ON THE ACCURACY OF SOME APPROXIMATE ANTIPLANE
HALF-SPACE STIFFNESSES

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ABSTRACT

The modeling with discrete methods of elastic media of infinite extent that are subjected to dynamic loads normally calls for the use of special transmitting (or nonreflecting) boundaries. One such discrete method is the thin layer method, which allows efficient computation of the Green's functions for layered soils of finite depth; its application to elastic half-spaces, however, requires that the infinite medium be represented by means of approximations that are analogous to transmitting boundaries. In this article, we explore the accuracy of two of these approximations in the context of the Green's functions for antiplane (or SH) line loads. We find that the paraxial approximation of Engquist–Majda gives good results, provided that a "buffer layer" with the same material properties as the half-space separates the computational domain from the transmitting boundary. While these results were studied from the point of view of the thin layer method, they apply equally well to models with finite elements or finite differences.

INTRODUCTION

The Green's functions for a layered elastic body are essential solutions for soil dynamics and soil-structure interaction problems and are also important for understanding earthquake source mechanisms. In seismology, these functions lie at the heart of mathematical formulations of seismic dislocations and are essential for computing the motions that occur after the ground ruptures. In general, these functions are evaluated by the Haskell–Thomson Method (Haskell, 1953; Harkrider, 1964), in which the transfer matrices for displacements and stresses at the interfaces of each layer are established in the frequency-wavenumber domain, and free surface conditions, rigid base or radiation conditions, and applied force conditions are prescribed. Finally, the Green's functions in the frequency-space domain are constructed by Fourier or Fourier–Bessel inverse transformations in space.

These transfer matrices have as parameters the horizontal wavenumber \( k \) and frequency \( \omega \). The Green's functions are expressed as an improper integral over wavenumbers, so that an infinite path integral with singularities in the integrand must be carried out along the wavenumber axis. This integral must be evaluated by numerical methods, which entail a large number of operations to obtain accurately the Green's functions. The principal difficulties encountered in the numerical integration are (a) the fluctuations and singularities of the integrand and (b) the error associated with the truncation of the infinite integration path; these errors are discussed by Xu and Mal (1985, 1987).

Another method of constructing the Green's functions is the thin layer method. In this approach, the layered elastic body is modeled by a sequence of thin layers that are characterized mathematically by appropriate stiffness matrices (Kausel, 1981; Kausel and Peek, 1982). The thin layer method approach is very attractive for the layered elastic body with fixed base (rigid rock),
because it does not require numerical integration over wavenumber. The reason for this is that the stiffness matrices for the layers, which are obtained by applying the principle of virtual work to the wave equations while using interpolation functions for the displacement across the thicknesses of the layers, are algebraic in the wavenumber. This implies that the integral transformation can be carried out in closed form, as shown in references (Kausel, 1981; Kausel and Peek, 1982). The advantage of avoiding numerical integration, however, is gained at the expense of having to compute an eigenvalue problem so as to be able to carry out a similarity transformation. This eigenvalue problem requires in turn that the form of the global stiffness matrix be polynomial in the wavenumber. Hence, to apply this method to a layered elastic body with a flexible half-space underneath, it is necessary to represent the half-space’s true impedance (stiffness) by means of polynomial approximations that match the structure of the stiffness matrices for the layers. In this article, we shall explore the accuracy and applicability to the Green’s function problem of two approximations for the half-space stiffness, namely the paraxial approximation (Engquist and Majda, 1977; Seale, 1985) and the doubly asymptotic approximation (Mathews and Geers, 1987). Briefly, a paraxial approximation is obtained by expanding the exact stiffness function in Taylor series in the wavenumber, while a doubly asymptotic approximation is obtained by combining the low- and high-frequency limits of the impedance function, as will be shown.

The use of paraxial approximations to represent elastic half-spaces had been proposed earlier (Engquist and Majda, 1977; Seale, 1985), but no indications on their accuracy and applicability have been given. Since the paraxial half-space stiffnesses are similar in form to the layer stiffness matrices in the thin layer method, they can be incorporated into the global stiffness matrix without affecting the solvability of the eigenvalue problem. It can be shown that when paraxial approximations are used for plane-strain problems, severe limitations are encountered for certain values of Poisson’s ratio, which are related to generations of negative energy, that is, to a supply rather than dissipation of energy. This problem occurs because the imaginary part of the approximate stiffness is not a positive definite function of the wavenumber. Thus, a paraxial approximation alone is not sufficient to represent a half-space. It can be shown that improvements can be achieved by means of overlying buffer layers (Seale, 1985). However, neither the necessity of implementing buffer layers had been recognized before nor have appropriate thicknesses for the overlying layers been evaluated. Thus, this issue is taken up in this article.

Another alternative to avoid the problem of negative energies is to use the doubly asymptotic approximation referred to earlier. This approximation is composed of two terms: One is the asymptotic value of the exact stiffness at \( \omega = 0 \), and the other is the asymptotic value at \( \omega = \infty \). This approximation is expected to work well at very low frequencies and at very high frequencies (equivalently, low horizontal wavenumbers). However, the applicability with regard to moderate frequencies must be examined.

The objective of this article is to examine the accuracy and applicability to the Green’s function problem of the paraxial and doubly asymptotic approximations of the antiplane half-space stiffness. This article is composed of three parts, namely (a) construction of the approximate half-space stiffnesses, (b) their characteristics, and (c) their effects on the accuracy with which surface displacements due to surface line loads can be computed.
In the first part, the approximate stiffnesses are defined. In the second part, they are examined with respect to reflection coefficients and to average energy dissipated in one cycle of motion; also, the possibility of improving the approximate stiffnesses by overlaying a buffer layer is investigated. In the third part, surface displacements due to surface line loads are computed for a half-space and for a one-layered half-space using both the exact half-space stiffness and the approximate one.

In this study, it is found that approximate half-space stiffnesses should be used with a buffer layer having the same material properties as the half-space. A relatively thin buffer layer can minimize the shortcomings of the approximate stiffnesses in the region of evanescent waves, e.g., energy supply by the paraxial approximation and inappropriate reflection coefficients in both approximations. The doubly asymptotic approximation behaves well in the region of evanescent waves and can be used without buffer layer when a load of very low- or very high-frequency is applied or a very stiff half-space exists underneath the layers.

**Solutions for Layered System**

Consider an elastic system of parallel layers over a homogeneous elastic half-space as shown in Figure 1. The Cartesian coordinate system is defined as indicated in this figure, with the vertical axis positive up. This system is subjected to an antiplane, harmonic line source at the origin. Each layer is identified by the subindex \( j \), while the half-space is denoted by the index \( n \) (where \( n - 1 \) is the number of layers).

Each layer in this system is characterized by the well-known acoustic (antiplane shear) wave equation; the system must satisfy continuity conditions on stresses and displacements across the layer interfaces, free stress conditions on the surface, and radiation conditions at minus infinity (\( z = -\infty \)).

The solution of this system of partial differential equations is usually solved in the frequency-wavenumber domain by the transfer matrix method of

![Figure 1. A model and a coordinate system for the antiplane problem.](image)
Haskell-Thomson (Haskell, 1953; Harkrider, 1964) or the stiffness matrix method of Kausel-Roesset (1981). Following the latter method, it can be shown that each layer has a stiffness matrix

\[ \mathbf{K}_j^{ex} = \frac{k_{sj} G_j^c}{\sin h(k_{sj} H_j)} \begin{bmatrix} \cos h(k_{sj} H_j) & -1 \\ -1 & \cos h(k_{sj} H_j) \end{bmatrix} \]  

(1)

that relates the external tractions at the boundary interfaces \([\tau_j, \tau_{j+1}]^T\) with the observed displacements there \([U_j, U_{j+1}]^T\) (see Fig. 1). In this equation, \(G_j^c = G_j (1 + 2i\beta)\) is the complex shear modulus of the \(j\)th layer, \(\beta\) is the damping factor (default value = 0.01), \(H_j\) is the thickness of the \(j\)th layer, \(k_{sj} = \sqrt{k^2 - (k^c_{sj})^2}\), \(k\) is the horizontal wavenumber, \(k^c_{sj} = \omega \sqrt{\rho_j / G_j^c}\), \(\omega\): frequency of excitation, and \(\rho_j\): mass density of \(j\)th layer. For details, the reader is referred to the aforementioned references. On the other hand, the elastic half-space is characterized by the stiffness

\[ K_n^{ex} = G_n^c \sqrt{k^2 - (k^c_{0n})^2} \]  

with \(k^c_{0n} = \omega \sqrt{\rho_n / G_n^c}\).  

(2)

The entire system of layers satisfies the equilibrium equations

\[ \mathbf{P} = \mathbf{K}\mathbf{U}, \]  

(3)

where \(\mathbf{P}\) and \(\mathbf{U}\) are the vectors of the interface tractions and displacements and \(\mathbf{K}\) is the symmetric global stiffness matrix. This matrix is assembled with the individual layer matrices and half-space stiffness by overlaying appropriate terms. As can be seen from (1), the stiffness matrix is a function of the horizontal wavenumber \(k\), which appears hidden as argument in transcendental functions. This means that the solution to (3) (formally \(\mathbf{U} = \mathbf{K}^{-1}\mathbf{P}\)) for each wavenumber must be done numerically, which is computationally very intensive.

An alternative to obviate this problem can be found in the thin layer method of Lysmer-Waas-Kausel (Lysmer and Waas, 1972; Wass, 1972; Kausel, 1974). In this method, the soil profile is discretized into thin layers and the global stiffness matrix is formed by a weighted residual formulation. When this is done, it is found that the global matrix (in the antiplane case) has the polynomial structure

\[ \mathbf{K} = \mathbf{A} k^2 + \mathbf{G} - \omega^2 \mathbf{M}, \]  

(4)

where the matrices \(\mathbf{A}, \mathbf{G}, \) and \(\mathbf{M}\) are only a function of the material properties of the layers. From this equation, it transpires that the stiffness matrix is polynomial in the wavenumber and, indeed, linear in the squared wavenumber. Hence, its inversion can be accomplished by a similarity transformation (spectral decomposition) requiring the solution of a linear eigenvalue problem that is valid for all wavenumbers (i.e., it is independent of \(k\)). Furthermore, the inverse thus obtained is explicit in the wavenumber, so that integral transforms can readily be carried out in closed form. It is this feature that makes the thin layer method attractive (for details, see the articles referred to earlier).
Comparison of equation (4) for the layers and equation (2) for the half-space shows that the latter does not have a polynomial structure. This implies that when the system of layers is resting on an elastic half-space as opposed to a rigid base, the advantage of the thin layer method breaks down. However, it is possible to derive approximations for the half-space stiffness (Seale and Kausel, 1989) that are polynomial, and do permit to carry out the inversion of the global matrix by the similarity transformation referred to earlier.

Several polynomial approximations are possible, of which we examine here only two, namely the paraxial approximation (PA) of Engquist and Majda (1977) and the doubly asymptotic approximation (DAA) of Mathews and Geers (1987). In a nutshell, the paraxial approximation is obtained by expanding the exact stiffness (equation 2) in Taylor series in the horizontal wavenumber around \( k = 0 \), and truncating after the term in \( k^2 \) (more precisely, Engquist and Majda use Padé approximations, which coincide with the Taylor series for low order approximations). The result is

\[
K_n^{pa} = iG_n^c \left( \frac{k^0_n}{k^c_n} - 0.5 \frac{k^2}{k^0_n} \right) = i\omega\rho_n c_n \left( 1 - 0.5 \frac{k^2}{(k^0_n)^2} \right).
\]

Since this equation has the same structure as equation (4), it is obvious that the half-space can be included without destroying the capability of solving a linear eigenvalue problem.

The doubly asymptotic approximation, on the other hand, is based on adding the asymptotic values of equation (2) for very low- and very high-frequencies (\( \omega \rightarrow 0, \omega \rightarrow \infty \)). The resulting stiffness is then

\[
K_n^{da} = G_n^c (k + ik^c_0).
\]

While this equation is also polynomial in the wavenumber, it contains a linear term that is absent in the equations for the layers. Thus, use of this approximation leads to a quadratic eigenvalue problem that requires substantially more computational effort. This, however, does not invalidate the use of the thin layer method.

**Characteristics of Approximate Half-Space Stiffnesses**

Figures 2a and b show a comparison of the exact half-space stiffness (equation 2) and the paraxial (equation 5) and the doubly asymptotic (equation 6) approximations. As can be seen, the paraxial approximation provides a good fit at low horizontal wavenumbers, deteriorates rapidly as \( k \) approaches \( k^0_n \) and is useless thereafter. The doubly asymptotic approximation, by contrast, deteriorates faster as the horizontal wavenumber increases, but it provides acceptable results for large wavenumbers. Hence, both approximations are expected to work well only in the range of wavenumbers \([0 \leq k < 0.6k^0_n]\) for the paraxial approximation and \([0 \leq k < 0.2k^0_n, k \gg k^0_n]\) for the doubly asymptotic approximation. If we incorporate these approximate stiffnesses into the global stiffness matrix and solve it as in the thin layer method, this implies the assumption that the approximations are valid for all wavenumbers. Thus, it is important to investigate the effect of the errors in the half-space stiffnesses on the computed
Green's functions. Such a task is taken up in the following sections, considering first the spurious reflections by the approximations as well as their energy absorbing characteristics. Thereafter, the accuracy of the displacements is examined, and solution strategies using buffer layers are proposed.
Reflection Coefficients

When two identical half-spaces are joined together, and a plane wave propagates from one into the other, no reflections take place at the union, since they are equivalent to a full homogeneous space. When one of these spaces is represented by an approximation, however, spurious reflection may develop that depend on the angle of incidence of the waves. To assess this false reflection, it suffices to consider the equilibrium problem of the two half-spaces with impedances (stiffnesses) $K_1$ and $K_2$ for the upper and lower media, respectively. Using standard equilibrium considerations involving the propagation of waves across the interface separating these media, it can be shown that the reflection coefficient is given by the equation,

$$\kappa = \frac{K_1 - K_2}{K_1 + K_2}.$$  \hfill (7)

Clearly, if the two half-spaces are modeled exactly, then $K_1 = K_2$ and $\kappa = 0$, that is, no reflection takes place. However, if $K_1$ and $K_2$ are given, respectively, by equations (2) and (5) (or 6), then the associated reflection coefficients are (PA)

$$\kappa = \frac{K_1^{ex} - K_2^{pa}}{K_1^{ex} + K_2^{pa}}$$ \hfill (8a)

and (DAA)

$$\kappa = \frac{K_1^{ex} - K_2^{da}}{K_1^{ex} + K_2^{da}}.$$ \hfill (8b)

These reflection coefficients are shown in Figure 3. Notice that the largest wavenumber that can be considered for plane waves is $k = k_0 n$, since larger wavenumbers would correspond to inhomogeneous waves.

It can be observed that for wavenumbers that are smaller than approximately $k = 0.7 k_0 n$ (corresponding to waves with angle of incidence less than $45^\circ$), the paraxial approximation provides a substantially better fit to the exact condition than the doubly asymptotic approximation; this holds true both for the case of two homogeneous media shown in Figure 3 (no theoretical reflections) and for two dissimilar media (having a 1.5 ratio in shear-wave velocities) depicted in Figure 4. However, in the vicinity of $k = k_0 n$, and particularly for larger values of $k$, the paraxial approximation has extremely poor fit while the doubly asymptotic approximation displays acceptable fit, particularly in the case of dissimilar media. These results seem to indicate that the paraxial approximation will provide better results when used for modeling low-contrast transitions, while the doubly asymptotic approximation will perform better when modeling larger transitions in soil stiffness. This observation suggests in turn that the paraxial approximation could still be used fruitfully in the latter case with recourse to buffer layers having the same material properties as the half-space represented by the paraxial approximation.

Energy Considerations

An alternative measure of the quality of the approximations is the energy dissipated in one cycle of motion by the system represented by the approximate stiffness (equation 5 or 6).
Fig. 3. (a) Reflection coefficients, absolute value ($C_{s2}/C_{s1} = 1.0$, $k_{o1} = k_{o2}$); (b) Reflection coefficients, phase angle ($C_{s2}/C_{s1} = 1.0$, $k_{o1} = k_{o2}$).
Fig. 4. (a) Reflection coefficients, absolute value \(C_{s2}/C_{s1} = 1.5, k_{o1} = 1.5k_{o2}\); (b) Reflection coefficients, phase angle \(C_{s2}/C_{s1} = 1.5, k_{o1} = 1.5k_{o2}\).
It can be shown that the average energy dissipated by a mechanism with symmetric impedance matrix $K$ is given by (Waas, 1972)

$$E_j(k) = \frac{\omega}{2} V_j^*(k) \text{Im}[K_j(k)] V_j(k),$$

where $V_j(k)$: vector of surface displacements, $V_j^*(k)$: transpose of $V_j(k)$ with complex conjugate elements, and $K_j(k)$: layer stiffness matrix of $j$th layer or a half-space stiffness. If we calculate the average energy density dissipated in a half-space without layers, the sign of the energy density reflects only the sign of the imaginary part of the half-space stiffness. Therefore, both the exact and the doubly asymptotic half-space stiffnesses imply always positive energy, i.e., energy dissipation; the paraxial approximation, on the other hand, has a transition wavenumber $k = \sqrt{2}k_{on}$, beyond which it exhibits negative energy, i.e., energy supply. Thus, as far as the sign of energy density is concerned, we cannot avoid energy supply for the paraxial approximation, which could potentially be a serious drawback for the entire system.

The average energy density dissipated in a half-space without layers is shown in Figure 5 for a line load exerted at the origin. The exact solution has positive energy concentrated at $k = k_{on}$; the paraxial approximation, on the other hand, follows the exact solution reasonably well in the low wavenumbers, but it deteriorates rapidly after $k = k_{on}$ and exhibits large negative energy dissipation (energy supply) for $k > \sqrt{2}k_{on}$. Finally the doubly asymptotic approximation results only in energy dissipation, but it deviates substantially from the

![Figure 5](image-url)
exact solution and provides very modest energy dissipation characteristics. This corroborates the strong reflection coefficients associated with the approximations that were found in the previous section.

**Effects of a Buffer Layer**

When a layer of arbitrary thickness $H_1$ is laid on a half-space with the same material properties, the combination constitutes again a homogeneous half-space. However, when the half-space is represented by means of a polynomial approximation, the combination is only an approximate representation of a homogeneous half-space.

The effective impedance $K_{\text{eff}}$ of the combination of a layer and a half-space can be obtained by adding the half-space’s impedance to the second diagonal element in (1) (which gives the stiffness of the layer) and eliminating the degree of freedom associated with the half-space’s interface. The result is

$$K_{\text{eff}} = K_{\text{ex}} \frac{1 + R \cdot \cot h(k \sigma H_1)}{R + \cot h(k \sigma H_1)},$$

where $K_{\text{ex}} = $ exact half-space impedance (equation 2) and $R = K_{\text{app}} / K_{\text{ex}} = $ ratio of approximate and exact impedance.

Clearly, if $R = 1$, then $K_{\text{eff}} = K_{\text{ex}}$. However, if $K_{\text{app}}$ is given by equation (5) or (6) (i.e., $K_{\text{app}}$ represents the paraxial approximation or the doubly asymptotic approximation), then this ratio is not unity. On the other hand, since the Fourier transform of a line load is constant with wavenumber, then the ratio $F = K_{\text{ex}} / K_{\text{eff}} = $ flexibility ratio represents the ratio of approximate to exact displacements on the surface of the combination. Ideally, this ratio should be as close as possible to unity over all wavenumbers.

Figure 6 shows the absolute value of the ratio $F$ for the two approximations considered for the case of a half-space without a buffer layer. Figures 7 and 8, on the other hand, show this ratio for various depths of a buffer layer when using the paraxial approximation and the doubly asymptotic approximation, respectively. The depth of the layer in these figures has been normalized by the wavelength of shear waves $L_s = C_s / f$ (with $f = \omega / 2\pi$). As can be seen, the presence of the buffer layer improves the results substantially, particularly in the case of the paraxial approximation. Moreover, the improvement is most significant for $k > k_{0n}$, which represents the region of evanescent waves. Comparison of Figures 7 and 8 reveals that the buffer layer is far more effective in removing the shortcomings of the paraxial approximation, and not so effective for the doubly asymptotic approximation, which shows large oscillations for values $k < k_{0n}$. Since use of a buffer layer confers to the paraxial approximation the same good behavior at high wavenumbers exhibited by the doubly asymptotic approximation, and in addition it has better behavior at low and intermediate wavenumbers, it follows that the paraxial approximation is the method of choice for representing the half-space in the thin layer method.

Figure 9 shows the total average energy dissipated in one cycle of motion by a harmonically loaded buffer layer over a half-space modeled exactly or with the approximations considered in this article (compare to Fig. 5). As can be seen, the buffer layer is very effective in correcting the spurious generation of energy at high wavenumbers exhibited by the paraxial approximation. The reason is
FIG. 6. |(Approximate disp.)/(Exact disp.)| at surface of half-space, no buffer layers.

FIG. 7. |(Approximate disp.)/(Exact disp.)| at surface of half-space modeled by a buffer layer and a paraxial approximation.
Fig. 8. \(|(\text{Approximate disp.})/(\text{Exact disp.})|\) at surface of half-space modeled by a buffer layer and a doubly asymptotic approximation.

Fig. 9. Average energy dissipated in one cycle of motion for a half-space modeled by a buffer layer and half-space stiffnesses \((H/L_s = 0.5)\).
that high wavenumbers correspond to inhomogeneous waves that decay exponentially within the buffer layer; hence, the waves do not “touch” the paraxial approximation. Numerical experiments with other ratios of $H_1/L_s$ show similar improvements in behavior.

On the basis of the flexibility ratio $F$ and energy dissipation studied, it appears that a buffer layer is most effective in correcting the deficiencies of the paraxial approximation.

**DISPLACEMENTS DUE TO A LINE LOAD (GREEN’S FUNCTION)**

In this section, we consider the displacements on the surface of a layer of thickness $H_1$ welded to an elastic half-space. The excitation consists of a harmonic antiplane line load at the origin of coordinates on the free surface. The half-space is modeled alternatively by means of the exact impedance, the approximate impedances, or the approximate impedances in combination with a buffer layer. The results are presented in Figures 10 through 13, in which the label “exact” refers to the use of the exact half-space impedance, not to the accuracy of the computed Green’s functions.

While the thin layer method could have been used for the last two cases referred to previously, the computations in this study were all carried out in the frequency-wavenumber domain, representing the soil layer and the buffer layer with the exact stiffness matrices (equation 1). The resulting equations were then inverted numerically and converted to the space domain by carrying out the associated Fourier transform with the modified Clenshaw–Curtis method (MCC) with Chebyshev polynomials proposed by Xu and Mal (1985, 1987). The motivation for this computational strategy relates to the need to carry out consistent comparisons, that is, to avoid the small additional error introduced by discretization of the medium in the thin layer method.

**Homogeneous Half-space**

We consider first the case of a homogeneous half-space without an overlying layer. Figures 10a and b show the displacements on the surface in terms of the dimensionless horizontal distance $x/L_s$, where $L_s = C_s/f = $ wavelength of shear waves. In the first of these figures, the displacements were computed directly with the exact impedance and with the paraxial approximation, while in the second, a buffer layer of thickness $H_1/L_s = 0.25$ has been added. As can be seen, the solution without the buffer layer is very poor, or even useless, while the relatively thin buffer layer improves the results dramatically.

Figures 11a and b depict the corresponding results using the doubly asymptotic approximation. Again, a substantial improvement is achieved with a buffer layer, although not nearly as much as with the paraxial approximation.

**Layer over Elastic Half-space**

Next, we consider a layer over an elastic half-space with the same material density, but with a shear-wave velocity that is $\frac{2}{3}$ of that of the half-space. The depth of the layer is chosen as $H_1 = 0.1L_{s1}$.

Figures 12a and b show the displacements computed with the two approximations without using a buffer layer, while Figures 13a and b depict the case with a buffer layer of thickness $H_2 = 0.75L_{s1} = 0.5L_{s2}$.

Comparison of Figures 12a and b shows that the doubly asymptotic approximation outperforms the paraxial approximation when no buffer layer is used.
Fig. 10. (a) Surface displacements due to a surface line load for a half-space modeled by a paraxial approximation, no buffer layer ($H_1/L_s = 0$); (b) Surface displacements due to a surface line load for a half-space modeled by a buffer layer and a paraxial approximation ($H_1/L_s = 0.25$).
Fig. 11. (a) Surface displacements due to a surface line load for a half-space modeled by a doubly asymptotic approximation, no buffer layer ($H_1/L_s = 0$); (b) Surface displacements due to a surface line load for a half-space modeled by a buffer layer and a doubly asymptotic approximation ($H_1/L_s = 0.25$).
FIG. 12. (a) Surface displacements due to a surface line load for a one-layered half-space ($C_{2}/C_{1} = 1.5$) modeled by the layer and a doubly asymptotic approximation, no buffer layer ($H_{1}/L_{s1} = 0.1$); (b) Surface displacements due to a surface line load for a one-layered half-space ($C_{2}/C_{1} = 1.5$) modeled by the layer and a doubly asymptotic approximation, no buffer layer ($H_{1}/L_{s1} = 0.1$).
Fig. 13. (a) Surface displacements due to a surface line load for a one-layered half-space ($C_s2/C_s1 = 1.5$) modeled by the layer, a buffer layer, and a paraxial approximation ($H_1/L_{s1} = 0.1$, $H_2/L_{s1} = 0.75$); (b) Surface displacements due to a surface line load for a one-layered half-space ($C_s2/C_s1 = 1.5$) modeled by the layer, a buffer layer, and a doubly asymptotic approximation ($H_1/L_{s1} = 0.1$, $H_2/L_{s1} = 0.75$).
However, introduction of such layer improves the behavior of the paraxial approximation to the extent that virtually exact results are obtained, as shown in Figures 13a and b.

**Conclusions**

In this article, we have examined the behavior of two approximations in the antiplane half-space impedance, namely the Paraxial Approximation of Engquist and Majda and the Doubly Asymptotic Approximation of Mathews and Geers. The basic characteristics of the approximate stiffnesses were explored from the points of view of spurious reflections of waves, energy dissipation in one cycle of motion, and accuracy of displacements caused by an antiplane line load. From the results obtained, we can derive the following general conclusions.

As far as absorption of energy of traveling waves is concerned (i.e., low horizontal wavenumbers), the paraxial approximation is much better than the doubly asymptotic approximation.

If evanescent waves are important (e.g., a layer over a much stiffer half-space), then the doubly asymptotic approximation outperforms significantly the paraxial approximation.

Neither the paraxial approximation nor the doubly asymptotic approximation are sufficient by themselves to represent an elastic half-space.

Addition of a buffer layer having the same material properties as the half-space represented by the approximation improves the performance of the approximation to a very substantial degree. This is particularly true in the case of the paraxial approximation. Hence, it appears that the use of buffer layers is a prerequisite for accurate results. The thickness of such a buffer layer should be $H > 0.25L_s$ for acceptable results, or $H > 0.5L_s$ for accurate results. The implication of this conclusion on the computational burden depends, in part, on the geometry, material properties, and load characteristics of the problem being solved. In the context of the thin layer method, this implies adding a buffer layer having some two to four sublayers since the thickness of the sublayers should not exceed $\frac{1}{3}$ of the shortest wavelength $L_s$ being considered. This entails only a modest increase in computational effort.

It should be noted that while these results were studied from the point of view of the thin layer method, they apply equally well to models with finite elements or finite differences.

**References**


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