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for Infinite Domain
Media-Structure Interaction

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ANTHONY J. KALINOWSKI
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CONTENTS

Elastic Wave Scattering by Embedded Structures — A Survey of the T-Matrix Approach
V. V. Varadan and V. K. Varadan .............................................. 1

Structural and Acoustic Response of Submerged Axisymmetric Shells
A. J. Bronowicki and R. B. Nelson ........................................... 37

A Method of Successive Approximations for Structure-Medium Interaction Problems
I. S. Sandler ................................................................. 67

Solution of Complex Electromagnetic Penetration and Scattering Problems in Unbound Regions
A. Taflove and K. R. Umashankar ........................................... 83

Numerical Methods for Unbounded Field Problems and a New Infinite Element Formulation
U. C. Zienkiewicz, P. Bettess, T. C. Chiam, and C. Emson ................. 115

Consistent Boundaries for Semi-Infinite Problems
J. L. Tassoulas, E. Kausel, and J. M. Roesset ................................ 149

An Evaluation of the Paraxial Boundary
M. Cohen ............................................................. 167

A Non-Reflecting Boundary for Explicit Calculations
R. R. Kumar and J. Marti .................................................. 183

Geometrically Corrected Viscous Boundaries for Steady State Acoustic Scattering and Radiation Problems
A. J. Kalinowski .......................................................... 205

List of Related Titles ....................................................... 225
CONSISTENT BOUNDARIES FOR SEMI-INFINITE PROBLEMS

J. L. Tassoulas
University of Texas at Austin, Texas

E. Kause
Massachusetts Institute of Technology, Massachusetts

M. Roesset
University of Texas at Austin, Texas

ABSTRACT

The formulation of a consistent boundary matrix to simulate the effect of a semi-infinite medium surrounding a core region discretized with finite elements is presented. Other solutions to this problem are briefly reviewed first, in order to put the method into proper perspective. The consistent boundary is derived in detail for the simple two dimensional case using both a discrete and a continuous displacement expansion in the horizontal direction. The extension of the formulation to a three dimensional problem in cylindrical coordinates is discussed next. Further applications of this basic formulation to problems involving propagation of waves through soil deposits, as well as the determination of foundation stiffnesses, or hydrodynamic forces due to fluid structure interaction are finally discussed.

INTRODUCTION

The discretization of a continuum through the use of finite elements (or finite differences) requires the existence of a finite domain with well defined edges, where conditions are specified for forces or displacements. If the edges of the domain do not exist naturally but are created artificially for modeling purposes it is necessary to impose appropriate boundary conditions which will simulate the physical behavior of the actual problem. This situation is encountered in the solution of geotechnical and seismological problems involving the propagation of waves in soil deposits, or in fluid structure interaction problems involving the determination of hydrodynamic forces due to waves on structures. The boundary conditions imposed at the edges of the finite model to simulate the infinite domain are referred to as transmitting, non-reflecting, energy absorbing or silent boundaries, or also as infinite elements. The purpose of this paper is to present the conceptual basis for a solution to the problem known as consistent boundaries. While the formulation of these boundaries has been reported in a number of papers and research reports (1, 2, 3, 4, 5) it appears that it is not yet well known or properly understood. Additional applications for these boundaries are discussed.
TRANSMITTING BOUNDARIES

Three types of boundary conditions are normally used to simulate the existence of an infinite or semi-infinite domain outside the region of interest: elementary (non-transmitting) boundaries; local (imperfect transmitting) or viscous boundaries; and consistent, nonlocal or perfect transmitting boundaries.

Elementary Boundaries

In this alternative either forces or displacements (velocities or accelerations) are prescribed at the boundary. The first case corresponds to a fixed or Dirichlet boundary condition, while the second constitutes a free or Neumann boundary. Combinations of forces and displacements are also specified in some cases. For instance, when dealing with shear waves propagating vertically through a horizontally stratified soil deposit, rollers may be placed at the boundary nodes corresponding to zero vertical displacements and zero horizontal forces. For problems where the excitation is located outside of the core region the prescribed forces or displacements are those that would occur in the free field, that is to say those that would be observed at the location of the boundaries if the core region had the same properties as the far field, without any inclusions or discontinuities. This has been referred to as the "soil inland" approach.

Elementary boundaries are perfect reflectors in the sense that no energy is absorbed or transmitted. Some energy may get trapped, however, in the form of surface waves that travel along the boundary and dissipate when internal damping is present. Thus the echo may not fully develop if the angle of the incident wave exceeds some critical value. More importantly, if the boundaries are located at a sufficient distance from the region of interest and the medium has internal damping, the amplitude of the waves that reach the core region after reflection at the boundary may be very small.

An attempt to refine the results obtained with elementary boundaries was presented by Smith (6), who proposed solving the problem twice; once with Dirichlet boundary conditions in the normal direction and Neumann's conditions in the tangential direction, then with the reciprocal conditions. The basis for this approach is the fact that plane waves are reflected with equal amplitude and phase in the first situation but with opposite phase in the second. Addition of the two solutions would thus cancel the reflection. For a solution of a general problem with a boundary points this method would require the computation of 2^n solutions, which is clearly economically prohibitive. A variation to Smith's original idea was introduced by Cunali et al (7), solving the problem in the time domain with an explicit integration scheme, and averaging the solutions for the two boundaries at each time step (actually every four time steps) in order to prevent multiple reflections (reverberations), replacing the conditions on displacements by equivalent conditions or velocities, and using boundary zones consisting of four grid columns instead of a single line boundary.

Viscous Boundaries

This solution was first used by Tsai (8) for the analysis of shear waves propagating vertically through a layered soil (one dimensional problem) and by Kuhlmeier (9) and Lysmer and Kuhlmeier (10) for two dimensional, plane strain problems. In both cases the boundary conditions corresponded to the placement of viscous dashpots with constant properties at each boundary node. This provides an exact solution for Tsai's one dimensional case (as does the Smith boundary) and for a train of plane waves impinging on the boundary at normal incidence in general. The solution is only approximate, however, for arbitrary combinations of wave trains.

Attempts to improve the absorbing characteristics of these boundaries, although still for trains of plane waves, were presented by White, Vailappan and Lee (11), while Akiyoshi (12) introduced a correction to account for the discretization scheme.
Kuhlemeyer (9) did also develop a viscous boundary that absorbs perfectly specific types of surface waves. The main disadvantages of this solution over the previous one are that this boundary can only be defined in the frequency domain, that it requires considerably more computational effort and that it does not offer a guarantee of improved results for other types of waves.

Ang and Newmark (13) developed a very similar model for finite difference idealizations of the soil using the concept of the transmittal of the O’Alember force. The solution provides again perfect absorption of plane waves arriving normally at the boundary but it is only approximate for other types of waves. A very general family of local boundary conditions was presented by Engquist and Majda (14) and Clayton and Engquist (15). These conditions were developed for finite difference models using mathematical approximations which can be implemented with varying degrees of accuracy. The first order approximation has absorbing characteristics similar to those of the Lysmer-Kuhlemeyer and Ang-Newmark solutions. The second order approximation increases substantially the absorptions capacity but the higher the order of the approximation the larger the number of points near the boundary which must be involved (the boundaries lose then their local characteristic). The possible connection, with the boundary zones in Gundall’s modification of the Smith boundary seems to deserve some further scrutiny. A similar type of approach has been presented more recently by Cohen (16).

A solution which has received surprisingly little attention from researchers in this topic and which would seem to provide substantial improvement over other viscous boundaries is the application of the complex, frequency dependent springs derived by Novak (17, 18, 19). These equivalent springs are only strictly applicable in the dynamic case but they can be generalized by extrapolating their values in the low frequency range. They are applicable to cylindrical boundaries and they are still spatially local (a spring can be lumped at each node), but they are only defined in the frequency domain. To use them with a time domain solution would require to obtain impulse response functions (calculating their Fourier transform) and storing therefore the results for various time steps. (Notice again the connection to the higher order approximations of Engquist.)

**Consistent Boundaries**

These boundaries constitute perfect absorbers of any kind of waves impinging with arbitrary incidence. Their realization requires however, conditions which are essentially non-local in nature. For non-stationary in the frequency domain the consistent boundaries couple all the boundary points and are frequency dependent; on the other hand since they are perfect absorbers they can be placed immediately adjoining the region of interest, so that the size of the model can be held reasonably small. Thus the added expense in the formulation of the boundary is compensated for by the reduction in the total number of degrees of freedom needed to describe the problem. For solutions in the time domain it would again be necessary to find the Fourier transform of the boundary matrix, obtaining a matrix of impulse response functions.

The first consistent boundary was proposed by Lysmer and Waas (1) for layered strata over rigid rock involving antiplane loads; the concept was extended by Waas (2) for general in plane motion in layered soils and by Chang Liang (3) to the case where the excitation was outside of the core region. Kasagi (4, 5) developed the boundary for three dimensional problems exhibiting cylindrical geometry by expanding the solution in a Fourier series in the circumferential direction and obtaining the boundary matrix for each term of the series.

More general transmitting boundaries for horizontally layered soils can be developed using the appropriate Green function for the problem in connection with the Boundary Integral Method (20). In the general case the Green function must be evaluated numerically by expanding the point load in a normal Fourier
series for the plane case, a double Fourier series for a three-dimensional problem in cartesian coordinates and a Hankel series for a three-dimensional problem in cylindrical coordinates. For each term of the series the solution of the layered system with a rigid base or an elastic half space underlying the soil strata is obtained using transfer matrices, or more conveniently, stiffness matrices. For the simpler case of a homogeneous half space an alternative, used by Dominguez (21) is to use the Green's function for a full space which is available in closed analytical form. A tradeoff is made in this case between the simplicity of this function, avoiding the use of the series computation and the need to set boundary elements along the free surface of the soil. Dominguez found, however, that the number of elements needed to obtain a good solution was relatively small.

Comparison of Various Boundaries

A comparison of elementary boundaries, the viscous boundaries of Kuhlemeyer, and the consistent boundary was performed by Ettouney (22) for a plane strain two-dimensional case. In order to evaluate the different solutions Ettouney compared the results for the compliance function (real and imaginary part) of a strip-footing as a function of frequency, and the response spectra at the top of a structure resting on the spread footing. He found that the elementary and viscous boundaries studied provided relatively poor results unless they were placed at a sufficient distance from the footing (function of the internal soil damping), while the consistent boundary could be placed directly at the edge of the footing with excellent results.

A more extensive comparison, using closed form solutions, was performed by Kassaj and Tassoulas (23) for the case of an antiplane line load (again a two-dimensional situation).

FORMULATION OF CONSISTENT BOUNDARIES

In order to understand better the concepts behind the Wasa boundary it is convenient to start looking first at the simple two dimensional, plane strain, case. For this problem two alternative formulations can be derived and their equivalence can be clearly visualized. The same ideas can then be easily extended to cases with cylindrical geometry or more general three dimensional situations.

Plane Strain Case

Consider a soil profile consisting of several horizontal layers of soil with constant properties in the horizontal direction and resting on a much stiffer, rock like material which can be considered as a rigid base. Let \( x \) be the horizontal coordinate axis in the plane of the profile, \( z \) the vertical axis and \( y \) the horizontal axis perpendicular to the plane of the model (Figure 1). Let \( u, w, v \) be the displacements at a point along these three directions.

For a plane condition \( u, v \) and \( w \) are only functions of \( x \) and \( z \). It is then possible to study separately the case of Love waves (only motion out of the plane, \( v \), takes place) and the case of generalized Rayleigh waves with only \( u \) and \( w \) displacements.

Assume that the soil has been discretized using columns of finite elements of equal width as shown in Figure 2. Ideally these columns should continue till infinity, but if the soil has some internal damping good solutions could be obtained by extending the model to a sufficient distance from the core region. In order to minimize the number of degrees of freedom of the problem, and consequently the computational expense, the width of the finite element columns is sometimes increased (making elements which are very elongated in the horizontal direction) to reach a sufficient distance with only a few columns. This approach can produce acceptable results when the width of the columns is gradually increased but it may be dangerous when the elongated elements are placed close to the foundation. A better solution to reach an adequate distance without increasing the number of degrees of freedom is to use substructuring techniques as shown in ref. 24.
Figure 1 - Geometry and Definitions

Figure 2 - Finite Element Discretization
While the substructuring approach will produce good results for soils with some internal damping it does not truly account for the infinite nature of the medium. Considering instead one column of finite elements, calling \( U_A \) the displacements of the nodes on the left side of the column, \( U_B \) the displacements of the right boundary, \( P_A \), \( P_B \) the corresponding vectors of forces and \( K_{11}, K_{12}, K_{21}, K_{22} \) the dynamic stiffness submatrices for the column (resulting from \( K_{11} \) or \( K_{22} \) of \( K_{11} \) is the matrix if the material had viscous damping), the dynamic equilibrium equations of the column are

\[
\begin{align*}
K_{11} U_A + K_{12} U_B & = P_A \\
K_{21} U_A + K_{22} U_B & = P_B
\end{align*}
\]  
\( (1) \)

If the bottom boundary is fixed and the soil deposit has \( n \) elements (or layers) in the vertical direction, the vectors \( U, P \) will have \( n \) components for the Love wave case and \( 2n \) components for the Rayleigh wave problem. The dynamic stiffness submatrices will be \( n \times n \) or \( 2n \times 2n \) respectively. Considering now several columns of finite elements and using the subscripts \( i, i-1, i, i+1 \) to indicate forces or displacements at the nodes along a vertical line, the equations of motion for line \( i \), resulting from assembling the matrices for the columns immediately to the left and to the right of the line are

\[
K_{i1} U_{i-1} + (K_{11} + K_{22}) U_i + K_{i2} U_{i+1} = 0
\]  
\( (2) \)

if there are no external forces applied at the nodes of line \( i \).

This is a system of recurrence equations. Its solution can be expressed in the form

\[
U_i = \sum a_j r_j^i x_j
\]  
\( (3) \)

where \( r_j \) are the roots (eigenvalues) of the equation

\[
(r^2 K_{12} + r(K_{11} + K_{22}) + K_{21}) U = 0
\]  
\( (4) \)

and \( x_j \) are the corresponding eigenvectors.

For the case of out of plane motion this problem will have \( 2n \) eigenvalues, and for the in plane motion \( 4n \). Although it is a quadratic eigenvalue problem, its solution is relatively easy because of the special form of the matrices. Noting that \( K_{21} = K_{12}^{-1} \) it can be seen that if \( r_j \) is an eigenvalue with eigenvector \( x_j \), \( r_j^{-1} \) is also an eigenvalue with the same eigenvector.

Denoting by \( r \) the eigenvalues with modulus smaller than 1, and \( s \) their reciprocals equation (3) could be rewritten as

\[
U_i = \sum (a_j r_j^i + b_j s_j^i) x_j
\]  
\( (5) \)

It can be seen that the terms with \( a_j r_j \) represent a solution where the amplitude of displacements decreases with increasing \( i \). It corresponds to waves propagating in the positive \( x \) direction. The terms with \( b_j s_j \) represent the other waves propagating in the negative \( x \) direction, the amplitude of displacements increasing with increasing \( i \). If the excitation is assumed to be located in the core region the first series of terms should be used for a right boundary (the soil extending to the right of the boundary) and the second series for a left boundary.

Taking a right boundary (the formulation is analogous for the left boundary) and calling \( U_o, P_o \) the displacements and forces at the nodes along the boundary

\[
P_o = X_{11} U_o + K_{12} U_i
\]  
\( (6) \)
\[ U_1 = Q R A \]
\[ U_0 = Q A \]
\[ U_0 = Q R Q^{-1} U_0 \] (8)

where \( Q \) is the modal matrix with the eigenvectors \( x_i \) as columns, \( R \) is a diagonal matrix with \( r_j \) as the \( j \)th diagonal elements and \( A \) is the vector with \( a_i \) as its \( j \)th component. The coefficients \( a_j \) can be interpreted as the participation factors of the various modes.

Thus
\[ A = Q^{-1} U_0 \]
\[ U_1 = Q R Q^{-1} U_0 \]

and
\[ P_0 = (K_{11} + K_{12} Q R Q^{-1}) U_0 = B U_0 \] (9)

The forces acting on the core region, exerted by the far field to the right would be then equal and opposite to \( P_0 \) and can be expressed as
\[ P = -B U \] (10)

where \( B \) is the boundary matrix replacing the far field.

If the excitation were applied in the far field, calling \( P_f, U_f \) the forces and displacements that would occur on the right edge of the core region as part of the free field Equation 10 would become
\[ P - P_f = -B (U - U_f) \] (11)
or \[ P = P_f + B U_f - B U \]

Notice that the matrix \( B \) and the vector \( P_f, U_f \) must be computed for each frequency of interest.

The actual formulation of the consistent boundary as developed by Weiss (2, 3) starts by expressing the displacements in each layer in the form
\[ u = [u_1(1 - z/h) + u_2 z/h^2] f(x) \] (12)
\[ w = [w_1(1 - z/h) + w_2 z/h^2] f(x) \]
for in-plane motions (Generalized Rayleigh waves), and
\[ v = [v_1(1 - z/h) + v_2 z/h^2] g(x) \] (13)
for out-of-plane motions (Generalized Love waves).

In these expressions \( h \) is the thickness of the layer and \( z \) varies from 0 at the top of the layer to \( h \) at the bottom. This assumes a linear displacement expansion in the \( z \) direction. One could, however, use a quadratic or higher order expansion in the \( z \) direction if these were the displacement functions used for the finite element discretization of the core region, the main purpose being to have consistent displacement functions for the core region and the far field.
The functions \( f(x) \) and \( g(x) \) are assumed of the form \( \exp(-irx) \) which is the general solution of the homogeneous governing equation, \( r \) being the wave number.

The strains become then

\[
\varepsilon_x = -ir \left[ u_1 (1 - \frac{2}{h}) + u_2 \frac{x}{h} \right] f(x)
\]
\[
\varepsilon_z = \frac{1}{h} \left[ -w_1 + w_2 \right] f(x)
\]
\[
\gamma_{xz} = \left[ \frac{1}{h} (-u_1 + u_2) - ir (\frac{w_1}{h} (1 - \frac{2}{h}) + \frac{w_2}{h}) \right] f(x)
\]

for the in plane case and

\[
\gamma_{xy} = -ir \left[ v_1 (1 - \frac{2}{h}) + v_2 \frac{x}{h} \right] g(x)
\]
\[
\gamma_{yz} = \frac{1}{h} \left[ -v_1 + v_2 \right] g(x)
\]

for the out of plane problem.

Using standard finite element procedures it is then possible to derive a stiffness matrix for a layer extending to infinity of the form

\[
k = k_0 + ir k_1 + r^2 k_2
\]

where

\[
k_0 = \begin{bmatrix}
\frac{G}{h} & 0 & -\frac{G}{h} & 0 \\
0 & \frac{\lambda+2G}{h} & 0 & -\frac{\lambda+2G}{h} \\
-\frac{G}{h} & 0 & \frac{G}{h} & 0 \\
0 & -\frac{\lambda+2G}{h} & 0 & \frac{\lambda+2G}{h}
\end{bmatrix}
\]

\[
k_1 = \frac{h}{2} \begin{bmatrix}
0 & G+\lambda & 0 & G+\lambda \\
\lambda-G & 0 & \lambda+G & 0 \\
0 & -\lambda-G & 0 & \lambda+G \\
-\lambda-G & 0 & \lambda+G & 0
\end{bmatrix}
\]

and

\[
k_2 = \begin{bmatrix}
\frac{h}{3} (\lambda+2G) & 0 & \frac{h}{6} (\lambda+2G) & 0 \\
0 & \frac{G}{3} h & 0 & \frac{G}{6} h \\
\frac{h}{6} (\lambda+2G) & 0 & \frac{h}{3} (\lambda+2G) & 0 \\
0 & \frac{G}{3} h & 0 & \frac{G}{3} h
\end{bmatrix}
\]
for the Rayleigh case

\[
k_0 = \begin{bmatrix}
    \frac{G}{h} & \frac{G}{h} \\
    \frac{G}{h} & \frac{G}{h}
\end{bmatrix}
\]

(20)

\[
k_1 = \begin{bmatrix}
    0 & 0 \\
    0 & 0
\end{bmatrix}
\]

(21)

\[
k_2 = \begin{bmatrix}
    \frac{Gh}{3} & \frac{Gh}{3} \\
    \frac{Gh}{6} & \frac{Gh}{3}
\end{bmatrix}
\]

(22)

for the Love problem.

In the same way one can define for each layer a mass matrix

\[
m = \frac{\rho h}{6} \begin{bmatrix}
    2 & 0 & 1 & 0 \\
    0 & 2 & 0 & 1 \\
    1 & 0 & 2 & 0 \\
    0 & 1 & 0 & 2
\end{bmatrix}
\]

or

\[
m = \frac{\rho h}{6} \begin{bmatrix}
    2 & 1 \\
    1 & 2
\end{bmatrix}
\]

(23)

In these expressions \(\rho\) is the mass density of the soil, \(\lambda,\ G\) the Lamé parameters (\(G\) is the shear modulus, \(\lambda + 2G\) the constrained modulus) and \(h\) is again the layer thickness.

The layer matrices \(k_0, \ k_1, \ k_2, \ \rho, \ h\) can then be assembled for the complete system of layers forming matrices \(K_0, \ K_1, \ K_2, \ M\) corresponding to the soil profile, the displacements at the bottom being eliminated when assuming a rigid base. For the Love problem the total matrices are \(n\times n\) if there are \(n\) layers. While for Rayleigh's problem there are \(2n\times 2n\).

Imposing the condition of no external forces the dynamic equilibrium equations become

\[
(r^2K_2 + \text{tr}K_1 + K_0 - \omega^2M) \ \bar{U} = 0
\]

(24)

This equation is the continuous equivalent of Equation (4). It represents again a quadratic eigenvalue problem with \(2n\) on \(4n\) eigenvalues and eigenvectors. The mathematical properties of the solution are analogous to those of the discrete problem. Half of the eigenvalues have a positive imaginary term and correspond to waves propagating in the positive x direction (their amplitude decreases in this direction), while the other half correspond to waves in the opposite direction (having the imaginary part negative).

A more detailed discussion of the properties of the eigenvalues (wave numbers) can be found in References 2, 25, including the study of special cases, in the absence of damping, where the eigenvectors are not linearly independent (for frequencies where the equation has double roots).

Selecting the eigenvalues corresponding to the boundary of interest, calling \(R\) a diagonal matrix with \(r_i\) as the \(i\)th diagonal element, \(Q\) the modal matrix with the corresponding eigenvectors as columns, and \(A\) a vector with the \(i\)th component \(a_i\) being the participation factor of the \(i\)th mode, the displacements and forces at the nodes of the boundary become
\[ U = QA \]

\[ \mathbf{P} = i k_2 Q A + k' U \quad \text{for case 1} \]
\[ \mathbf{P} = i k_2 Q A \quad \text{for case 2} \]

where the matrix \( k' \) is assembled from individual stiffness matrices for the layers

\[
k' = \frac{1}{2}
\begin{bmatrix}
0 & \lambda & 0 & -\lambda \\
\lambda & 0 & -\lambda & 0 \\
0 & \lambda & 0 & -\lambda \\
\lambda & 0 & -\lambda & 0
\end{bmatrix}
\]  

From the first system of Equations (25)

\[ \lambda = Q^{-1} U \]

and

\[ \mathbf{P} = (i k_2 Q Q^{-1} + k') U = BU \quad \text{for case 1} \]
\[ \mathbf{P} = (i k_2 Q Q^{-1}) U = B U \quad \text{for case 2} \]

where \( B \) is the appropriate boundary matrix.

It is important to notice that this solution is fundamentally different from the so-called "infinite elements." In these the same type of displacement expansion is used, but the relation between forces and displacements on the vertical boundary is obtained independently for each layer (element) assuming a value of the wave number \( r \). The boundary matrix is obtained by assembling the matrices

\[ i r k_2 + k' \quad \text{for in plane motion} \]
\[ i r k_2 \quad \text{for out of plane motion} \]

for each layer. As a result equilibrium of stresses and compatibility of displacements along the horizontal interfaces of the layers are violated. Notice that this approach is in fact closer to that of the viscous boundaries.

In the consistent boundary on the other hand full compatibility is insured by determining, for each frequency, the wave numbers of all the waves that can propagate through the soil deposit within the approximation introduced by the piecewise linear displacement expansion in the vertical direction (this is the solution of the quadratic eigenvalue problem).

**Cylindrical Coordinates**

The substructure approach described in the previous section can be extended to problems formulated in cylindrical coordinates or to more general three-dimensional situations (in which case the condition of a rigid bottom boundary may be dropped but it is replaced by an assumption of homogeneous properties outside a curved or polygonal boundary), as long as the finite elements within consecutive layers satisfy similarity conditions (24).

Dasgupta (26) has shown that the recurrence equations leading to the quadratic eigenvalue problem can also be extended to these other cases under the same conditions.
The formulation of the consistent boundary matrix for soil deposits with a rigid base in cylindrical coordinates was developed by Kausel (4, 5). The displacement at each point has then a radial component \( u \), a tangential or circumferential component \( v \), and a vertical component \( w \). Displacement expansions are then selected by:

- expanding the solution in a Fourier series in the circumferential direction. The dependence of the displacements on the angular coordinate \( \phi \) is then given by terms of the form \( \cos n\phi \) for \( u \) and \( w \) and \( -\sin n\phi \) for \( v \) (symmetric condition) or \( \sin n\phi \) for \( u \) and \( w \) and \( \cos n\phi \) for \( v \) (antisymmetric modes). The boundary matrix must be developed for each term of the Fourier series independently. It should be noticed, however, that in many practical cases only one term is necessary. So, for instance in the determination of the dynamic stiffnesses of circular foundations (surface or embedded) only the \( n=0 \) term is needed for vertical or torsional excitation and only the \( n=1 \) term must be used for coupled rocking-pinning. This expansion in a Fourier series allows also to discretize the core region with toroidal finite elements, leading thus to a two dimensional model for each term.

- using in the radial direction Hankel functions of the second kind and of order \( n \) instead of the simple exponentials of the plane strain case. These are the solutions of Bessel's equation

\[
\frac{d^2}{d\zeta^2} C_n^*(\zeta) + \frac{1}{\zeta} \frac{d}{d\zeta} C_n^*(\zeta) + \left( 1 - \frac{n^2}{\zeta^2} \right) C_n^*(\zeta) = 0
\]

(30)

which are applicable for a radial distance \( \rho \) varying from \( \rho_o \) (radius of cylindrical core region) to infinity.

\[
C_n^*(\zeta) = H_n^{(2)}(\rho_o)
\]

(31)

where \( n \) represents again the wave number.

It should be noted that these Hankel functions tend asymptotically to decaying exponentials as the argument increases.

- using again a polynomial displacement expansion in the vertical direction consistent with the one used for the core region. Normally a linear displacement expansion is used over each element but one could equally formulate the problem with a quadratic or higher order expansion.

One of the most important characteristics of this formulation is the fact that in order to determine the wave numbers and corresponding mode shapes it is only necessary to solve the two quadratic eigenvalue problems of the plane strain case (one for in plane motion, Rayleigh waves, the other for out of plane motions, Love waves), irrespective of the order of the term of the Fourier expansion or whether a symmetric or antisymmetric mode is being considered. This part of the problem must be solved therefore, only once, even if various terms of the series are desired.

Calling \( K \) a 3Nx3N diagonal matrix (\( N \) is the number of layers) containing the eigenvalues of the Rayleigh problem as the first \( 2N \) diagonal elements, and those of the Love problem as the last \( N \), \( W_j \), \( W_{j+2} \), the \( j \)th horizontal and vertical displacement components of the \( j \)th mode for the Rayleigh problem (\( j=1 \) to \( 2N \)) and \( v_j \), the \( j \)th component of the out of plane \( j \)th mode (\( j=2N+1 \) to \( 3N \)) one can define 3 modal matrices \( z, v, w \) given by

\[
\begin{align*}
\dot{z}_{j-1} & = -\dot{z}_j, C_{n-1}^*(r_o \lambda) \quad j = 1 \text{ to } 2N \\
\dot{z}_{j-1} & = -\dot{w}_j, C_n^*(r_o \lambda) \quad j = 1 \text{ to } N \\
\dot{z}_j & = 0
\end{align*}
\]

(32)
\[ \delta_{j,2,k} = 0 \quad \ell = 2N+1 \text{ to } 3N \]
\[ \delta_{j,1,k} = 0 \quad j = 1 \text{ to } N \]
\[ \delta_{j,1,\ell} = -V_{j,\ell} C_{n-1}(r_{\ell,0}) \quad j = 1 \text{ to } N \]
\[ \gamma_{j,2,k} = V_{j,\ell} C_{n-1}(r_{\ell,0}) \quad \ell = 1 \text{ to } 2N \]
\[ \gamma_{j,1,\ell} = -i r_{j,\ell} C_{n-1}(r_{\ell,0}) \quad j = 1 \text{ to } N \]
\[ \gamma_{j,1,\ell} = 0 \]
\[ \gamma_{j,2,\ell} = 0 \quad \ell = 2N+1 \text{ to } 3N \]
\[ \gamma_{j,1,\ell} = 0 \quad j = 1 \text{ to } N \]
\[ \gamma_{j,2,\ell} = V_{j,\ell} C_{n}(r_{\ell,0}) \]
\[ Q_{j,2,\ell} = r_{\ell} U_{j,\ell} C_{n}(r_{\ell,0}) \quad \ell = 2 \text{ to } 3N \]
\[ Q_{j,1,\ell} = -i r_{j,\ell} W_{j,\ell} C_{n}(r_{\ell,0}) \quad \ell = 1 \text{ to } N \]
\[ Q_{j,1,\ell} = 0 \]
\[ Q_{j,2,\ell} = 0 \quad \ell = 3N+1 \text{ to } 3N \]
\[ Q_{j,1,\ell} = 0 \quad j = 1 \text{ to } N \]

These matrices involve therefore weighted combinations of the mode shapes of the two plane strain problems.

Defining stiffness matrices \( K_{0}, K_{1}, K_{2}, K_{3}, K_{4}, K_{5} \), assemble on the basis of matrices \( k, k_{1}, k_{2}, k_{3}, k_{4}, k_{5} \) for each individual layer, applying the condition of a rigid base by suppressing the last 3 rows and columns with

\[
K_{0} = \frac{h}{6}
\left[
\begin{array}{cccccc}
2(\lambda+2\mu) & 0 & 0 & \lambda+2\mu & 0 & 0 \\
0 & 2\mu & 0 & 0 & \mu & 0 \\
0 & 0 & 2\mu & 0 & 0 & \mu \\
\lambda+2\mu & 0 & 0 & 2(\lambda+2\mu) & 0 & 0 \\
0 & \mu & 0 & 0 & 2\mu & 0 \\
0 & 0 & \mu & 0 & 0 & 2\mu
\end{array}
\right]
\]

\[
K_{1} = \frac{h}{2}
\left[
\begin{array}{cccccc}
\sigma & \lambda & 0 & 0 & -\lambda & 0 \\
0 & \mu & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{array}
\right]
\]

160
\[ k_2 = \frac{\Gamma_h}{3 e_o} \]
\[
\begin{bmatrix}
2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 1 \\
1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 \\
\end{bmatrix}
\]

\[ k_3 = \frac{\Gamma_h}{6 e_o} \]
\[
\begin{bmatrix}
0 & 0 & 4 & 0 & 0 & 2 \\
0 & 2 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & 2 & 0 \\
2 & 0 & 0 & 4 & 0 & 0 \\
\end{bmatrix}
\]

\[ k_4 = \frac{2\Gamma_h}{3 e_o^2} \]
\[
\begin{bmatrix}
2 & 0 & -2 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 2 & -1 & 0 & 1 \\
1 & 0 & -1 & 2 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -2 & 0 & 2 \\
\end{bmatrix}
\]

\[ k_5 = \frac{\alpha}{2 e_o^n} \]
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

the consistent boundary matrix is given by

\[ B = \delta_0 \left[ K_o \mathbf{y}_R + \left( K_o - K_2 + n K_3 \right) \mathbf{y}_R - \frac{n(n+1)}{2} K_4 + n K_5 \right] \mathbf{q}^{-1} \]

Further details of this formulation can be found in references 4, 25.

OTHER APPLICATIONS

The consistent boundary described in the previous section has been primarily used in soil structure interaction problems for the determination of dynamic stiffnesses of surface and embedded foundations and for the direct seismic
analysis of structures including the soil. In these applications there has been normally a core region discretized with finite elements, immediately below the structure or foundation, and the boundary matrix has been developed for the vertical boundaries around the core region. There are, however, a number of possible interesting extensions of the basic formulation, as well as other applications in the general area of geotechnical engineering and in wave propagation problems, either in solids or in fluids. Some of these extensions and further applications are briefly discussed in this section.

Representation of a Finite Domain

While the consistent boundary is most often used to reproduce the behavior of a semi-infinite domain (the far field), it is equally possible with this formulation to study the dynamic response of a finite region. This leads to what has been called a hyperelement (27).

For the two-dimensional, plane strain, case it is necessary to combine a left and right boundaries. Calling \( U_1 \), \( U_2 \) the displacements and forces on the left boundary of the region, \( U_{1,2} \), the corresponding quantities on the right boundary, for the in plane problem

\[
U_1 = \alpha A_1 + TQ A_2 \\
U_2 = \beta A_1 + TQ A_2
\]

where \( T \) is an \( 2N \times 2N \) matrix given by

\[
T_{2j-1,2j-1} = 1 \\
T_{2j,2j} = -1
\]

and \( E \) is a diagonal matrix with \( \exp(-ir_j d) \) as the \( j \)-th diagonal element, \( d \) being the width of the domain.

Equations 42 must be solved to find the unknown vectors of participation factors \( \lambda_1 \lambda_2 \) in terms of the displacements \( U_1 U_2 \). The resulting expressions must then be introduced in

\[
F_1 = (iK_2 Q R + K' Q) \lambda_1 + (-iK_2 T Q E R + K' T Q E) \lambda_2 \\
F_2 = (-iK_2 Q E R - K' Q S) \lambda_1 + (iK_2 T Q R - K' T Q) \lambda_2
\]

(44)

to obtain

\[
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix} = \mathbf{K} \begin{bmatrix}
U_1 \\
U_2
\end{bmatrix}
\]

(45)

For the out of plane problem the corresponding equations are

\[ U_1 = \alpha A_1 + \beta A_2 \]
\[ U_2 = -\beta A_1 + \alpha A_2 \]

(46)

and

\[
F_1 = iK_2 Q R A_1 - iK_2 Q E A_2 \\
F_2 = -iK_2 Q E A_1 - iK_2 Q R A_2
\]

(47)

For the cylindrical case, on the other hand, the expression for the stiffness matrix of a region extending from \( r = 0 \) to \( r = \rho \) is still given by Equation 42, using for the definition of the modal matrices (equations 31, 32 and 33).

167
\[ C_n(\rho r) = J_n(\rho r) \] (48)

where \( J_n \) is the Bessel function of order \( n \), and changing the sign of \( B \).

An approach similar to that described for the plane strain case must be used, however, to derive the stiffness matrix of a cylindrical region extending from \( \rho = \rho_1 \) to \( \rho = \rho_2 \). In this case both Hankel functions of the first and the second kind must be used and the modal matrices must be evaluated at both values of \( \rho \) for each of these functions (25).

An even more interesting extension of this formulation has been presented by Tassoulas (25). In all the above derivations it has been assumed that no forces were applied in the region under consideration, whether the far field or a finite domain. The relation between forces and displacements could then be obtained in terms of the solutions of a homogeneous differential equation. Considering on the other hand a problem where displacements are still fixed at the bottom of the soil profile, but where the homogeneous conditions on stresses at the free surface are replaced by known, imposed displacements, the stiffness matrix can be obtained by combining

- the solutions of the homogeneous problem with fixed displacements at the top and the bottom.

- any particular solution.

In this way the dynamic stiffness of surface or embedded strip footings, circular mats, or ring foundations can be obtained directly, without the need to discretize any part of the domain with finite elements.

**Dynamic Stiffness of Piles**

A direct application of the consistent boundary matrix in cylindrical coordinates was presented by Blaney (28) for the determination of the dynamic response of isolated piles. Using \( \rho \), the radius of the pile, the boundary matrix represents then the effect of the surrounding soil. The pile itself is reproduced by linear members, each segment having a length equal to the thickness of a soil sublayer. The displacements at the nodes of the soil region are again a radial displacement \( u \), a tangential displacement \( v \), and a vertical displacement \( w \), whereas the degrees of freedom at the pile nodes are the horizontal displacement, the rotation and the vertical displacement. It is thus necessary for each case (axial or lateral excitation), to apply an appropriate transformation to the boundary matrix, assuming a rigid cross section for the pile, before adding both stiffness matrices.

Since the consistent boundary is developed for the case of a soil stratum on a rigid base the application to the study of a pile is immediate when dealing with a point bearing pile. The situation of a floating pile can be simulated by assuming that the properties of the member below the pile tip are equal to those of the surrounding soil.

**Fluid-Structure Interaction**

Although the original derivation of the consistent boundary matrix was intended for the solution of soil dynamics problems it can clearly be extended to any wave propagation problem. The extension of the two dimensional formulation to determine the hydrodynamic pressures on dome was presented by Hall and Chopra (29, 30); assuming a perfect fluid. The state variables in this case are pressures and accelerations instead of displacements and forces.

In the same way the boundary matrix in cylindrical coordinates could be applied to determine wave forces on a flexible cylinder for an ideal fluid and linear wave theory (diffraction equation). This formulation is entirely analogous to that of the isolated pile. And the matrix corresponding to a finite cylindrical domain would be used to reproduce the dynamic behavior of a fluid inside a cylindrical tank, while the matrix for a ring would find immediate application in the study of a fluid between two concentric cylinders.
Further Applications

The consistent boundary matrix relates forces and displacements at the boundary nodes of a semi-infinite domain. It should be noticed, however, that for a given set of nodal forces one can compute not only the displacements at the boundary but at any point within the soil mass. The formulation can therefore be used to compute numerically the Green function for a stratified medium resting on a rigid base. This approach was used by Gonzalez (31) to determine the dynamic stiffnesses of square foundations and the interaction between two adjacent structures through the soil, and by Vardanega (32) to obtain dynamic stiffnesses of rectangular foundations.

In the previous cases it was assumed that the unit force was applied at the surface and was distributed over a circle of small radius. One could, however, obtain also the displacements caused by forces applied at any of the nodal points (that is to say at any depth). This approach could be used to determine the dynamic behavior of complete pile foundations (pile group effects) or hydrodynamic forces on various cylinders.

Final Comments

The consistent boundary matrix provides an efficient solution to wave propagation problems which can be conveniently formulated in the frequency domain. This implies basically a linear formulation. For problems where nonlinear behavior may be important in the core region the consistent boundaries can still be employed if use is made of equivalent linearization techniques, but they must be placed then at a sufficient distance to guarantee that the nonlinear effects are no longer important.

A second limitation of this approach is the assumption of a rigid base. This is not a serious drawback in many practical cases where a sharp discontinuity in material properties will exist at a certain depth. A vigorous solution considering an elastic half space below the base can be developed but it requires the solution of a transcendental eigenvalue problem, and it is therefore considerably more laborious. Approximate solutions considering viscous dashpots at the bottom, as suggested by Tsai (8), may also be tried, and should be expected to provide reasonable results.
REFERENCES


