \]

one can achieve an approximation

\[ h_j = f_j + e_j, \ g_j = f_j + e_j \]

where \( e_j \) is an error function which can be made arbitrarily small.
Surprisingly, it can be shown that this error function carries all the
information necessary to regenerate the higher order reflection of \( f \) in
combination with the active(?) boundary, which supplies the necessary
energy. In other words, no matter how small the amplitudes of \( e_j \), a
wave \( f \) is "generated" at the boundaries A, B after a time \( t = 2\lambda/c_j \)
equal to the travel time of \( e_j \) from A and B to C, and back: The
Phoenix bird reborn from its ashes! As it turns out, it is indeed
possible to "kill the bird" by averaging the solutions of branches A
and B every few time steps, since the phases of \( e_j \) in each branch can
be shown to have opposite signs. The averaging cannot be done every

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time step, however, because this would render the boundaries ineffective. An upper bound on the time delay between averagings is the travel time between the boundaries and the branch point C. With a central difference explicit integration scheme, the delay suggested by Cundall et al. of four time steps between averaging seems to work well.

References


