ELEMENTS FOR THE NUMERICAL ANALYSIS OF WAVE MOTION IN LAYERED STRATA

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SUMMARY
A technique is developed for the numerical analysis of wave motion in layered strata. Semidiscrete particular solutions satisfying inhomogeneous boundary conditions are calculated by the finite element method. These solutions are combined with semidiscrete modes of an appropriate eigenvalue problem. The boundary conditions corresponding to rigid and rough footings on a layered stratum are treated in detail. Applications are considered which demonstrate the validity of the technique.

INTRODUCTION
The analysis of wave motion in layered media has been of interest to both scientists and engineers for a long time. Classical treatises on the subject may be found, for example, in the books by Ewing, Jardetzky and Press and by Brekhovskikh. A problem which has received wide attention in the last two decades is that of determining the dynamic behaviour of a structure on a layered soil deposit. The source of excitation may be within the deposit, e.g. a seismic source, or within the structure itself, e.g. machine vibrations. In rather highly idealized situations, analytical methods may be used to obtain the solution; in cases of practical interest, however, the development of numerical methods becomes necessary.

A numerical technique for the analysis of wave motion in layered strata which accounts for the radiation into the far field was presented by Lysmer and Waas and Waas. Time-harmonic waves in plane strain or antiplane shear as well as axisymmetric waves in a layered stratum were considered. The technique is based on the calculation by the finite element method of semidiscrete solutions (modes of vibration) satisfying homogeneous boundary conditions on the surface and the base (bedrock) of the stratum. The solution in the far field is then written as a linear combination of these semidiscrete modes. It is combined with a fully discrete solution obtained using the finite element method for the region of the stratum with inhomogeneous boundary conditions. The technique was extended by Kausel to nonaxisymmetric waves in axisymmetric regions of a layered stratum (see also the papers by Kausel, Roësset and Waas and by Kausel and Roësset). It must be noted that, in all these developments, the semidiscrete solutions were obtained in regions of infinite extent. Kausel and Roësset generalized the technique to the analysis of wave motion in a finite region of a layered stratum with homogeneous boundary conditions, e.g. a surface free of tractions and a fixed base. In this paper a further extension of the technique to the analysis of wave motion in finite regions of a layered stratum with inhomogeneous boundary conditions is considered.
Semidiscrete particular solutions satisfying the inhomogeneous boundary conditions are calculated and then combined with semidiscrete modes satisfying the inhomogeneous boundary conditions. The technique is described through an example of significant practical interest, namely, the boundary conditions corresponding to a rigid and rough footing. The validity of the technique is demonstrated by considering applications to the analysis of vibrations of footings on layered strata.

**PLANE STRAIN**

*Homogeneous boundary conditions*

Time-harmonic vibrations of a layered stratum (Figure 1) in plane strain are considered first. It is assumed that interfaces of layers are planes parallel to the surface and the base of the stratum. Moreover, layers are taken as linearly viscoelastic solids. The governing differential equations in layer \( j, 1 \leq j \leq M, M \) being the number of layers in the stratum, are

\[
\begin{align}
\left( \lambda_j + 2G_j \right) \frac{\partial^2 u}{\partial x^2} + \lambda_j \frac{\partial^2 w}{\partial x \partial z} + G_j \left( \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} \right) + \omega^2 \rho_j u &= 0 \\
\left( \lambda_j + 2G_j \right) \frac{\partial^2 w}{\partial z^2} + \lambda_j \frac{\partial^2 u}{\partial x \partial z} + G_j \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z} \right) + \omega^2 \rho_j w &= 0
\end{align}
\]

\( u, w \) are the amplitudes of the displacements in the \( x \)-direction (horizontal direction) and the \( z \)-direction (vertical direction) respectively (\( x, z \) are rectangular Cartesian co-ordinates). \( \lambda_j, G_j \) are the Lamé moduli of layer \( j \). In general, these are complex functions of the frequency \( \omega \) of the time-harmonic vibrations. If dissipative behaviour is assumed to be identical in bulk and shear straining, the moduli may be taken as

\[
\lambda_j = \lambda_j^0 (1 + 2i\beta_j), \quad G_j = G_j^0 (1 + 2i\beta_j).
\]

\( \lambda_j^0, G_j^0 \) are the Lamé moduli of the corresponding linearly elastic solid (in general, real functions of the frequency). \( \beta_j \) is commonly referred to as the fraction of critical damping (also, in general, a real function of the frequency). \( \rho_j \) is the mass density of layer \( j \). The interfaces of layers are located at \( z = z_j, 2 \leq j \leq M, \) with

\[
0 = z_1 < z_2 < \ldots < z_M < z_{M+1} = h,
\]
$h$ being the depth of the layered stratum. The thickness of layer $j$ is denoted by $h_j$. It is given by

$$h_j = z_{j+1} - z_j \quad 1 \leq j \leq M.$$

Layers are assumed to be 'bonded' at interfaces. This implies continuity requirements on the amplitudes $\sigma_z, \tau_{xz}$ of the stress components acting on the interfaces and the amplitudes $u, w$ of the displacements there:

$$
\sigma_z \bigg|_{z = z_j} = \left( (\lambda_{j-1} + 2G_{j-1}) \frac{\partial w}{\partial z} + \lambda_{j-1} \frac{\partial u}{\partial x} \right)_{z = z_j^-} = \left( (\lambda_j + 2G_j) \frac{\partial w}{\partial z} + \lambda_j \frac{\partial u}{\partial x} \right)_{z = z_j^+} = \sigma_z \bigg|_{z = z_j^-} \quad (2a)
$$

$$
\tau_{xz} \bigg|_{z = z_j} = G_{j-1} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)_{z = z_j^-} = G_j \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)_{z = z_j^+} = \tau_{xz} \bigg|_{z = z_j^-} \quad (2b)
$$

$$
u(x, z_j^-) = u(x, z_j^+) \quad (2c)$$

$$w(x, z_j^-) = w(x, z_j^+) \quad (2d)$$

Boundary conditions must be specified on the surface and the base of the stratum, i.e. at $z = z_1 = 0$ and $z = z_{M+1} = h$. For a surface free of tractions and a fixed base, the boundary conditions are given by

$$
\sigma_z \bigg|_{z = 0} = \left( (\lambda_1 + 2G_1) \frac{\partial w}{\partial z} + \lambda_1 \frac{\partial u}{\partial x} \right)_{z = 0} = 0 \quad (3a)
$$

$$
\tau_{xz} \bigg|_{z = 0} = G_1 \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)_{z = 0} = 0 \quad (3b)
$$

$$u(x, h) = 0 \quad (3c)$$

$$w(x, h) = 0 \quad (3d)$$

Solutions of the differential equations (1a, b) satisfying the continuity conditions (2a–d) at interfaces of layers and the homogeneous boundary conditions (3a–d) on the surface and the base of the layered stratum are, in general, linear combinations of modes of the form:

$$u(x, z) = U(z) \exp(-ikx) \quad (4a)$$

$$w(x, z) = W(z) \exp(-ikx) \quad (4b)$$

The amplitudes $U$ and $W$ satisfy the differential equations

$$k^2(\lambda_j + 2G_j)U + ik(\lambda_j + G_j) \frac{dW}{dz} - G_j \frac{d^2U}{dz^2} - \omega^2 \rho_j U = 0 \quad (5a)$$

$$k^2G_jW + ik(\lambda_j + G_j) \frac{dU}{dz} - (\lambda_j + 2G_j) \frac{d^2W}{dz^2} - \omega^2 \rho_j W = 0 \quad (5b)$$
in layer \( j \), the continuity conditions

\[
\left( \left( \lambda_{j-1} + 2G_{j-1} \right) \frac{dW}{dz} - i k \lambda_{j-1} U \right)_{z = z_j^-} = \left( \left( \lambda_j + 2G_j \right) \frac{dW}{dz} - i k \lambda_j U \right)_{z = z_j^+}
\]

(6a)

\[
G_{j-1} \left( \frac{dU}{dz} \right)_{z = z_j^-} = G_j \left( \frac{dU}{dz} \right)_{z = z_j^+}
\]

(6b)

\[
U(z_j^-) = U(z_j^+)
\]

(6c)

\[
W(z_j^-) = W(z_j^+)
\]

(6d)

at interfaces of layers, i.e. at \( z = z_j, 2 \leq j \leq M \), and the boundary conditions

\[
\left( \left( \lambda_1 + 2G_1 \right) \frac{dW}{dz} - i k \lambda_1 U \right)_{z = 0} = 0
\]

(7a)

\[
\left( -i k W + \frac{dU}{dz} \right)_{z = 0} = 0
\]

(7b)

\[
U(h) = 0
\]

(7c)

\[
W(h) = 0
\]

(7d)

The differential equations (5a, b), the continuity conditions (6a–d) and the homogeneous boundary conditions (7a–d) define an eigenvalue problem on the interval \( 0 \leq z \leq h \). The values of \( k \), the wave number, for which nontrivial solutions \( U, W \) (eigenfunctions) exist are the eigenvalues of the problem. The frequency equation, i.e. the equation the roots of which are the eigenvalues for the given frequency \( \omega \) of time-harmonic vibrations, involves transcendental functions. In general, solving such an equation is not an easy task. Search methods are typically used. However, approximate eigenvalues and eigenfunctions may be obtained using a numerical method. Following Waas\(^4\) the finite element method may be applied. Each of the subintervals \([z_j, z_{j+1}^+], 1 \leq j \leq M\), corresponding to the layers, is subdivided into finite elements. Let \( N \) be the number of finite elements on the interval \([0, h]\). Linear interpolation is used for the amplitudes \( U, W \) in each element. In the sequel, it will be assumed that the interfaces of sublayers corresponding to the finite elements are located at \( z = z_j, 2 \leq j \leq N \), with

\[
0 = z_1 < z_2 < \ldots < z_N < z_{N+1} = h
\]

The element vector of nodal amplitudes, for element \( j \), is taken as

\[
\begin{pmatrix}
U_j \\
W_j \\
U_{j+1} \\
W_{j+1}
\end{pmatrix}
\]

with \( U_j = U(z_j), \ W_j = W(z_j), 1 \leq j \leq N + 1 \). Note that \( U_{N+1} = W_{N+1} = 0 \) as required by the boundary conditions (7c, d). The finite element method yields an algebraic eigenvalue problem corresponding to the eigenvalue problem defined by the differential equations (5a, b) and the conditions (6a–d), (7a–d):

\[
[k^2 A + i k B + G - \omega^2 M] \Delta = 0
\]

(8)

\( A, G, M \) are \( 2N \times 2N \) symmetric matrices assembled from element matrices \( A^j, G^j, M^j \) which are given in Appendix I. \( B \) is a \( 2N \times 2N \) antisymmetric matrix assembled from \( B^j \) also given
in Appendix I. \( \Delta \) is the \( 2N \)-vector of nodal amplitudes:
\[
\Delta_{2j-1} = U_j \\
\Delta_{2j} = W_j \\
1 \leq j \leq N
\]

Details of the derivation of the algebraic eigenvalue problem may be found in References 3, 4 and 9. \( A, B, G, M \) being block tridiagonal matrices, the quadratic eigenvalue problem (8) is easily solved using one of the standard methods. Its solution yields \( 4N \) wave numbers \( k \) and the corresponding eigenvectors \( \Delta \). After the discrete solution for \( U \) and \( W \) is obtained, the semidiscrete modes of vibration of the layered stratum at frequency \( \omega \) corresponding to (4a, b) may be found as
\[

t(x, z_j) = U_j \exp (-ikx) \\
w(x, z_j) = W_j \exp (-ikx) \\
1 \leq j \leq N
\]

It is easily checked that if \( k \) is an eigenvalue with eigenvector \( \Delta \), then \( -k \) is also an eigenvalue with eigenvector \( \bar{\Delta} \) (or \( -\bar{\Delta} \)) given by
\[
\hat{\Delta}_{2j-1} = \Delta_{2j-1} \\
\hat{\Delta}_{2j} = -\Delta_{2j} \\
1 \leq j \leq N
\]

In fact, the modes may be split into those which transmit energy in the positive \( x \)-direction or vanish as \( x \to \infty \) and those which transmit energy in the negative \( x \)-direction or vanish as \( x \to -\infty \). Note that phase propagation and energy propagation are not always in the same direction (see, for example, References 10 and 11). Clearly, if the mode with wave number \( k \) and eigenvector \( \Delta \) transmits energy or decays in the positive \( x \)-direction, then the mode with wave number \( -k \) and eigenvector \( \bar{\Delta} \) transmits energy or decays in the negative \( x \)-direction.

The region \( x \geq 0 \) may be thought of as an element (Figure 2a) with nodes at \((0, z_j), 1 \leq j \leq N\). Alternatively, it may be referred to as a hyperelement or a macroelement in order to distinguish it from a finite element. The nodal forces corresponding to the semidiscrete mode with eigenvector \( \Delta \) and wave number \( k \) which are consistent with the interpolation of displacements in the sublayers are given by
\[
P = [ik \Delta + D] \Delta.
\]

\( D \) is assembled from element matrices \( D^j \) given in Appendix I. \( A \) is the same as in (8). Odd-numbered components of the \( 2N \)-vector \( P \) are the amplitudes of forces in the \( x \)-direction. Even-numbered components of \( P \) are the amplitudes of forces in the \( z \)-direction. If radiation and boundedness conditions are imposed for \( x \to \infty \), any (semidiscrete) solution in the region may be obtained as a linear combination of (semidiscrete) modes which transmit energy in the positive \( x \)-direction or vanish as \( x \to \infty \). Let \( k_j, \Delta^j, 1 \leq j \leq 2N \), be the wave numbers and eigenvectors corresponding to these modes. It is convenient to define the diagonal matrix \( K(2N \times 2N) \) which contains the wave numbers
\[
K = \text{diag}[k_j]
\]

and the modal matrix \( X(2N \times 2N) \) the columns of which are the corresponding eigenvectors:
\[
X = [\Delta^1, \Delta^2, \ldots, \Delta^{2N}].
\]
Let \( \mathbf{U} \) be the vector of nodal amplitudes:

\[
U_{2j-1} = u(0, z_j) \\
U_{2j} = w(0, z_j)
\]

\( 1 \leq j \leq N \)

It may be written as a linear combination of the eigenvectors:

\[
\mathbf{U} = \mathbf{X}\Gamma
\]

(13)
\[ \Gamma \] is a 2N-vector of participation factors. The vector of nodal forces corresponding to the mode with wave number \( k_j \) and eigenvector \( \Delta^j \) is, according to (10),

\[ P^j = [ik_j A + D] \Delta^j \]

The vector of nodal forces \( F \) corresponding to \( U \) is given by

\[ F = \sum_{j=1}^{2N} \Gamma_j P^j \]

\( \Gamma_j, 1 \leq j \leq 2N, \) being the components of the vector \( \Gamma \) of participation factors. It is easily seen that

\[ F = [iA \hat{X} X^* K + D] \Gamma \]  \hspace{1cm} (14)

The vector of participation factors \( \Gamma \) may be eliminated from (14) using (13):

\[ F = RU \]  \hspace{1cm} (15)

\( R \) is referred to as the dynamic stiffness matrix of the element. It is given by

\[ R = iA \hat{X} \hat{X}^{-1} + D \]  \hspace{1cm} (16)

Details about the development reviewed here may be found in the original work by Waas. A detailed review is given by Tassoulas. The element has become known as the consistent transmitting boundary.

An extension of the technique to the case of a finite region of a layered stratum with homogeneous boundary conditions on the surface and the base has been presented by Kausel and Roesset. The region \( x_1 \leq x \leq x_2 \) is considered. The surface \( (z = 0) \) is assumed free of tractions while the base \( (z = h) \) is fixed. The region is understood as a hyperelement (Figure 2b) with nodes at \( (x_1, z_j), (x_2, z_j), 1 \leq j \leq N. \) Any (semidiscrete) solution in this region may be written as a linear combination of the (semidiscrete) modes. Since the region is finite all the modes are admissible. Let \( U^1, U^2 \) be the vectors of nodal amplitudes corresponding to \( x = x_1 \) and \( x = x_2, \) respectively:

\[ U_{2j-1}^i = u(x_i, z_j) \]
\[ U_{2j}^i = w(x_i, z_j) \]
\[ 1 \leq j \leq N \quad i = 1, 2 \]

These vectors may be written as linear combinations of the vectors of nodal amplitudes corresponding to the 4N modes:

\[ U^1 = X \Gamma^1 + \hat{X} E \Gamma^2 \]  \hspace{1cm} (17a)
\[ U^2 = X \hat{E} \Gamma^1 + \hat{X} \Gamma^2 \]  \hspace{1cm} (17b)

\( X \) is the modal matrix in (12). \( \hat{X} \) is the modal matrix corresponding to the modes which transmit energy in the negative \( x \)-direction or vanish as \( x \to -\infty: \)

\[ \hat{X} = [\hat{\Delta}^1, \hat{\Delta}^2, \ldots, \hat{\Delta}^{2N}] \]  \hspace{1cm} (18)

The corresponding diagonal matrix of wave numbers is \( -K. \) \( E \) is a diagonal matrix \((2N \times 2N)\):

\[ E = \text{diag}[\exp(-ik_j L)] \]  \hspace{1cm} (19)

\( L \) being the length of the region: \( L = x_2 - x_1. \) \( \Gamma^1, \Gamma^2 \) are 2N-vectors of participation factors. Let \( F^1, F^2 \) be the vectors of consistent nodal forces corresponding to \( x = x_1 \) and \( x = x_2, \)
respectively. Equation (10) yields

\[ F^1 = (iAXK + DX)^1 + (-iAXK + D^2)^1 \]
\[ F^2 = -(-iAXK + D^2)^1 - (-iAXK + D^2)^1 \] (20a)
\[ F^2 = -(-iAXK + D^2)^1 - (-iAXK + D^2)^1 \] (20b)

Eliminating the vectors of participation factors \( \Gamma^1, \Gamma^2 \) from (20a, b), using (17a, b), gives

\[ \begin{pmatrix} F^1 \\ F^2 \end{pmatrix} = K \begin{pmatrix} U^1 \\ U^2 \end{pmatrix}, \] (21)

\( K \) being the dynamic stiffness matrix of the element. It is important to note that the computational effort involved in the calculation of this matrix is independent of the length \( L \) of the region. The dynamic stiffness matrices of this hyperelement and of the consistent transmitting boundary are symmetric.\(^4\)\(^9\) Details may be found in the paper by Kausel and Roëssév.\(^8\)

**Inhomogeneous boundary conditions**

An extension of the technique to the analysis of wave motion in regions of a layered stratum with inhomogeneous boundary conditions is now considered. An example of practical interest is that of the boundary conditions corresponding to a rigid and rough footing (Figure 2c) imposed on the surface of the region \(-L/2 \leq x \leq L/2\).

\[ u(x, 0) = A, \]
\[ w(x, 0) = Az - 8x, \]
\[ \Delta_1, \Delta_2, \theta \] being the amplitudes of the horizontal displacement, vertical displacement and rotation of the footing, respectively. The base is assumed fixed, i.e. the conditions (3c, d) apply. Any solution in the region may be obtained as a linear combination of a particular solution satisfying the inhomogeneous boundary conditions and modes of the eigenvalue problem resulting from the corresponding homogeneous boundary conditions. The homogeneous boundary conditions corresponding to (22a, b) and (3c, d) are

\[ u(x, 0) = 0 \]
\[ w(x, 0) = 0 \]
\[ u(x, h) = 0 \]
\[ w(x, h) = 0 \] (23a, b, c, d)

implying a fixed surface and a fixed base. The differential equations (1a, b) together with the continuity conditions (2a–d) at interfaces of layers and the homogeneous boundary conditions (23a–d) define an eigenvalue problem which is of the same type as that considered earlier for a layered stratum with free surface and fixed base. The algebraic eigenvalue problem which yields the discrete eigenfunctions \( U \) and \( W \) is obtained from that in (8) by deleting the first two rows and the first two columns in the matrices, since \( U(0) = W(0) = 0 \) by (23a, b). The
components of the eigenvector \( \Delta \) corresponding to eigenvalue \( k \) are

\[
\Delta_{2j-1} = U_{j+1} \\
\Delta_{2j} = W_{j+1} \\
1 \leq j \leq N - 1
\]

The algebraic eigenvalue problem yields \( 4N - 4 \) wave numbers \( k \) and the corresponding eigenvectors. Solutions satisfying the homogeneous boundary conditions (23a–d) may be written as linear combinations of the modes. The region \(-L/2 \leq x \leq L/2\) may be understood as an element with nodes at \((-L/2, z_j), (L/2, z_j), 2 \leq j \leq N, \) and a node at \((0, 0)\).

Let \( U^1, U^2 \) be the vectors of nodal amplitudes corresponding to \( x = -L/2, x = L/2, \) respectively:

\[
U^1_{2j-1} = u\left(\frac{-L}{2}, z_{j+1}\right) \quad U^2_{2j-1} = u\left(\frac{L}{2}, z_{j+1}\right) \\
U^1_{2j} = w\left(\frac{-L}{2}, z_{j+1}\right) \quad U^2_{2j} = w\left(\frac{L}{2}, z_{j+1}\right)
\]

\(1 \leq j \leq N - 1\)

A matrix relating the vectors of consistent nodal forces \( F^1, F^2 \) at \((-L/2, z_j), (L/2, z_j), 2 \leq j \leq N, \) respectively, to \( U^1, U^2, \) for a semidiscrete solution satisfying the homogeneous boundary conditions (23a–d), may be obtained as in the case of the hyperelement considered by Kausel and Roësset.\(^8\) \( U^1, U^2 \) are written as linear combinations of the vectors corresponding to the \( 4N - 4 \) modes. \( F^1, F^2 \) are then obtained as linear combinations of the vectors of consistent nodal forces corresponding to the modes. Finally:

\[
\begin{pmatrix}
F^1 \\
F^2
\end{pmatrix} =
\begin{pmatrix}
K^{11} & K^{12} \\
K^{21} & K^{22}
\end{pmatrix}
\begin{pmatrix}
U^1 \\
U^2
\end{pmatrix}.
\]

\( K^{11}, K^{22} \) are symmetric matrices while \( K^{21} \) is the transpose of \( K^{12} \).

(a) Horizontal vibrations. A particular solution is sought which satisfies the inhomogeneous boundary conditions (22a, b) with

\[
\Delta_x = 1 \quad \Delta_z = 0, \quad \theta = 0
\]

The vectors of consistent nodal forces and amplitudes of nodal displacements at \((-L/2, z_j), (L/2, z_j), 2 \leq j \leq N, \) are denoted by

\( F^{1,1}, F^{2,1}, U^{1,1}, U^{2,1} \)

The nodal degrees-of-freedom at \((0, 0)\) are \( \Delta_x, \Delta_z, \theta \) and the corresponding consistent loads are the horizontal force \( F_x, \) the vertical force \( F_z, \) and the moment \( M. \) For the particular solution under consideration, the loads are denoted by

\( F^1_x, F^1_z, M^1 \)

Substituting

\[
u(x, z) = U(z) \quad (25a)
\]

\[
w(x, z) = 0 \quad (25b)
\]
into the differential equations (1a, b), the conditions (2a, b) at interfaces of sublayers and the
boundary conditions (22a, b), (3c, d) it is easily obtained that \( U \) must satisfy the differential
equation, in sublayer \( j \),
\[
G_j \frac{d^2 U}{dz^2} + \rho \omega^2 U = 0 \quad (26a)
\]
the conditions, at \( z = z_j \), \( 2 \leq j \leq N \),
\[
G_{j-1} \frac{dU}{dz} \bigg|_{z = z_j} = G_j \frac{dU}{dz} \bigg|_{z = z_j} \quad (26b)
\]
and the boundary conditions
\[
U(0) = 1 \quad (26c)
\]
\[
U(h) = 0 \quad (26d)
\]
The discrete solution for \( U \) is easily obtained by the finite element method. It is noted that,
in general, for a particular solution of the form
\[
\begin{align*}
  u(x, z) &= U(z) \exp(-ikx) \\
  w(x, z) &= W(z) \exp(-ikx)
\end{align*}
\]
the corresponding discrete solution for \( U \) and \( W \) is calculated from
\[
[k^2A + ikB + G - \omega^2M] \Delta = \mathbf{f} \quad (27)
\]
in which \( A, B, G, M, \Delta \) are the same as in (8) and \( \mathbf{f} \) is a \( 2N \)-vector with components
\[
\begin{align*}
  f_1 &= -G_1 \left( \frac{dU}{dz} - ikW \right)_{z=0} \\
  f_2 &= - \left( (\lambda_1 + 2G_1) \frac{dW}{dz} - ik\lambda_1 U \right)_{z=0} \\
  f_j &= 0, \quad 3 \leq j \leq 2N,
\end{align*}
\]
i.e. \( f_1, f_2 \) are the amplitudes of the shear and normal tractions at the surface. In fact, the
algebraic eigenvalue problem (8) is obtained from (27) by setting \( \mathbf{f} = 0 \), i.e. imposing the
condition that the surface is free of tractions. For the particular solution (25a, b), \( k = 0 \),
\( W(z) = 0, 0 \leq z \leq h \). Thus
\[
[G - \omega^2M] \Delta = \mathbf{f} \quad (28)
\]
with \( \Delta_1 = 1, \Delta_2 = 0, \Delta_{2j-1} = U_j, \Delta_{2j} = 0, 2 \leq j \leq N, \)
\[
\begin{align*}
  f_1 &= -G_1 \frac{dU}{dz} \bigg|_{z=0} \\
  f_2 &= 0 \\
  f_{2j-1} &= f_{2j} = 0 \quad 2 \leq j \leq N
\end{align*}
\]
It may be seen that only odd-numbered rows and columns in matrices \( G \) and \( M \) are used in
the calculation of the discrete solution for \( U \) since horizontal and vertical displacements are
uncoupled in these matrices. Thus the calculation involves symmetric tridiagonal matrices and
equation (28) is easily solved. Let \( \mathbf{Y} \) be a \( (2N - 2) \)-vector with components
\[
\begin{align*}
  Y_{2j-1} &= U_{j+1} \\
  Y_{2j} &= 0 \\
  1 \leq j \leq N - 1
\end{align*}
\]
Then

\[ U^{1,1} = Y \quad (29a) \]
\[ U^{2,1} = Y \quad (29b) \]

The forces are given by

\[ F^{1,1} = D\Delta \quad (29c) \]
\[ F^{2,1} = -D\Delta \quad (29d) \]

\( D \) is the same as in (14) except that the first two rows are deleted since the forces in (29c, d) correspond to nodes at \( z = z_j, 2 \leq j \leq N \). For this particular solution, the horizontal forces are equal to zero. The consistent loads at \( (0, 0) \) are given by

\[ F_x^1 = f_1 L \quad (29e) \]
\[ F_z^1 = 0 \quad (29f) \]
\[ M_1 = \frac{L}{2} G_1 (1 - \Delta_3) \quad (29g) \]

\( f_1, \Delta_3 \) are obtained from (28).

(b) Vertical vibrations. Working similarly, a particular solution is obtained which satisfies the boundary conditions (22a, b) with

\[ \Delta_x = 0 \quad \Delta_z = 1 \quad \theta = 0 \]

The vectors of consistent nodal forces, amplitudes of nodal displacements and consistent loads at \( (0, 0) \) are denoted by

\[ F^{1,2}, F^{2,2}, U^{1,2}, U^{2,2}, F_x^2, F_z^2, M_2 \]

Substituting

\[ u(x, z) = 0 \quad (30a) \]
\[ w(x, z) = W(z) \quad (30b) \]

into the differential equations (1a, b), the conditions (2a, b) at interfaces of sublayers and the boundary conditions (22a, b), (3c, d) shows that \( W \) must satisfy the differential equation, in sublayer \( j \),

\[ (\lambda_j + 2G_j) \frac{d^2 W}{dz^2} + \rho_j \omega^2 W = 0 \quad (31a) \]

the conditions, at \( z = z_j, 2 \leq j \leq N \),

\[ (\lambda_{j-1} + 2G_{j-1}) \frac{dW}{dz} \bigg|_{z=z_j} = (\lambda_j + 2G_j) \frac{dW}{dz} \bigg|_{z=z_j} \quad (31b) \]

and the boundary conditions

\[ W(0) = 1 \quad (31c) \]
\[ W(h) = 0 \quad (31d) \]
The discrete solution for $W$ is easily obtained from (27) with $k = 0$:

$$[G - \omega^2 M]\Delta = f$$

with

$$\Delta_0 = 0, \Delta_2 = 1, \Delta_{2j-1} = 0, \Delta_{2j} = W_j, \quad 2 \leq j \leq N,$$

$$f_1 = 0, f_2 = -(\lambda_1 + 2G_1) \frac{dW}{dz} \bigg|_{z=0}, \quad f_{2j-1} = f_{2j} = 0, \quad 2 \leq j \leq N.$$

In this case only even-numbered rows and columns in matrices $G$ and $M$ are used in the solution of (32). Again, the calculation involves symmetric tridiagonal matrices and solving (32) is an easy task. Let $Y$ be a $(2N - 2)$-vector with components

$$Y_{2j-1} = 0 \quad Y_{2j} = W_{j+1}$$

$$1 \leq j \leq N - 1.$$

Thus

$$U^{1,2} = Y \quad (33a)$$

$$U^{2,2} = Y \quad (33b)$$

The forces are given by

$$F^{1,2} = D\Delta \quad (33c)$$

$$F^{2,2} = -D\Delta \quad (33d)$$

$D$ is the same as in (29c, d). For this particular solution, the vertical forces in (33c, d) are equal to zero. The consistent loads at $(0, 0)$ are given by

$$F_x^1 = 0 \quad (33e)$$

$$F_z^2 = f_1L \quad (33f)$$

$$M_2 = 0 \quad (33g)$$

$$f_1$$ is obtained from (32).

(c) Rocking. Finally, a particular solution is sought which satisfies the inhomogeneous boundary conditions (22a, b) with

$$\Delta_x = 0 \quad \Delta_z = 0 \quad \theta = 1$$

The vectors of consistent nodal forces, amplitudes of nodal displacements and consistent nodal loads at $(0, 0)$ are denoted by

$$F^{1,3}, F^{2,3}, U^{1,3}, U^{2,3}, F_x^2, F_z^2, M_3$$

Substituting

$$u(x, z) = U(z) \quad (34a)$$

$$w(x, z) = -xW(z) \quad (34b)$$
NUMERICAL ANALYSIS OF WAVE MOTION

into the differential equations (1a, b), the conditions (2a, b) at interfaces of sublayers and the boundary conditions (22a, b), (3c, d) it is seen that $U$ and $W$ must satisfy the differential equations, in sublayer $j$,

$$G_j \frac{d^2 U}{dz^2} - (\lambda_j + G_j) \frac{dW}{dz} + \rho \omega^2 U = 0$$

$$\lambda_j + 2G_j \frac{d^2 W}{dz^2} + \rho \omega^2 W = 0$$

the conditions at $z = z_j$, $2 \leq j \leq N$,

$$\left. (\lambda_j - 1 + 2G_{j-1}) \frac{dW}{dz} \right|_{z = z_j} = \left. (\lambda_j + 2G_j) \frac{dW}{dz} \right|_{z = z_j}$$

$$G_{j-1} \left( -W + \frac{dU}{dz} \right)_{z = z_j} = G_j \left( -W + \frac{dU}{dz} \right)_{z = z_j}$$

and the boundary conditions

$$U(0) = 0$$

$$W(0) = 1$$

$$U(h) = 0$$

$$W(h) = 0$$

The discrete solution for $U$ and $W$ may be calculated by the finite element method:

$$S \Delta = f$$

$S$ is a $2N \times 2N$ matrix assembled from element matrices $S^i$ given in Appendix I. $\Delta$, $f$ are $2N$-vectors with components

$$\Delta_1 = 0 \quad \Delta_2 = 1 \quad \Delta_{2j-1} = U_j \quad \Delta_{2j} = W_j \quad 2 \leq j \leq N$$

$$f_1 = -G_1 \left( -W + \frac{dU}{dz} \right)_{z = 0} \quad f_2 = -(\lambda_1 + 2G_1) \left. \frac{dW}{dz} \right|_{z = 0}$$

$$f_{2j-1} = 0 \quad f_{2j} = 0 \quad 2 \leq j \leq N$$

Note that the amplitudes of the normal and shear tractions on the surface are given by $xf_2$, $f_1$, respectively. Thus

$$U^{1,3}_{2j-1} = \Delta_{2j+1} \quad U^{1,3}_{2j} = \frac{L}{2} \Delta_{2j+2}$$

$$U^{2,3}_{2j-1} = \Delta_{2j+1} \quad U^{2,3}_{2j} = -\frac{L}{2} \Delta_{2j+2}$$

The consistent nodal forces are

$$F^{1,3} = H^{-L/2} \Delta$$

$$F^{2,3} = -H^{L/2} \Delta$$
The matrix $H^*$ is obtained by assembling the sublayer matrices $H^{x,j}$ given in Appendix I (here $H^*$ is evaluated at $x = -L/2$ and $x = L/2$ and the first two rows are deleted since the forces in (37c, d) correspond to nodes at $z = z_j, 2 \leq j \leq N$). The consistent loads at $(0, 0)$ are given by

$$F_x^3 = f_1 L + \frac{L}{2} \lambda_1 (1 - \Delta_4)$$

$$F_z^3 = 0$$

$$M_3 = \frac{1}{12} f_2 L^3 + (\frac{1}{3} G_1 h_1 - \frac{1}{2} G_1 \Delta_3 + \frac{1}{6} G_1 h_1 \Delta_4) L$$

$f_1, f_2, \Delta_3, \Delta_4$ are obtained from (36).

Clearly, a particular solution satisfying the inhomogeneous boundary conditions (22a, b) may be obtained as a linear combination of the particular solutions derived above. Thus any solution in the region is a linear combination of these particular semidiscrete solutions and the semidiscrete modes which satisfy the (homogeneous) boundary conditions (23a-d). Using (24), (29a–g), (33a–g), (37a–g) and exploiting the fact that the dynamic stiffness matrix of the element is symmetric it is easily obtained that

$$\begin{pmatrix} F^1 \\ F_x \\ F_z \\ M \end{pmatrix} = \begin{pmatrix} K^{11} & K^{1x} & K^{1z} & K^{1\theta} \\ K^{x1} & K_{xx} & K_{xz} & K_{x\theta} \\ K^{z1} & K_{zx} & K_{zz} & K_{z\theta} \\ K^{\theta1} & K_{\theta x} & K_{\theta z} & K_{\theta\theta} \end{pmatrix} \begin{pmatrix} U^1 \\ \Delta_x \\ \Delta_z \\ \theta \end{pmatrix}$$

(38)

Note that, since the horizontal translation and the rotation are uncoupled from the vertical translation,

$$K_{xx} = K_{zz} = 0 \quad K_{x\theta} = K_{\theta z} = 0$$

It is important to emphasize the fact that the computational effort required to calculate the stiffness matrix in (38) is independent of $L$, the length of the element. Details on this development may be found in Reference 9.

Application

In order to verify the derivations above, a simple application is considered: time-harmonic vibrations of a rigid and rough strip footing on the surface of a homogeneous stratum in plane strain. The boundary conditions on the surface of the stratum are

$$u(x, 0) = \Delta_x \quad \left| x \right| \leq b$$

$$w(x, 0) = \Delta_z - \theta x$$

$$\sigma_x \big|_{z=0} = 0$$

$$\sigma_z \big|_{z=0} = 0$$

$$\left| x \right| > b$$

the width of the footing being equal to $2b$. The base of the stratum is taken fixed. Let $F_x, F_z, M$ be the amplitudes of the horizontal force, vertical force and moment applied on the footing. Then

$$\begin{pmatrix} \Delta_z \\ \Delta_x \\ \theta \end{pmatrix} = \begin{pmatrix} F_{zz} & 0 & 0 \\ 0 & F_{xx} & F_{x\theta} \\ 0 & F_{x\theta} & F_{\theta\theta} \end{pmatrix} \begin{pmatrix} F_z \\ F_x \\ M \end{pmatrix}$$
The nondimensional compliances
\[ GF_{zz}, GF_{xx}, Gb^2F_{\theta\theta}, GbF_{x\theta} \]
are functions of the nondimensional quantities \((\omega h)/C_T (C_T = (G/\rho)^{1/2}\) being the velocity of transverse waves), \(h/b, \nu, \beta\). For given values of these quantities the compliances may be calculated by combining (Figure 3) the element (modelling the region \(-b \leq x \leq b, 0 \leq z \leq h\))

![Diagram of a strip footing on a stratum]

Figure 3. Scheme for the calculation of the stiffness of a strip footing on a stratum

developed above with the transmitting boundaries modelling the regions \(x < -b\) and \(x > b, 0 < z < h\) developed by Waas\(^4\) and reviewed earlier in this paper. The results were compared with those obtained using a conventional finite element mesh under the footing combined with the transmitting boundaries (such results have been reported by Chang-Liang\(^{12}\)). Figures 4a, b, c show plots of the nondimensional compliances \(GF_{zz}, GF_{xx}, Gb^2F_{\theta\theta}\) versus the nondimensional frequency \(1/(2\pi)(\omega h)/C_T\), for \(h/b = 2, \nu = 0.30, \beta = 0.05\). The stratum was divided into 10 sublayers of equal depth. For each frequency the computations take approximately 5.0 sec on IBM 370/165. It must be pointed out that the memory requirements are very low compared with those of a conventional finite element mesh with several columns of elements for accurate results. The agreement of the results obtained using the element developed above with those reported in Reference 12 is excellent; in fact, the difference between the results cannot be resolved within the scale of the drawings.

The technique is applicable to problems involving a variety of inhomogeneous boundary conditions. For example, an element with time-harmonic displacements specified at the base is easily obtained and so is an element with boundary conditions at the surface corresponding to a rigid and smooth footing.\(^9\) In all cases, semidiscrete particular solutions are found which satisfy the inhomogeneous boundary conditions and are then combined with semidiscrete modes satisfying the corresponding homogeneous boundary conditions.

ANTIPLANE SHEAR

Time-harmonic vibrations of a layered stratum in antiplane shear are now briefly considered. In sublayer \(j\), the governing differential equation is
\[
G_i \frac{\partial^2 v}{\partial x^2} + G_i \frac{\partial^2 v}{\partial z^2} + \rho \omega^2 v = 0
\]
Figure 4(a). The nondimensional vertical compliance of a strip footing on a stratum ($\nu = 0.30$, $\beta = 0.05$, $h/b = 2$)

Figure 4(b). The nondimensional horizontal compliance of a strip footing on a stratum ($\nu = 0.30$, $\beta = 0.05$, $h/b = 2$)
Figure 4(c). The nondimensional rocking compliance of a strip footing on a stratum ($\nu = 0.30$, $\beta = 0.05$, $h/b = 2$)

$v$ is the amplitude of the displacement in the $y$-direction (perpendicular to the $x$-$z$ plane). For antiplane shear the other two displacement components are equal to zero. Continuity conditions are imposed at interfaces of sublayers:

$$
t_{yz} \bigg|_{z = z_j} = G_{j-1} \frac{\partial v}{\partial z} \bigg|_{z = z_j} = G_{j} \frac{\partial v}{\partial z} \bigg|_{z = z_j} = t_{yz} \bigg|_{z = z_j}
$$

$$
v(x, z_j^-) = v(x, z_j^+)
$$

$$
2 \leq j \leq N
$$

$t_{yz}$ is the amplitude of the stress component acting on horizontal planes. Boundary conditions are specified on the surface and the base of the stratum. For a traction-free surface and a fixed base

$$
t_{yz} \bigg|_{z = 0} = G_{1} \frac{\partial v}{\partial z} \bigg|_{z = 0} = 0
$$

$$
v(x, h) = 0
$$

Solutions of the differential equation (39) which satisfy the continuity conditions (40a, b) and the boundary conditions (41a, b) are linear combinations of modes of the form

$$
v(x, z) = V(z) \exp (-ikx)
$$

The amplitude $V$ satisfies the differential equation

$$
k^2 G_1 V - G_1 \frac{d^2 V}{dz^2} - \rho \omega^2 V = 0
$$
in sublayer $j$, the conditions

$$ G_{j-1} \frac{dV}{dz} \bigg|_{z=z_j^-} - G_j \frac{dV}{dz} \bigg|_{z=z_j^+} = 0 $$

(44a)

$$ V(z_j^-) = V(z_j^+) $$

(44b)

at $z = z_j$, $2 \leq j \leq N$, and, finally, the boundary conditions

$$ \frac{dV}{dz} \bigg|_{z=0} = 0 $$

(45a)

$$ V(h) = 0 $$

(45b)

Following Lysmer and Waas$^3$ and Waas$^4$ the finite element method may be used to calculate discrete eigenfunctions $V$. It yields

$$ [k^2A + G - \omega^2M] \Delta = 0 $$

(46)

$A$, $G$, $M$ are $N \times N$ symmetric tridiagonal matrices assembled from element matrices $A^i$, $G^i$, $M^i$ given in Appendix I. $\Delta$ is an $N$-vector with components

$$ \Delta_j = V_j = V(z_j) \quad 1 \leq j \leq N $$

The algebraic eigenvalue problem (46) is linear in $k^2$. It may easily be solved by any of the standard methods. Its solution yields $2N$ wave numbers $k$ and the corresponding eigenvectors $\Delta$. Note that if $k$ is an eigenvalue with eigenvector $\Delta$ then $-k$ is another eigenvalue with eigenvector $\Delta$. The discrete eigenfunctions obtained from (46) lead to semidiscrete modes of the form

$$ v(x, z_j) = V_j \exp(-ikx) $$

(47)

Any (semidiscrete) solution in a region of a layered stratum with homogeneous boundary conditions may be written as a linear combination of the semidiscrete modes. Thus a dynamic stiffness matrix for the region $x \geq 0$ (consistent transmitting boundary) may be obtained (see the work by Lysmer and Waas$^3$ and Waas$^4$) as in the case of plane strain. It is also simple to calculate the dynamic stiffness matrix of the region $x_1 \leq x \leq x_2$ (see the paper by Kausel and Roesset$^8$), the computational effort being independent of the length $x_2 - x_1$ of the hyperelement. If inhomogeneous boundary conditions are specified, the procedure described earlier in this paper may be followed. For example, let the boundary condition corresponding to a rigid strip footing be imposed:

$$ v(x, 0) = \Delta, $$

(48)

$$ |x| \leq \frac{L}{2} $$

while the base is assumed fixed. The modes are found which satisfy the corresponding homogeneous boundary conditions

$$ v(x, 0) = 0 $$

(49a)

$$ v(x, h) = 0 $$

(49b)

To obtain these modes, an eigenvalue problem of the type considered above for a free surface and a fixed base must be solved. A particular solution may be obtained which satisfies the
inhomogeneous boundary condition (48). Substituting

\[ v(x, z) = V(z) \]  

(50)

into the differential equation (39), the conditions (40a, b) and the boundary conditions (48), (41b) it is seen that \( V \) must satisfy the differential equation

\[ G_j \frac{d^2 V}{dz^2} + \rho \omega^2 V = 0 \]  

(51a)

in sublayer \( j \), the conditions

\[ G_{j-1} \frac{dV}{dz} \bigg|_{z = z_j^-} = G_j \frac{dV}{dz} \bigg|_{z = z_j^+} \]  

(51b)

\[ V(z_j^-) = V(z_j^+) \]  

(51c)

at \( z = z_j \), \( 2 \leq j \leq N \), and the boundary conditions

\[ V(0) = \Delta, \]  

(51d)

\[ V(h) = 0 \]  

(51e)

Clearly, it is a very simple task to obtain the discrete solution for \( V \) using the finite element method. Any solution in the region may be written as a linear combination of this particular solution and the semidiscrete modes which satisfy the (homogeneous) boundary conditions (49a, b). Details may be found in Reference 9.

**AXISYMMETRIC ELEMENTS**

*Homogeneous boundary conditions*

Time-harmonic vibrations in axisymmetric regions of a layered stratum will now be considered. We use cylindrical co-ordinates \((r, \theta, z)\). The amplitudes of the radial, tangential and axial (vertical) displacements are denoted by \( u, v \) and \( w \), respectively. The modes of wave motion are given by (see the work by Kausel) \(^5\)

\[ u(r, \theta, z) = kU(z)C_n(kr) \begin{pmatrix} \cos(n\theta) \\ \sin(n\theta) \end{pmatrix} \]  

(52a)

\[ w(r, \theta, z) = -ikW(z)C_n(kr) \begin{pmatrix} \cos(n\theta) \\ \sin(n\theta) \end{pmatrix} \]  

(52b)

\[ v(r, \theta, z) = -rU(z)C_n(kr) \begin{pmatrix} -\sin(n\theta) \\ \cos(n\theta) \end{pmatrix} \]  

(52c)

and

\[ u(r, \theta, z) = \frac{n}{r} V(z)C_n(kr) \begin{pmatrix} \cos(n\theta) \\ \sin(n\theta) \end{pmatrix} \]  

(53a)

\[ w(r, \theta, z) = 0 \]  

(53b)

\[ v(r, \theta, z) = k V(z)C_n'(kr) \begin{pmatrix} -\sin(n\theta) \\ \cos(n\theta) \end{pmatrix} \]  

(53c)

\[ n = 0, 1, 2, \ldots \]
For symmetric modes \( \cos(n\theta) \) must be used for \( u \) and \( w \), while \(-\sin(n\theta)\) must be chosen for \( v \) (symmetry of the displacement field with respect to the plane \( \theta = 0 \)). For antisymmetric modes \( \sin(n\theta) \) must be used for \( u \) and \( w \) and \( \cos(n\theta) \) for \( v \). \( C_n \) is any solution of Bessel's equation of order \( n \):

\[
C''_n + \frac{1}{\xi} C'_n + \left(1 - \frac{n^2}{\xi^2}\right) C_n = 0
\]  

(54)

the prime indicating differentiation with respect to the argument \( \xi \). It is easily shown that the wave number \( k \) and the eigenfunctions \( U \) and \( W \) in (52a–c) satisfy the same eigenvalue problem as the wave number and eigenfunctions corresponding to a mode of vibration of the stratum in plane strain, i.e. \( k, U \) and \( W \) satisfy the differential equations (5a, b), the continuity conditions (6a–d) at interfaces of sublayers and, if the surface is assumed free and the base is taken fixed, the boundary conditions (7a–d). Similarly, it is easily derived that the wave number \( k \) and the eigenfunction \( V \) in (53a–c) satisfy the same eigenvalue problem as the wave number and eigenfunction corresponding to a mode of vibration of the stratum in antiplane shear, i.e. \( k, V \) satisfy the differential equation (43), the continuity conditions (44a, b) and, if the surface is traction-free and the base is fixed, the boundary conditions (45a, b). Thus the algebraic eigenvalue problems considered in the discussion of plane elements apply to the analysis of time-harmonic wave motion in axisymmetric regions and semidiscrete modes corresponding to (52a–c) and (53a–c) are easily obtained.

Waas\(^4\) considered time-harmonic vibrations in the region \( r \geq r_0 > 0 \) (Figure 5a) for the Fourier number \( n = 0 \), i.e. axisymmetric vibrations, and derived a dynamic stiffness matrix for the region imposing radiation and boundedness conditions as \( r \to \infty \) (a consistent transmitting axisymmetric boundary). Later, Kausel\(^5\) generalized this development for arbitrary Fourier number \( n \). The solution of Bessel's equation to be used in this region is the Hankel function of the second kind of order \( n \).

\[ H_n^{(2)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{1/2} \exp \left(-ikr + i\frac{n\pi}{2} + \frac{\pi}{4}\right) \]

indicates that the wave numbers for which the radiation and boundedness conditions are satisfied are the same as those for the transmitting boundaries in plane strain and antiplane shear. To obtain the dynamic stiffness matrix of the consistent transmitting boundary the vector of amplitudes of nodal displacements is written as a linear combination of the vectors corresponding to the semidiscrete modes and the vector of consistent nodal forces as the same linear combination, i.e. using the same participation factors, of the vectors of forces corresponding to the modes and the participation factors are eliminated. Details may be found in the works by Waas\(^4\) and Kausel\(^5\). The development is also reviewed in Reference 9.

Kausel and Roësset\(^8\) considered the regions \( 0 \leq r \leq r_0 \) (Figure 5b) and \( 0 < r_1 \leq r \leq r_2 \) (Figure 5c). The solution of Bessel's equation to be used in the region \( 0 \leq r \leq r_0 \) is the Bessel function of order \( n \), \( J_n \), since any solution of Bessel's equation of order \( n \) that is nonsingular at zero argument is a multiple of \( J_n \). For the region \( 0 < r_1 \leq r \leq r_2 \) any two linearly independent solutions of Bessel's equation are appropriate, e.g. \( H_n^{(1)} \) and \( H_n^{(2)} \) or \( H_n^{(1)} \) and \( J_n \), etc. Dynamic stiffness matrices for these elements were developed working as for the plane elements. Details of the derivation may be found in the paper by Kausel and Roësset.\(^8\) Note that the computational effort required to obtain the stiffness matrices is independent of the extent of the elements in the radial direction, i.e. independent of \( r_0 \) for the region \( 0 \leq r \leq r_0 \) and \( r_2 - r_1 \) for the region \( 0 < r_1 \leq r \leq r_2 \).
Inhomogeneous boundary conditions

The developments reviewed above involved homogeneous boundary conditions on the surface and the base of the elements. If inhomogeneous boundary conditions are specified, the procedure described for the plane elements may be used. For example, consider the boundary conditions corresponding to a rigid and rough circular or ring footing.

(a) Torsional vibrations. The boundary condition at $z = 0$ (surface) is

$$ v(r, \theta, 0) = r \phi, $$

(55a)

$$ 0 \leqslant r \leqslant r_0 \quad \text{or} \quad 0 < r_1 \leqslant r \leqslant r_2 $$
The base is taken fixed:

\[ v(r, \theta, h) = 0 \]  \hspace{2cm} (55b)

\( \phi \), is the rotation of the footing. Solutions may be found which satisfy the corresponding homogeneous boundary conditions

\[ v(r, \theta, 0) = 0 \]  \hspace{2cm} (56a)

\[ v(r, \theta, h) = 0 \]  \hspace{2cm} (56b)

Such solutions are linear combinations of antisymmetric modes, with \( n = 0 \), of the form (53a–c). The eigenfunctions \( V \) and wave numbers \( k \) are obtained as in the corresponding plane problem, i.e. they satisfy the differential equation (43), the conditions (44a, b) and the boundary conditions indicating a fixed surface and a fixed base. A particular solution satisfying the inhomogeneous boundary condition (55a) may be found as

\[ v(r, \theta, z) = r V(z) \]  \hspace{2cm} (57)

It is easily shown that \( V \) satisfies the differential equation (51a), the conditions (51b, c) and the boundary conditions

\[ V(0) = \phi \]  \hspace{2cm} (58a)

\[ V(h) = 0 \]  \hspace{2cm} (58b)

Therefore, \( V \) is a multiple of that in (51a–d) which was obtained in our discussion of plane elements in connection with wave motion in antiplane shear. Having found the corresponding semidiscrete modes and the semidiscrete particular solution, the dynamic stiffness matrix may be obtained for antisymmetric, \( n = 0 \) (i.e. independent of \( \theta \)), vibrations in the regions \( 0 \leq r \leq r_0 \) and \( 0 < r_1 \leq r \leq r_2 \).

(b) Vertical vibrations. Vertical vibrations will now be considered:

\[ u(r, \theta, 0) = 0 \]  \hspace{2cm} (59a)

\[ w(r, \theta, 0) = \Delta_z \]  \hspace{2cm} (59b)

\[ 0 \leq r \leq r_0 \] \hspace{2cm} or \hspace{2cm} \[ 0 < r_1 \leq r \leq r_2 \]

\( \Delta_z \) is the amplitude of vertical vibrations of the footing. The modes satisfying the corresponding homogeneous boundary conditions are symmetric solutions with \( n = 0 \) of the form (52a–c). A particular solution satisfying the boundary conditions is obtained as

\[ u(r, \theta, z) = 0 \]  \hspace{2cm} (60a)

\[ w(r, \theta, z) = W(z) \]  \hspace{2cm} (60b)

The solution for \( W \) is a multiple of the solution obtained for vertical vibrations of a strip footing in (30a, b). Thus, again, by combining a particular solution with the modes, the dynamic stiffness matrix of the elements for symmetric \( n = 0 \) (i.e. independent of \( \theta \)) vibrations is obtained.

(c) Horizontal vibrations and rocking. For horizontal vibrations and rocking the boundary conditions are

\[ u(r, \theta, 0) = \Delta_x \cos \theta \]  \hspace{2cm} (61a)

\[ w(r, \theta, 0) = -r \phi, \cos \theta \]  \hspace{2cm} (61b)
\[ u(r, \theta, 0) = -\Delta_0 \sin \theta \] (61c)

\[ 0 \leq r \leq r_0 \quad \text{or} \quad 0 < r_1 \leq r \leq r_2 \]

Solutions satisfying the corresponding homogeneous boundary conditions are linear combinations of symmetric modes with \( n = 1 \) of the form (52a–c), (53a–c). The eigenvalues \( k \) and the corresponding eigenfunctions \( U, W \) are obtained from the problem considered earlier for wave motion in plane strain. The eigenvalues \( k \) and the corresponding eigenfunctions \( V \) are found from the eigenvalue problem obtained for antiplane shear. A particular solution satisfying (61a–c) with \( \phi_s = 0 \) may be calculated as

\[ u(r, \theta, z) = U(z) \cos \theta \] (62a)

\[ w(r, \theta, z) = 0 \] (62b)

\[ v(r, \theta, z) = -U(z) \sin \theta \] (62c)

The solution for \( U \) is a multiple of the solution obtained for horizontal vibrations of a strip footing in (25a, b). Another particular solution satisfying (61a–c) with \( \Delta_0 = 0 \) may be found as

\[ u(r, \theta, z) = U(z) \cos \theta \] (63a)

\[ w(r, \theta, z) = -rW(z) \cos \theta \] (63b)

\[ v(r, \theta, z) = -U(z) \sin \theta \] (63c)

Again, the solution for \( U \) and \( W \) is a multiple of that obtained for rocking of a strip footing in (34a, b). These particular solutions, together with the modes, suffice for the calculation of the dynamic stiffness matrix of the elements in the case of horizontal vibrations and rocking of the footing which is imposed on the elements. Detailed derivations may be found in Reference 9. It must be noted, once more, that the computational effort involved in the calculation of the stiffness matrix is independent of the extent of the elements in the radial direction.

**Application**

In order to verify the developments above, the dynamic stiffness of a rigid and rough circular footing on the surface of a homogeneous stratum is calculated. The boundary conditions on

![Figure 6. Scheme for the calculation of the stiffness of a circular footing on the surface of a stratum](image-url)
the surface are

\[
\begin{align*}
  u(r, \theta, 0) &= \Delta_x \cos \theta \\
  w(r, \theta, 0) &= \Delta_z - r\phi, \cos \theta \\
  v(r, \theta, 0) &= r\phi - \Delta_x \sin \theta
\end{align*}
\]

0 \leq r \leq R

\( R \) is the radius of the footing. Tractions are equal to zero for \( r > R \). The base of the stratum is fixed. Let \( F_x, F_z, M_n, M_t \) be the amplitudes of the horizontal force, vertical force, rocking

Figure 7(a). Torsional stiffness coefficients (circular footing, \( \beta = 0.05, h/R = 2 \))

Figure 7(b). Vertical stiffness coefficients (circular footing, \( \beta = 0.05, \nu = 1/3, h/R = 2 \))
moment and torsional moment, respectively:
\[
\begin{pmatrix}
M_t \\
F_z \\
F_x \\
M_r
\end{pmatrix} =
\begin{pmatrix}
K_{tt} & 0 & 0 & 0 \\
0 & K_{zz} & 0 & 0 \\
0 & 0 & K_{xx} & K_{xt} \\
0 & 0 & K_{rx} & K_{rr}
\end{pmatrix}
\begin{pmatrix}
\phi_t \\
\Delta_z \\
\Delta_z \\
\phi_r
\end{pmatrix}
\]

For the calculations the elements considered above (with a rigid and rough circular footing imposed on the surface) were combined (Figure 6) with the consistent transmitting boundaries.
developed by Waas\textsuperscript{4} and Kausel.\textsuperscript{5} The results, reported in References 4, 5, and 13, were obtained by combining the transmitting boundaries with a conventional finite element mesh in the region under the footing. Figures 7a–d show plots of the stiffness coefficients versus the nondimensional frequency $1/(2\pi)[(\omega R)/C_T]$ for $h/R = 2$, $\nu = 1/3$, $\beta = 0.05$. Note that the stiffness coefficients are related to the stiffnesses by

$$\frac{K_{nt}}{GR^3} = \frac{K_{nt}^0}{GR^3} \left( k_{nt} + \frac{\omega R}{C_T} c_n \right)(1 + 2i\beta)$$

e etc.

The static stiffnesses are

$$\frac{K_{nt}^0}{GR^3} = 5.79$$

$$\frac{K_{zz}}{GR} = 10.37$$

$$\frac{K_{zz}}{GR} = 6.36$$

$$\frac{K_{nn}}{GR^3} = 4.63$$

The stratum was divided into 12 sublayers of equal depth. For each frequency the computation of the torsional stiffness takes approximately 0.8 sec on IBM 370/165, the computation of the vertical stiffness 6.0 sec and the computation of the horizontal and rocking stiffnesses 9.5 sec. Again, a significant advantage of using the elements considered in this paper is that the memory requirements are very low compared to those of a conventional finite element mesh which is fine enough for accurate results. The agreement of the results obtained using the elements described in this paper with those reported in References 4, 5 and 13 is excellent.

It is not possible to resolve the difference within the scale of the drawings.

CONCLUSIONS

In this paper some elements were presented for the numerical analysis of wave motion in layered strata. The technique developed in earlier works\textsuperscript{3,4–8} has been extended to the analysis of wave motion in regions of a layered stratum with inhomogeneous boundary conditions. Semidiscrete particular solutions are obtained satisfying the inhomogeneous boundary conditions. Moreover, semidiscrete modes satisfying the corresponding homogeneous boundary conditions are easily calculated. In previous works\textsuperscript{4,5} a semidiscrete solution (satisfying homogeneous boundary conditions) for part of the region was combined with a fully discrete solution (satisfying inhomogeneous boundary conditions) for the rest of the region. In this paper it was demonstrated that a semidiscrete solution may be obtained by combining semidiscrete particular solutions and semidiscrete modes. It was noted that the computational effort becomes independent of the extent of the region in the horizontal direction. Moreover, the memory requirements for the elements considered here are low compared to those of a conventional finite element mesh. Furthermore, another advantage of the technique is the fact that the displacements, stresses and strains at any point in the region may be expressed in terms of relatively few parameters, namely, the participation factors of the semidiscrete modes and particular solutions.
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APPENDIX I

Assuming linear interpolation, the matrices appearing in the development of the plane elements considered in this paper are given by the following.

*Plane strain*

\[
A' = \frac{1}{\delta h_j} \begin{pmatrix}
2(\lambda_j + 2G_j) & 0 & \lambda_j + 2G_j & 0 \\
0 & 2G_j & 0 & G_j \\
\lambda_j + 2G_j & 0 & 2(\lambda_j + 2G_j) & 0 \\
0 & G_j & 0 & 2G_j
\end{pmatrix}
\]

\[
B' = \frac{1}{2} \begin{pmatrix}
0 & -(\lambda_j - G_j) & 0 & \lambda_j + G_j \\
\lambda_j - G_j & 0 & \lambda_j + G_j & 0 \\
0 & -(\lambda_j + G_j) & 0 & \lambda_j - G_j \\
-(\lambda_j + G_j) & 0 & -(\lambda_j - G_j) & 0
\end{pmatrix}
\]

\[
G' = \frac{1}{h_j} \begin{pmatrix}
G_i & 0 & -G_i & 0 \\
0 & \lambda_j + 2G_i & 0 & -(\lambda_j + 2G_i) \\
-G_i & 0 & G_i & 0 \\
0 & -(\lambda_j + 2G_i) & 0 & \lambda_j + 2G_i
\end{pmatrix}
\]

\[
M' = \rho h_j \begin{pmatrix}
\frac{1}{3} & 0 & \frac{1}{6} & 0 \\
0 & \frac{1}{3} & 0 & \frac{1}{6} \\
\frac{1}{6} & 0 & \frac{1}{3} & 0 \\
0 & \frac{1}{6} & 0 & \frac{1}{3}
\end{pmatrix}
\]

\[
D' = \frac{1}{2} \begin{pmatrix}
0 & \lambda_j & 0 & -\lambda_j \\
G_i & 0 & -G_i & 0 \\
0 & \lambda_j & 0 & -\lambda_j \\
G_i & 0 & -G_i & 0
\end{pmatrix}
\]

\[
S' = \begin{pmatrix}
\frac{G_i}{h_j} - \frac{1}{3} \omega^2 \rho h_j & -\frac{1}{3}(\lambda_j - G_i) & -\frac{G_i}{h_j} - \frac{1}{3} \omega^2 \rho h_j & \frac{1}{3}(\lambda_j + G_i) \\
0 & -\frac{\lambda_j + 2G_i}{h_j} + \frac{1}{3} \omega^2 \rho h_j & 0 & \frac{\lambda_j + 2G_i}{h_j} + \frac{1}{3} \omega^2 \rho h_j \\
-\frac{G_i}{h_j} - \frac{1}{3} \omega^2 \rho h_j & -\frac{1}{3}(\lambda_j + G_i) & \frac{G_i}{h_j} - \frac{1}{3} \omega^2 \rho h_j & \frac{1}{3}(\lambda_j - G_i) \\
0 & \frac{\lambda_j + 2G_i}{h_j} + \frac{1}{3} \omega^2 \rho h_j & 0 & -\frac{\lambda_j + 2G_i}{h_j} + \frac{1}{3} \omega^2 \rho h_j
\end{pmatrix}
\]
J. L. TASSOULAS AND E. KAUSEL

\[ \mathbf{H}^{'ij} = \begin{pmatrix}
0 & -(x/2)\lambda_j & 0 & (x/2)\lambda_j \\
\frac{1}{2}G_i & \frac{1}{2}G_jh_j & -\frac{1}{2}G_i & \frac{1}{2}G_ih_j \\
0 & -(x/2)\lambda_j & 0 & (x/2)\lambda_j \\
\frac{1}{2}G_i & \frac{1}{2}G_jh_j & -\frac{1}{2}G_i & \frac{1}{2}G_ih_j
\end{pmatrix} \]

**Antiplane shear**

\[ \mathbf{A}' = G_ih_j \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6}
\end{pmatrix} \]

\[ \mathbf{G}' = \frac{G_i}{h_j} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} \]

\[ \mathbf{M}' = \rho_ih_j \begin{pmatrix}
\frac{1}{3} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6}
\end{pmatrix} \]

**REFERENCES**