Numerical dispersion in the thin-layer method

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Abstract

The thin-layer method (TLM) is an effective numerical tool for the analysis of wave motions in laminated media. In a nutshell, the TLM combines the finite element method in the direction of layering together with analytical solutions for the remaining directions. This partial discretization introduces some numerical dispersion in the TLM, the degree of which depends on the refinement of the model. In this paper, we first characterize this numerical dispersion for both anti-plane (SH) and in-plane (SV–P) body waves in an unbounded medium. We then develop optimal tuning factors, with the aid of which the numerical dispersion error is minimized and the accuracy of the solution improved. Finally, we verify the effectiveness of the tuning factors by comparing the numerical results obtained with the TLM against the exact results of canonical models for guided waves in plates, and for Love and for Rayleigh waves in semi-infinite media.

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1. Introduction

The thin-layer method (TLM) is a semi-discrete numerical tool for the dynamic analysis of laminated media. It combines the advantages of finite elements with the rigor of analytical solutions, and consists in a partial discretization of the medium in the direction of lamination. When an elastic medium subjected to propagating waves is modeled with the TLM, or for that matter with finite elements [3,5], the spatial discretization introduces numerical dispersion, which slightly alters the path of waves as well as their velocity of propagation. This shows up as a discrepancy between the solutions for the discrete and continuous models, the intensity of which increases with the frequency of the waves. In as much as this numerical artifact degrades the accuracy of the wave fields obtained with the discrete model, it behooves to minimize this degradation to the extent possible. With this objective in mind, we first characterize the numerical dispersion in the TLM for various wave types, and then determine optimal tuning factors to minimize the numerical dispersion error.

Since its inception in the 1970s [18,23], the TLM has evolved into an efficient technique for the analysis of wave motion in laminated plates, layered soils and other viscoelastic media. Because the TLM is a partial-discretization technique, it requires relatively small computational effort in comparison with other full-discretization methods, such as the finite differences and finite element methods, provided that the medium is sufficiently regular in the remaining coordinate directions that analytical solutions can be found. However, irregularly shaped domains can also be approached by coupling the TLM with finite elements. This allows solving, for example, problems of soil-structure interaction where the soil island is irregular either in shape or in material conditions [6,7,11,19]. More importantly, the TLM can provide the fundamental solutions or Green’s functions for arbitrarily multi-layered media [14,17], which allow application of the TLM to the boundary
element method [15]. More recently, [16] presented a formulation of the TLM in the time domain, which avoids error-prone inverse Fourier transformations into the time domain. In the intervening years, the TLM has become an essential tool for the solution of practical problems in engineering, such as non-destructive testing via ultrasonic vibrations, moving loads on pavements and bridge decks, or the dynamic response of foundations on layered soils. Nevertheless, there remain unexplored aspects, one of which is the assessment of the numerical dispersion and the general accuracy of the TLM.

Over the past several decades, numerous studies on the numerical dispersion and accuracy of discrete models have appeared in the technical literature. For example, Belytschko and Mullen [2] derived the spectrum equations for one-dimensional wave motion using both linear and quadratic finite elements, and proposed an averaging of lumped and consistent mass matrices to increase the accuracy of the results. Also, Mullen and Belytschko [20] showed that the diagonalization of the mass matrix and under-integration of the stiffness matrix both markedly increase dispersive numerical errors in quadrilateral elements. Ahmadian et al. [1] constructed stiffness and mass matrices by using an inverse parametric model approach, whose parameters are similar, but not identical to the optimal tuning factors considered herein. Galan and Abascal [24] applied the TLM to study the scattering of Lamb waves in both homogeneous and sandwich plates, and applied a mesh criterion to minimize discretization errors. Also, they found that a TLM based on a quadratic expansion is more efficient and accurate than one based on a linear expansion. This result agrees with that of Park [22], who estimated the accuracy of the TLM for both displacements and for stresses. Harari [8] designed a Galerkin least square (GLS) resolution-dependent method parameter for two-dimensional computation based on the dispersion analysis of the one-dimensional, square bilinear element. By means of the GLS modification, Harari and Nogueira [9] achieved improvements in the application of finite element to time-harmonic acoustics at high wavenumbers.

2. Numerical wave dispersion in the TLM

In the sections that follow, we explore the numerical dispersion of the TLM for both linear and quadratic expansions using for this purpose the exact solution to the spectrum equation for the discrete wave equation. Following Belytschko and Mullen [2], we obtain generalized matrices by linear combinations of lumped and consistent matrices via optimized tuning factors, but unlike these authors, we apply this procedure to both the mass and stiffness matrices involved in the TLM. To validate the effectiveness of the optimal tuning factors found, we compare the eigenvalues for the exact solution against the response obtained with the TLM. To this end, we solve canonical problems of guided-waves in a homogeneous plate with mixed boundary conditions, body and Rayleigh waves in a homogeneous half-space, and Love waves in an elastic layer over a homogeneous half-space. Because of their infinite size, the latter two canonical problems are strictly unsolvable in the classical finite element sense. However, using the general discrete solutions expounded in this paper, we find formal solutions to these problems as well.

To assess the dispersion, we first obtain the wavenumber–frequency spectra for these canonical problems modeled with the TLM, solve the requisite discrete wave equations, and then determine the numerical error entailed in the models. Not surprisingly, we find that a limiting relationship exists between the wavelength in the direction of discretization and the characteristic measure of discretization. On the basis of these results, and to avoid discretization errors in practical applications, we establish a lower bound to the number of thin-layers per wavelength needed in a model together with optimal tuning factors for the matrices involved.

2.1. Brief review of the TLM

Consider a horizontally layered, viscoelastic system, such as a plate or soil, subjected to dynamic sources somewhere in the medium. Within each layer, the material properties are constant across the thickness. While the elastic constants may not necessarily be isotropic, in this paper we restrict attention to such materials only. The medium is bounded by an upper and lower interface at which either stresses or displacements are prescribed (i.e. free, fixed or mixed boundary conditions). Within the body of the medium, and for plane strain conditions, the dynamic equilibrium is governed by the scalar and/or vector wave equations, depending on whether anti-plane or in-plane load conditions are being considered.

Let \( x, y, z \) be respectively, the two horizontal and vertical coordinates, with \( x \) from left to right and \( z \) pointing up. If the layers are thin in the finite element sense, then it is possible to carry out a discretization of the displacement field in the direction of layering (i.e. in the \( z \)-direction), to which the method of weighted residuals can be applied. For plane strain conditions, this changes the wave equation involving partial derivatives with respect to \( x, z \) into a system of ordinary differential equations in \( x, t \) of the form [10,12,13,25]

\[
p(x,t) = \mathbf{M}\ddot{u} + \mathbf{K}u + \mathbf{p}_0(t)
\]

(2.1)

in which the load vector \( \mathbf{p}_0(t) \) contains the consistent external sources (tractions) applied at the layer
interfaces, \( \mathbf{u} = \mathbf{u}(x, t) \) is the vector of interface displacements, primes denote differentiation with respect to \( x \), double dots indicate time derivatives, and \( \mathbf{M}, \mathbf{A}, \mathbf{B} \) and \( \mathbf{G} \) are narrowly banded matrices that solely depend on the material constants and layer thicknesses, and except for \( \mathbf{B} \) that is anti-symmetric and exists only in the in-plane case, they are symmetric. Coupling between horizontal and vertical degrees of freedom takes place only through the \( \mathbf{B} \) matrix.

For harmonic waves of the form \( \mathbf{u}(x, t) = \mathbf{U}(k_x, \omega) \exp(i(\omega t - k_x x)) \), Eq. (2.1) can be transformed into the fully symmetric form

\[
\mathbf{P}(k_x, \omega) = [\mathbf{A}k_x^2 + \mathbf{B}k_x + \mathbf{G} - \omega^2 \mathbf{M}]\mathbf{U} = \mathbf{KU}
\]

which requires inserting a factor \( i = \sqrt{-1} \) in front of each vertical component of both \( \mathbf{U} \) and \( \mathbf{P} \). Here, \( \mathbf{K} \) can be interpreted as the dynamic stiffness (or impedance) matrix of the layered system in \( k - \omega \) space. This is the discrete wave equation for the system of layers in the frequency–wavenumber domain, and constitutes the starting point for the equations presented in this paper.

2.2. Tuning factors \( \mu, \alpha \)

In general, the formulation of dynamic problems by means of discrete methods leads to consistent element mass matrices \( \mathbf{M} \) that have the same bandwidth as the element stiffness matrices. For computational efficiency, \( \mathbf{M}_c \) is often replaced with the lumped mass matrix \( \mathbf{M}_l \), which can be derived from \( \mathbf{M}_c \) by simple addition of the off-diagonal terms to the diagonal terms. In the context of modal analysis, the use of a consistent mass matrix generally leads to eigenvalues (i.e. natural frequencies) that exceed the exact eigenvalues of the continuous model. By contrast, the use of a lumped mass matrix generally produces eigenvalues that are smaller than those of the continuous model. This observation suggests that a hybrid between the consistent and lumped mass matrices may improve the results of discrete models, and this is indeed the case [2].

Concerning the TLM, when Eq. (2.2) is examined in detail, it is found that the two matrices \( \mathbf{A} \) and \( \mathbf{M} \) have the same internal block–tridiagonal structure and element arrangement. This suggests that it may be possible to tune optimally not only the mass matrix \( \mathbf{M} \), but also the stiffness component matrix \( \mathbf{A} \) so as to further improve the wave propagation characteristics, especially for anti-plane problems. In this paper, the two matrices \( \mathbf{A} \) and \( \mathbf{M} \) are thus constructed by linear combinations of their lumped and consistent representations via tuning factors \( \alpha \) and \( \mu \), respectively:

\[
\mathbf{M} = (1 - \mu)\mathbf{M}_l + \mu\mathbf{M}_c \quad \text{with } 0 \leq \mu \leq 1
\]

\[
\mathbf{A} = (1 - \alpha)\mathbf{A}_l + \alpha\mathbf{A}_c \quad \text{with } 0 \leq \alpha \leq 1
\]

2.3. Discrete wave equations (DWE)

Consider a homogeneous layered system composed of identical layers in both thickness and material properties. Numbering the layers in ascending order from the bottom up (i.e. following the positive \( z \)-direction), then each individual layer \( n \) of thickness \( h \) is characterized by an equation of the form \( \mathbf{P}_n = \mathbf{K}_n\mathbf{U}_n \). Interface nodes are also identified by this index \( n \), while internal nodes for quadratic expansions are assigned half-order indices of the form \( n \pm 1/2 \), which agree with their location at the center of the layers. The discrete wave equation (DWE) for the system of layers is then obtained by overlapping two contiguous element impedance matrices \( \mathbf{K}_n \) and setting the source term (i.e. the loads or interface tractions) to zero. At an external stress or Neumann boundary condition (NBC), however, no overlapping is required. On the other hand, since all layers are identical, the elements of the impedance matrix \( \mathbf{K}_n \) can be normalized appropriately by the shear modulus and element thickness, which renders them dimensionless. This requires introduction of dimensionless horizontal and vertical wavenumbers \( \zeta_x, \zeta_z \) and dimensionless frequency \( \Omega \)

\[
\zeta_x = k_x h, \quad \zeta_z = k_z h, \quad \text{and } \Omega = \frac{\omega h}{C_s}
\]

Omitting details, the discrete wave equations are found to be as follows (see [22]):

2.3.1. Anti-plane (\( \text{SH} \)) waves

\[
\mathbf{P}_n = [\mathbf{A}k_x^2 + \mathbf{G} - \omega^2 \mathbf{M}]\mathbf{U} = \mathbf{K}_n\mathbf{U}_n
\]

Linear expansion:

\[
\mathbf{P}_n = \{T_\tau \}^T \quad \mathbf{U}_n = \{T_v \}^T
\]

\[
\begin{align}
\alpha v_{n+1} + 2bv_n + \alpha v_{n-1} &= 0 & \text{DWE, interface node} \\
bv_0 + \alpha v_{-1} &= 0 & \text{NBC, upper boundary, say } n = 0
\end{align}
\]

with auxiliary parameters

\[
\begin{align}
a &= \alpha \zeta_x^2 \frac{2}{\zeta_x^2} - 6 - \mu \Omega^2, \\
b &= (3 - \alpha) \zeta_z^2 \frac{2}{\zeta_z^2} + 6 - (3 - \mu) \Omega^2
\end{align}
\]

Quadratic expansion:

\[
\mathbf{P}_n = \{T_\tau \}^T \quad \mathbf{U}_n = \{T_v \}^T
\]

\[
\begin{align}
\alpha v_{n+1} + 2bv_n + \alpha v_{n-1} &= 0 & \text{DWE, interface node} \\
bv_0 + \alpha v_{-1} &= 0 & \text{NBC, upper boundary, say } n = 0
\end{align}
\]

with auxiliary parameters

\[
\begin{align}
a &= \alpha \zeta_x^2 \frac{2}{\zeta_x^2} - 6 - \mu \Omega^2, \\
b &= (3 - \alpha) \zeta_z^2 \frac{2}{\zeta_z^2} + 6 - (3 - \mu) \Omega^2
\end{align}
\]
2.3.2. In-plane (SV-P) waves

\[ P_l = [A\kappa^2 + BK + G - \omega^2 M]U = K_p U_l \]

\[ r = \frac{\lambda + 2G}{G} = \left( \frac{C_p}{C_s} \right)^2 = \frac{2(1-v)}{1-2v} \]

Linear expansion:

\[ P_l = \{ \tau_l \quad i\sigma_l \quad \tau_{l-1} i\sigma_{l-1} \}^T \]

\[ U_l = \{ u_l \quad iw_l \quad u_{l-1} \quad iw_{l-1} \}^T \]

\[ a^T u_{l+1} + 2b u_l + a u_{l-1} = 0 \quad \text{DWE, interface node} \] 

\[ b u_0 + a u_{-1} = 0 \quad \text{NBC, upper boundary} \]

with auxiliary parameters

\[ a = \begin{bmatrix} a_{11} & a_{12} \\ -a_{12} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} \\ . \\ b_{22} \end{bmatrix} \]

\[ b_0 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} \]

\[ a_{11} = x r \xi^2 - \mu \Omega^2 - 6, \quad b_{11} = r(5 - \alpha)\xi^2 - (3 - \mu)\Omega^2 + 6 \]

\[ a_{22} = x \xi^2 - \mu \Omega^2 - 6r, \quad b_{22} = (5 - \alpha)\xi^2 - (3 - \mu)\Omega^2 + 6r \]

\[ a_{12} = -3(r - 1)\xi^2, \quad b_{12} = 3(r - 3)\xi^2 \]

(2.13a–i)

Quadratic expansion:

\[ P_l = \{ \tau_l \quad i\sigma_l \quad \tau_{l-1/2} \quad i\sigma_{l-1/2} \quad \tau_{l-1} \quad i\sigma_{l-1} \}^T, \]

\[ U_l = \{ u_l \quad iw_l \quad u_{l-1/2} \quad iw_{l-1/2} \quad u_{l-1} \quad iw_{l-1} \}^T \]

\[ a^T u_{l+1} + a^T u_{l+1/2} + 2b u_l + a u_{l-1/2} + a u_{l-1} = 0 \]

DWE, interface node

\[ \hat{a}^T u_l + 2b u_{l-1/2} + \hat{a} u_{l-1} = 0 \]

DWE, internal node

\[ b_0 u_0 + \hat{a} u_{-1} + \hat{a} u_{-1} = 0 \quad \text{NBC, upper boundary} \]

Here, the matrices and parameters are:

\[ a = \begin{bmatrix} a_{11} & a_{12} \\ -a_{12} & a_{22} \end{bmatrix}, \quad \hat{a} = \begin{bmatrix} \hat{a}_{11} \\ -\hat{a}_{12} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} \\ . \\ b_{22} \end{bmatrix}, \quad b_0 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} \]

\[ a_{11} = x r \xi^2 - \mu \Omega^2 + 10 \]

\[ b_{11} = r(5 - \alpha)\xi^2 - (3 - \mu)\Omega^2 + 70 \]

\[ a_{22} = x \xi^2 - \mu \Omega^2 + 20r \]

\[ b_{22} = (5 - \alpha)\xi^2 - (3 - \mu)\Omega^2 + 70r \]

\[ a_{12} = 5(r - 1)\xi^2, \quad b_{12} = 15(r - 3)\xi^2 \]

\[ \hat{a}_{11} = 2(x r \xi^2 - \mu \Omega^2 - 40) \]

\[ \hat{b}_{11} = 2(r(5 - \alpha)\xi^2 - (3 - \mu)\Omega^2 + 40) \]

\[ \hat{a}_{22} = 2(x \xi^2 - \mu \Omega^2 - 40r) \]

\[ \hat{b}_{22} = 2((5 - \alpha)\xi^2 - (3 - \mu)\Omega^2 + 40r \]

\[ \hat{a}_{12} = -20(r - 1)\xi^2 \]

(2.16)

2.4. Solution of the discrete wave equations

Consider a homogeneous, viscoelastic medium that has been divided into thin, discrete layers, and which is subjected to free waves, that is to waves that propagate in the absence of sources. Internal points in the medium are characterized by the discrete wave equations just presented, while external stress-free boundaries are defined by the matching NBC boundary conditions. In the case of a full space, there is an infinite number of layers, so such a system would involve an infinite matrix equation. However, since all layers are equal, the solution for free waves can be found by simply solving the homogeneous difference equations, i.e. the discrete wave equations.

The general solution of the discrete wave equations can be obtained by assuming trial solutions or ansatz of the form
The two solutions of (3.1) are then
discrete wave equations, in which

\[ u_l \] (2.17a)

and

\[ u_{l+1/2} = Z_l^{1/2} \phi \] (2.17b)

in which \( l \) is the layer index, \( Z \) yields the roots of the discrete wave equations, \( \phi, \phi \) are amplitudes of \( SH \) waves, and the vectors \( \phi, \phi \) give amplitudes and polarization of \( P \) and \( SV' \) waves. Of course, Eqs. (2.17b) and (2.18b) apply only to quadratic elements. We shall demonstrate application of this ansatz to various canonical problems in turn. We first establish the relevant equations and solutions, and then discuss the results obtained.

3. Dispersion of body waves

3.1. Dispersion of SH body waves

3.1.1. Linear expansion

Substituting the trial solution (2.17a) into the DWE (2.5) and dividing by \( Z_l \), we obtain

\[ aZ + 2b + aZ^{-1} = 0 \] (3.1)

Defining \( Y = \frac{1}{2}(Z + Z^{-1}) \), which admits two solutions of the form \( Z = \gamma \mp \sqrt{\gamma^2 - 1} \), Eq. (3.1) can be written as

\[ Y = \frac{b}{a} = \frac{(3 - \mu)(z^2 + 6 - (3 - \mu)\Omega^2)}{z^2 - 6 - \mu\Omega^2} \] (3.2)

The two solutions of (3.1) are then

\[ Z_{1,2} = \gamma \mp \sqrt{\gamma^2 - 1} = \cos \xi \mp i \sin \xi \] (3.3)

Clearly, \( Y = \cos \xi \), and \( \xi \) can be recognized as the dimensionless vertical wavenumber. (Eq. (3.3) includes the case \( |Y| > 1 \), which simply makes the vertical wavenumber imaginary and changes the trigonometric functions into hyperbolic ones). In combination with the implicit factor \( \exp(\lambda t) \), the two roots of (3.3) represent waves propagating in the positive and negative \( z \)-directions, respectively. Hence, the general solution is of either one of the two forms

\[ v_l = Ae^{-i\xi z} + Be^{i\xi z} \] (3.4a)

\[ v_{l+1/2} = Z_l^{1/2} \phi \] (2.17b)

\[ v_{l+1/2} = Z_l^{1/2} \phi \] (2.18a)

\[ v_{l+1/2} = Z_l^{1/2} \phi \] (2.18b)

of the half angle, we obtain after brief algebra the dispersion equation as

\[
\left(1 - \frac{2}{3}z \sin^2 \frac{1}{2} \xi_z^2 + 4 \sin^2 \frac{1}{2} \xi_z^2 \right) \frac{\Omega^2}{C_0^2} = 0
\] (3.5)

\[
\left(1 - \frac{2}{3} \mu \sin^2 \frac{1}{2} \xi_z^2 \right) \Omega^2 = 0
\] (3.5)

It can be shown that for very low vertical wavenumbers (i.e. very long wavelengths in the \( z \)-direction), the spectrum equation (3.5) approaches the exact dispersion for a continuous medium, which is \( (\omega/C_0)^2 = k_z^2 + k_x^2 \), or in dimensionless form, \( \Omega^2 = \xi_z^2 + 4 \xi_z^2 \). This equation can also be written in terms of the direction of propagation \( \theta \) with respect to the horizontal and wavelength \( \lambda \) along the wave front. This is accomplished by expressing the wavenumbers in (3.5) as

\[ \xi_x = k_x h = k_0 h \cos \theta = \xi_0 \cos \theta = 2\pi \frac{h}{\lambda} \cos \theta \] (3.6a)

\[ \xi_z = k_z h = k_0 h \sin \theta = \xi_0 \sin \theta = 2\pi \frac{h}{\lambda} \sin \theta \] (3.6b)

On the other hand, the wave speed along the direction of propagation is \( V = \omega/k_0 \) or in dimensionless form, \( V/C_0 = \Omega/\xi_0 \), that is, \( \Omega = \xi_0 V/C_0 \). Using this in (3.5), we obtain

\[
V = \sqrt{\frac{(1 - \frac{2}{3}z \sin^2 \frac{1}{2} \xi_z^2) \cos^2 \theta + \left(\frac{\sin \xi_z}{\xi_z}ight)^2 \sin^2 \theta}{1 - \frac{2}{3} \mu \sin^2 \frac{1}{2} \xi_z^2}}
\] (3.7)

where, of course, the vertical wavenumber is given by (3.6b). As can be seen, when the waves propagate horizontally, \( \theta = 0, \xi_z = 0, V = C_0 \), and no dispersion takes place. This is so because the horizontal direction is not discretized. Conversely, the largest dispersion occurs when the waves propagate vertically, in which case the tuning factor \( z \) has no effect. In that case, the wave speed reduces to the well-known equation

\[
V = \frac{\sin \frac{1}{2} \xi_z}{\frac{1}{2} \xi_z \sqrt{1 - \frac{2}{3} \mu \sin^2 \frac{1}{2} \xi_z}}
\] (3.8)

3.1.2. Quadratic expansion

Substitution of (2.17a,b) into Eq. (2.8a,b) and division by \( Z_1 \) yields the system of equations

\[
\begin{align*}
\{a(Z + Z^{-1}) + 2b & \quad \bar{a}(Z^{1/2} + Z^{-1/2}) \} \{\phi \} = \{0\} \\
\bar{a}(Z^{1/2} + Z^{-1/2}) & \quad 2b \end{align*}
\] (3.9)

in which the coefficients from (2.10 a–d) must be used. Introducing the auxiliary variable \( Y = (Z^{1/2} + Z^{-1/2})/2 \), which satisfies \( Y^2 = (Z + Z^{-1} + 2)/4 \), we can write this equation as
\[
\begin{align*}
\left\{ a(2Y^2 - 1) + b Y \frac{\partial}{\partial Y} \right\} \left\{ \varphi \right\} &= \left\{ 0 \right\} \\
\left\{ \varphi \right\} &= \left\{ 0 \right\}
\end{align*}
\] (3.10)

This homogeneous set of equations has a non-trivial solution only if the determinant of the coefficient matrix vanishes. Satisfying this condition, we obtain the two eigenvalues and eigenvectors (indices \( j = 1, 2 \)) as

\[
Y_j = \pm \sqrt{\frac{(a - b) \bar{b}}{2ab - a}}
\]

\[
= \pm \sqrt{\frac{4(1 + 1/\bar{b}(\xi_j^2 - \Omega^2)) (1 + 1/\bar{b}(\xi_j^2 - \Omega^2) - \bar{\mu}^2(\xi_j^2 - \mu\Omega^2))}{(1 - \bar{\mu}^2(\xi_j^2 - \mu\Omega^2)) (1 - 1/\bar{b}(\xi_j^2 - \Omega^2)) + 3}}
\] (3.11a)

\[
\left\{ \varphi_1 \right\} = \left\{ \frac{1}{b} Y_j \right\} = \left\{ c_j \right\}
\]

\[
\left\{ \varphi_2 \right\} = \left\{ -\frac{1}{b} Y_j \right\} = \left\{ c_j \right\}
\] (3.11b)

\[
Z_j_{1/2} = Y_j \mp i\sqrt{1 - Y_j^2} = \exp \left( \mp \frac{1}{2} i\xi_j \right)
\] (3.11c)

The first solution is referred to as the acoustical branch \((c_1 > 0)\), while the second is the optical branch \((c_2 < 0)\) [2.4]. The complete solution is then either

\[
v_l = A e^{-i\xi_l} + B e^{i\xi_l} \quad \text{nodal interfaces}
\] (3.12a)

\[
v_{l+1/2} = (A e^{-i(1/2)\xi_l} + B e^{i(1/2)\xi_l}) \quad \text{internal node}
\] (3.12b)

or

\[
v_l = A \cos l\xi_l + B \sin l\xi_l \quad \text{nodal interfaces}
\] (3.12c)

\[
v_{l+1/2} = c \left[ A \cos \left( l \pm \frac{1}{2} \right) \xi_l + B \sin \left( l \pm \frac{1}{2} \right) \xi_l \right] \quad \text{internal node}
\] (3.12d)

in which \( A \) and \( B \) are again arbitrary constants, and \( c \) is chosen so that \( c = c_1 > 0 \) (Eq. (3.11b)). The dispersion relation is obtained by expressing Eq. (3.11a) in terms of the vertical wavenumber. The result is

\[
\left[ 2\mu \sin^2 \frac{1}{2} \xi + 5 - 3\mu \right] \Omega^2 - \left[ 120 - 2\sin^2 \frac{1}{2} \xi \right. \\
\left. \xi (10 + 6\mu - (\alpha + \mu) \xi_l^2) + (10 - 3\xi - 3\mu) \xi_l^2 \right] \\
\Omega^2 + 2\sin^2 \frac{1}{2} \xi (240 - (10 + 6\gamma) \xi_l^2 + \xi_l^4) \\
\left. + \xi_l^2 (120 + (5 - 3\gamma) \xi_l^4) = 0 \right]
\] (3.13)

In particular, if the horizontal wavenumber vanishes, that is, if the waves propagate vertically, this reduces to

\[
\left[ 5 - \mu \left( 3 - 2\sin^2 \frac{1}{2} \xi \right) \right] \Omega^2 - 2 \left[ 60 - 2(5 + 3\mu) \sin^2 \frac{1}{2} \xi \right] \Omega^2 + 480 \sin^2 \frac{1}{2} \xi = 0
\] (3.14)

from which we obtain the wave speed, using for this purpose \( \Omega = \xi \nu \nu^{-1} C_5 = \xi \nu \nu^{-1} C_5 \). Again, it can be shown that for very low vertical wavenumbers (i.e. very long wavelengths in the \( z \)-direction), the spectrum equation approaches the exact one for the continuous medium.

### 3.1.3. Optimal tuning factors for SH waves

Eqs. (3.5) and (3.13) provide the dispersion relations for SH waves in a thin-layer system modeled with linear and quadratic elements, respectively. These dispersion relations are functions of the frequency and wavelength of the waves, and of the tuning factors \( \alpha, \mu \) for the matrices \( A \) and \( M \), which define their degree of lumped-consistent combination. Optimal tuning factors can be found by minimizing the integrated squared error \( I = \int_0^\infty \left[ V/C_5 - 1 \right]^2 d\xi \) between the numerical and exact dispersion equations over a range of vertical wavenumbers. This range is chosen to extend from zero to the limit of wave propagation, which occurs when the wavelength equals twice the element thickness, that is, when \( \xi = \pi \). While the integrated squared error defines a surface in the \( \alpha, \mu \) space, the largest dispersion takes place when the waves propagate vertically, in which case matrix \( A \) has no effect whatsoever. Hence, it is clear that the \( \mu \) factor can be optimized independently from the \( \alpha \) factor by first considering vertical wave propagation.

Fig. 1(a) and (b) display the apparent wave velocity (i.e. Eqs. (3.8) and (3.14)) as a function of the vertical wavenumber \( \xi \), and tuning factor \( \mu \) for shear waves propagating vertically in systems with linear and quadratic elements. Notice that the dispersion curves for a quadratic expansion are discontinuous at \( \xi = \pi \), an observation to which we shall return to in the next section. Fig. 2(a) and (b), on the other hand, show the integrated squared error function for the linear and quadratic elements as function of \( \mu \). As can be seen, the minimum error occurs for \( \mu = 0.55 \) and \( \mu = 0.67 \) for the linear and quadratic elements, respectively.

Having found the optimal \( \mu \), we can proceed to optimize \( \alpha \). This is accomplished by fixing \( \mu \) and determining the integrated squared error for a range of angles of wave propagation other than 0 (for which there is no dispersion) and 90° (for which \( \alpha \) has no effect). While the optimal value for \( \alpha \) depends on the angle of propagation, it is found that the optimal tuning factors are as follows [22]:

\[
\text{Minimum error occurs for quadratic elements as function of }
\]
As can be seen from Eq. (3.7), when \( a = l \), the factor of the first term under the square root cancels with the denominator and makes it exact.

3.1.4. Dispersion spectra for \( SH \) body waves

Fig. 3(a)–(d) compare two frequency–wavenumber spectra for \( SH \) body waves obtained with the TLM (dashed lines) against the exact analytical spectrum (solid lines) for a fixed horizontal wavenumber \( (\xi_x = 0.5\pi) \). On the left are the spectra for fully consistent matrices, that is, \( a = \mu = 1 \), while on the right are the spectra for optimal tuning factors. Also, the top row displays the results for linear elements, while the bottom row shows the results for quadratic elements. As can be seen, the numerical dispersion grows with frequency and vertical wavenumber, as was already hinted at earlier in connection with Eq. (3.7).

The two intersecting dashed lines in the quadratic element are the acoustic and optical branches, which due to the periodicity in \( \xi_z \) exhibit symmetry with respect to \( \text{Re}(\xi_z) = \pi \). When these two branches are plotted simply for real, vertical wavenumbers as in Fig. 4, it appears as

<table>
<thead>
<tr>
<th>Tuning factor</th>
<th>Linear</th>
<th>Quadratic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>0.55</td>
<td>0.67</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.55</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Fig. 1. Dispersion of \( SH \) body waves propagation vertically: (a) linear element (b) quadratic element.

Fig. 2. Integrated squared error for \( SH \) body waves: (a) linear element; (b) quadratic element.
if the branches did not intersect, but simply approach each other and then diverge (the vertical gap between these branches is referred to as a *stopping band*). However, as the magnified detail of the complex wavenumber spectrum in Fig. 5 demonstrates, this is illusory. As can be seen, there are two complex loops that connect the apparent minimum and maximum of the two branches. The acoustic branch ascends to its maximum, loops around the complex branch, and then continues along the higher branch, and vice versa for the optical branch. Indeed, when the spectra are computed for a lightly damped medium, it is seen that the real acoustic and optical branches turn slightly complex, separate completely, and the loops do not intersect. Thus, we allege that the acoustic and optical branches are discontinuous at $\xi = \pi$ and resume at what seems to be the continuation of the other branch. They exhibit throughout either positive or negative group velocity (i.e. slope), except at the apparent minimum/maximum where the group velocity vanishes. This is the reason for the discontinuous dispersion curves shown earlier in Fig. 1(b). (An interesting discussion on spectra that exhibit an intersection of two branches at which the group velocity is zero is given by Tassoulas and Akylas [21]. When the displacement patterns associated with the acoustic and optical branches are computed via Eqs. (3.11) and (3.12), it is found that a root (say, point A in Fig. 3(d)) on the high frequency–low wavenumber optical branch

![Fig. 3. Dispersion spectra for $SH$ body waves with $\xi = 0.5\pi$. Left: consistent matrices. Right: optimally tuned matrices (segmented lines: TLM; solid line: exact solution).](image-url)
is indistinguishable from the symmetrically located root (D) lying on the low frequency and low wavenumber acoustic branch. For this reason, the optical branch can be said to have no physical meaning, but to be simply a mathematical artifact. Indeed, when propagation modes are sought for a discrete, finite layered system, the eigenvalue problem yields no modes that may correspond to optical branches.

Finally, notice that the discrete models exhibit a non-physical high-frequency evanescent branch, which for the linear element correspond to wavelengths of shear waves shorter than twice the element thickness \( h \), and half that much for the quadratic element, a result that will also hold for \( SV-P \) waves. These wave modes do not propagate, but represent standing waves that decay exponentially. Still, these non-physical evanescent waves can be avoided by limiting the frequency content of the excitation so that the wavelength is always longer than twice the element length. This can be accomplished by choosing the model refinement so that the combined thickness of \( N \) thin layers always exceeds the shortest expected wavelength, that is, \( N \cdot h \geq \lambda_{\text{min}} = C_0 / f_{\text{max}} \), where \( f_{\text{max}} \) is the highest frequency of excitation (Hz), and \( N \) is the minimum acceptable number of thin layers per wavelength. We have found that a value \( N = 4 \) for linear elements and \( N = 2.5 \) for quadratic elements provides reasonably accurate responses up to the maximum frequency.

3.2. Dispersion of \( SV-P \) body waves

3.2.1. Linear expansion

Substituting the trial solution (2.18a) into the DWE (2.11) and dividing by \( Z_l \), we obtain

\[
(a^T Z + 2b + a Z^{-1}) \psi = 0
\]

Defining \( Y = (Z + Z^{-1})/2 \) and \( \Upsilon = (Z - Z^{-1})/2 \), which satisfy \( Y^2 = Y^2 - 1 \), and in the light of Eqs. (2.13a,b), this reduces to

\[
\begin{align*}
Y a_{11} + b_{11} - \Upsilon a_{12} & \quad \{ \varphi_1 \} = \{ 0 \} \\
\Upsilon a_{12} + b_{22} & \quad \{ \varphi_2 \} = \{ 0 \}
\end{align*}
\]

A nontrivial solution requires the determinant of the coefficient matrix to vanish. This leads to the quadratic equation

\[
(a_{11} a_{22} + a_{12}^2) Y^2 + (a_{11} b_{22} + a_{22} b_{11}) Y + (b_{11} b_{22} - a_{12}^2) = 0
\]

with solutions \( Y_1 \) and \( Y_2 \) for each of which there is a pair of roots

\[
Z_j = Y_j \mp \sqrt{Y_j^2 - 1} = \cos \xi_j \mp i \sin \xi_j = e^{i \xi_j} \quad j = P, S
\]
These two roots represent either $P$ or $SV$ waves propagating along directions $\tan \theta_j = \xi_j / \phi$. Since both the horizontal wavenumber and frequency are fixed in (3.18), this fixes the horizontal phase velocity and wavelength. Hence, the propagation directions $\theta_1, \theta_2$ of $P$ and $SV$ waves will be different for each root of (3.18), so the eigenvectors will not be orthogonal to each other.

If $A, B, C, D$ are arbitrary constants, the complete solution is of either of the forms

\[
\begin{align*}
&u_j = \begin{cases} u_l \\ iw_j \end{cases} \\
&= \begin{cases} \phi_s \cos k_{12} \xi_{s,l} + \phi_{s \phi} \cos k_{12} \phi_{s \phi} \\
- \i \sin k_{12} \phi_{s \phi} \end{cases} A + \begin{cases} \phi_s \cos k_{12} \xi_{s,l} \\
- \i \sin k_{12} \phi_{s \phi} \end{cases} B + \begin{cases} \phi_s \cos k_{12} \xi_{s,l} + \phi_{s \phi} \cos k_{12} \phi_{s \phi} \\
- \i \sin k_{12} \phi_{s \phi} \end{cases} C + \begin{cases} \phi_s \cos k_{12} \xi_{s,l} \\
- \i \sin k_{12} \phi_{s \phi} \end{cases} D
\end{align*}
\]

or

\[
\begin{align*}
&u_j = \begin{cases} \cos k_{12} \xi_{s,l} \phi_{s \phi} \\
- \i \sin k_{12} \phi_{s \phi} \end{cases} A + \begin{cases} \cos k_{12} \xi_{s,l} \\
- \i \sin k_{12} \phi_{s \phi} \end{cases} B + \begin{cases} \cos k_{12} \xi_{s,l} + \phi_{s \phi} \cos k_{12} \phi_{s \phi} \\
- \i \sin k_{12} \phi_{s \phi} \end{cases} C + \begin{cases} \cos k_{12} \xi_{s,l} \\
- \i \sin k_{12} \phi_{s \phi} \end{cases} D
\end{align*}
\]

(3.19a,b)

Alternatively, Eq. (3.16) can also be expressed as an eigenvalue problem in the dimensionless frequency $\Omega = \xi_0 V / C_S$, while setting $Y = \cos \xi_0, \overline{Y} = \i \sin \xi_0$, which fixes the vertical wavenumber. Hence, this fixes the propagation direction $\tan \theta = \xi_0 / \phi$. The result is the symmetric, real, special eigenvalue problem

\[
\begin{align*}
&\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{cases} \phi_s \\
- \i \phi_{s \phi} \end{cases} = \left( V / C_S \right)^2 \begin{cases} \phi_s \\
- \i \phi_{s \phi} \end{cases}
\end{align*}
\]

(3.20)

in which

\[
\begin{align*}
k_{11} &= \frac{\beta \xi_0^2 - 6 \cos \xi_0 + \rho (3 - \rho) \xi_0^2 + 6}{\xi_0^2 (3 - \mu (1 - \cos \xi_0))} \\
k_{12} &= \frac{\beta \xi_0^2 - 6 \rho \cos \xi_0 + (3 - \rho) \xi_0^2 + 6 \rho}{\xi_0^2 (3 - \mu (1 - \cos \xi_0))} \\
k_{22} &= \frac{3 (\rho - 1) \xi_0 \sin \xi_0}{\xi_0^2 (3 - \mu (1 - \cos \xi_0))}
\end{align*}
\]

and $\{ u_l, w_j \}^T = \{ \phi_{s \phi} - \i \phi_{s \phi} \}^T$ is the $j$th modal shape for either $P$ or $SV$ waves propagating along a fixed direction $\theta$. The eigenvectors here satisfy the standard orthogonality condition, so particle motions for these two modes are perpendicular to each other, that is, the discrete medium admits both longitudinal and transverse motions at right angles to each other. However, unlike the continuum case, the direction of particle motions for the discrete medium deviate somewhat from the direction of propagation and its perpendicular, that is, they exhibit an aberration. The angle of vibration $\Theta$, i.e. the polarization, can be found from the components of the two associated eigenvectors as $\tan(\Theta) = w_j / u_l$, while the angle of propagation is $\tan \theta = \xi_0 / \phi$. After tedious algebra, the dispersion equation and the polarization angle implied by the eigenvalue problem (3.20) can be expressed compactly as

\[
\begin{align*}
&\left( V / C_S \right)^2 = \frac{r + 1}{2} \left( R^2 \cos^2 \theta + S^2 \sin^2 \theta \right) - \frac{r - 1}{2} \left( R^2 \cos^2 \theta - S^2 \sin^2 \theta \right)^2 + Q^2 S^2 \sin^2 \theta \cos^2 \theta \\
&S = \frac{\sin \frac{1}{2} \xi_0}{\frac{1}{2} \xi_0 \sqrt{1 - \frac{3}{2} \mu \sin^2 \frac{1}{2} \xi_0}}, \\
&\beta = \frac{1 - \frac{3}{2} \mu \sin^2 \frac{1}{2} \xi_0}{\sqrt{1 - \frac{3}{2} \mu \sin^2 \frac{1}{2} \xi_0}}, \\
&Q = \frac{2 \cos \frac{1}{2} \xi_0}{\frac{1}{2} \xi_0 \sqrt{1 - \frac{3}{2} \mu \sin^2 \frac{1}{2} \xi_0}}, \\
&\tan 2\Theta = \frac{QS \sin \theta \cos \theta}{R^2 \cos^2 \theta - S^2 \sin^2 \theta}
\end{align*}
\]

(3.22a–e)

in which the vertical wavenumber is again given by Eq. (3.6b). Observe that in the limit of an infinitesimal grid, $\xi_0 = k \hbar \to 0$, $S \to 1$, $R \to 1$, $Q \to 2$, so $V_l \to C_S$, $V_z \to rC_S = C_P$, and $\Theta \to \theta$.

3.2.2. Quadratic expansion

Substituting the trial solution (2.18a,b) into the DWE (2.14a,b) and dividing by $Z^2$, we obtain

\[
\begin{align*}
&\begin{bmatrix} a^T Z + a Z^{-1} + 2 b \quad a Z^{1/2} + a^{-T} Z^{-1/2} \quad 2 b \\
a Z^{1/2} + a Z^{-1/2} \quad 2 b \quad 0 \end{bmatrix} \begin{cases} \phi \\
0 \end{cases} = \begin{cases} 0 \\
0 \end{cases}
\end{align*}
\]

(3.23)

Applying once more the solution method employed for Eq. (3.9), we find four roots for $Z$, which are the acoustic and optical branches for $P$ and $SV$ waves. The general solution is then of the form

\[
\begin{align*}
&u_j = AZ^l \phi_{l1} + BZ^l \phi_{l2} + CZ^{1/2} \phi_{s1} + DZ^{1/2} \phi_{s2}
\end{align*}
\]

(3.24)

and a similar equation for the internal nodes (i.e. half indices), but using instead vectors $\phi$, as given by the eigenvalue problem.

The apparent wave velocity can be obtained by casting the eigenvalue problem given by Eq. (3.23) in terms of the frequency $\Omega = \xi_0 V / C_S$, setting $Z = \cos \xi_0 + i \sin \xi_0$ and expressing the wavenumbers in terms of Eqs. (3.6). The result is a symmetric eigenvalue of the form...
\[
\begin{pmatrix}
   k_{11} & k_{12} & k_{13} & k_{14} \\
   k_{22} & k_{14} & k_{24} & \phi_x \\
   k_{33} & 0 & k_{34} & -i\phi_x \\
   k_{44} & & & -i\phi_x
\end{pmatrix}
\]

\[
= \Omega^2 \begin{pmatrix}
   m_1 & 0 & m_2 & 0 \\
   m_1 & 0 & m_2 & 0 \\
   m_3 & 0 & m_3 & 0 \\
\end{pmatrix}
\]

\[\text{(sym)}\]

in which

\[
k_{11} = r_{S_1}^2 (1 - \frac{2}{5} \alpha \cos^2 \frac{1}{2} \xi_z) + 4 \cos^2 \frac{1}{2} \xi_z + 12
\]

\[
k_{12} = -2(r - 1) \xi_z \sin \frac{1}{2} \xi_z \cos \frac{1}{2} \xi_z
\]

\[\xi_z = \left( \frac{1}{2} \frac{2}{5} \alpha \cos^2 \frac{1}{2} \xi_z \right) + 4r \cos^2 \frac{1}{2} \xi_z + 12r
\]

\[
k_{13} = -2 \left( 8 - \frac{1}{5} \alpha \xi_z^2 \right) \cos \frac{1}{2} \xi_z
\]

\[
k_{14} = 4(r - 1) \xi_z \sin \frac{1}{2} \xi_z
\]

\[
k_{24} = -2 \left( 8r - \frac{1}{5} \alpha \xi_z^2 \right) \cos \frac{1}{2} \xi_z
\]

\[
k_{33} = 2r \left( 1 - \frac{1}{5} \alpha \right) \xi_z^2 + 16
\]

\[
k_{44} = 2 \left( 1 - \frac{1}{5} \alpha \right) \xi_z^2 + 16r
\]

\[
m_1 = 1 - \frac{2}{5} \mu \cos^2 \frac{1}{2} \xi_z
\]

\[
m_2 = \frac{2}{5} \mu \cos \frac{1}{2} \xi_z
\]

\[
m_3 = 2 - \frac{2}{5} H
\]

In as much as the dispersion equation is now a fourth order polynomial, it cannot be reduced to a simple formula; however, it can readily be evaluated numerically for any combination of parameters and direction of propagation. When this is done, it is found once more that the polarization of the waves does not coincide exactly with the direction of propagation and its transverse direction, but exhibits a small aberration.

Finally, it can once more be shown that for very small vertical wavenumbers (i.e. a very long wavelengths in the z-direction), the spectrum equations approach the exact spectra in dimensionless form, namely \( \Omega_P^2 = r(\xi_z^2 + \xi_z \alpha) \) and \( \Omega_S^2 = \xi_z^2 + \xi_z^2 \).

3.2.3. Optimal tuning factors for \( SV-P \) waves

It can be argued that the optimal tuning factor \( \mu \) for \( SV-P \) waves must be the same as for \( SH \) waves. The reason is that when \( SV \) and \( P \) waves propagate vertically, they uncouple from each other and the system behaves either as a discrete shear beam or as a rod. Considering that—except for the different propagation speed—the equations of motion for these two systems are identical, the optimal value for \( \mu \) for \( SV-P \) waves must be the same as for \( SH \) waves.

The optimal value for \( \alpha \) can be sought again while maintaining the optimal value \( \mu \) constant. While the squared error function is now a function not only of \( \alpha \) but also of Poisson’s ratio \( v \), numerical experiments demonstrate that its influence is small. When the optimization of the tuning factor \( \alpha \) is applied to \( SV-P \) waves, it is found that the optimal factor is the same as for \( SH \) waves. Thus, \( SH \) and \( SVP \) waves have identical tuning factors \( \alpha, \mu \).

3.2.4. Dispersion spectra for \( SV-P \) body waves

Fig. 6 compares the spectra obtained with the TLM (dashed lines) against the exact analytical spectrum (solid lines) for a fixed horizontal wavenumber \( (\xi_z = 0.5\pi) \) and a Poisson’s ratio \( v = 0.30 \). On the left is the spectrum for fully consistent matrices, that is, \( \alpha = \mu = 1.0 \), while on the right is the spectrum for optimal tuning factors. Also, the top row is for linear elements, the bottom for quadratic elements. In each case, two branches exist: one for \( S \) waves and one for \( P \) waves. As can be seen, for small to moderate vertical wavenumbers, the error of the discrete solution is small, but then grows as the wavelengths approach the limit of twice the element length.

Fig. 7 shows the wave velocity dispersion curves obtained with optimal tuning factors and Poisson’s ratio \( v = 0.30 \). The left column shows the results for linear elements, while the right depicts quadratic elements. The top row is for vertically propagating waves, and the bottom row for waves propagating at an angle with respect to the horizontal of \( \theta = 60^\circ \). Fig. 8, on the other hand, shows the aberration angle \( \theta - \theta \) for the latter case. As can be seen, the quadratic element exhibits discontinuities at stopping bands associated with the interlacing of acoustic and optical branches, as was also the case for \( SH \) waves. For small incidence angles, the numerical dispersion decreases, and it vanishes altogether when the body waves propagate horizontally.

4. Dispersion of guided waves in a finite depth medium

Consider a finite depth medium, such as a homogeneous plate or stratum, with unit shear wave velocity and density \( (C_S = \rho = 1) \) and also unit thickness \( H = 1 \). Because the medium has finite thickness, the dispersion of waves will also be a function of the grid refinement ratio \( h/H \), or alternatively, of the total number of thin layers. Moreover, the total number of propagation modes in the discrete model is directly proportional to the number of thin layers and element expansion order, while the continuous system has infinitely many such
modes—counting both the propagating and evanescent modes. To make the discretization errors obvious, we shall present herein results for a relatively small number of layers. Also, in all cases to be considered from now on, the dispersion results will be given only for the optimal tuning factors obtained earlier on the basis of body waves propagating in an infinite solid. Comparisons for other combinations of tuning factors can be found in [22].

4.1. SH waves in a plate

A free plate satisfies the boundary condition given by Eqs. (2.6) or (2.9) at both the upper surface, and similar equations at the lower interface, which are \( av_{1-N} + bv_{-N} = 0 \) for the linear element, and \( av_{1-N} + \frac{a}{2} v_{2-N} + bv_{-N} = 0 \) for the quadratic element. Combining these boundary conditions with the general solution for the unbounded medium, we can find the modes of vibration. Alternatively, and more easily, we can find the modes directly from the dispersion equations given earlier for the propagation speed in an unbounded medium. It suffices for us to set \( Q = \omega h / C_S = \xi / \xi_S = \xi / C_S \sin \theta, \xi_S = \xi_S \cos \theta \), and the depth of the plate equal to a multiple of one half of the vertical wavelength (so as to satisfy the boundary conditions). This is accomplished if \( \xi_S = 2 \pi h / \lambda_S = j \xi \). For the linear model, Eq. (3.5) then gives

![Fig. 6. Dispersion spectra for SV-P body waves with \( \xi_S = 0.5 \pi \). Left: consistent matrices. Right: optimally tuned matrices (segmented lines: TLM; solid line: exact solution).](image-url)
\[ \omega_j = \frac{C_S}{H} \sqrt{\frac{(1 - \frac{\mu}{2} \lambda \sin^2 \frac{j \pi}{N}) (k,j)^2 + 4N^2 \sin^2 \frac{j \pi}{N}}{(1 - \frac{\mu}{2} \lambda \sin^2 \frac{j \pi}{N})}}, \]

\[ j = 0, \ldots, N \]

which in the limit \( N \to \infty \) converges to the correct result for the continuum. Notice that if \( \mu = \lambda \) is chosen, the first term under the square root is rendered exact. The corresponding equation for the quadratic element is obtained using Eq. (3.13), which results in a somewhat more complicated expression, whose derivation is left to the reader.

Fig. 9a, b show the undamped modal spectra for linear and quadratic elements with optimal tuning factors. The linear model has ten layers, while the quadratic has only five, so both have a total of 11 interfaces and modes. Complex branches for low frequencies and imaginary horizontal wavenumbers are evanescent waves. The intersection of these branches with the vertical axis at zero wavenumber occurs at the resonant frequencies of the plate.

Observe that despite the application of optimal tuning factors, only the first 5 or 6 of the 11 modes are accurate. In fact, empirical results obtained for other models with many more layers and/or inhomogeneous properties demonstrate that roughly the first half of all guided modes obtained with the discrete model is accurate. A simple rule of thumb for the required number of thin layers consists then in choosing this number so that the resonant frequency associated with the \((N/2)\)th mode—which can easily be estimated a priori—does not exceed the highest frequency of the excitation. Alternatively, we have found that excluding the upper half of modes in the computation of response...
time histories or frequency response functions often improves the results of the computation in comparison to a calculation with all modes included.

4.2. SH waves in a stratum

Consider next a finite depth, homogeneous stratum with a stress free upper boundary, and a fixed lower boundary. Thus, the boundary conditions at the free surface are given by Eq. (2.6) for the linear model, and (2.9) for the quadratic model. At the bottom, on the other hand, the displacement must vanish, i.e. \( v_{z,0} = 0 \). Using the ad-hoc method described for the free plate, but choosing this time to satisfy the boundary conditions by taking the depth of the stratum equal to an odd multiple of the quarter vertical wavelength, that is, \( \xi_z = 2\pi h/\lambda_z = (2j-1)\pi/\lambda_z \), with \( j = 1, N \), the solution for the stratum can be found. In as much as the modal characteristics of strata are similar to those of plates, we omit here any further details or numerical results.

4.3. SV–P waves in a plate with mixed boundary conditions

We consider next a homogeneous plate subjected to \( SV–P \) waves, and choose the boundary conditions to be of the mixed type, that is, we require the normal displacements and tangential stresses to vanish simultaneously at each exterior surface. This has the advantage of uncoupling the discrete vector wave equation into two scalar equations for the symmetric and anti-symmetric modes with respect to the mid-plane, which can then be solved in closed form. These modes involve either pure \( P \) waves or pure \( SV \) waves. For the symmetric mode, the shearing stress and vertical displacement vanish at the mid-plane, while for the anti-symmetric mode, it is
the other way around, i.e. the horizontal displacement and vertical stress vanish at the mid-plane.

To obtain the solution, we could start with the DWE solutions (3.19) or (3.24), and combine these with the boundary conditions (2.12) or (2.15) (using only the horizontal component appropriate for the traction boundary condition) to arrive at the eigenvalue problem for the frequencies of the plate. However, the most expedite way is again to use the ad-hoc approach, relating the half-width of the plate to waves with wavelengths compatible with the mixed boundary conditions. For simplicity, we take an even number of thin layers, so that there is always an interface coinciding with the mid-plane at which the layer index \( l \) can be taken to be zero. The layer index for the upper exterior surface is then \( l = N/2 \).

For the symmetric mode, we set \( j \lambda_z = H = Nh \), that is, \( \xi_s = 2\pi h/\lambda_z = j\pi \), which for \( j > 0 \) gives at least one half wavelength from the mid-plane to the exterior surface (a cosine for the horizontal component, and a sine for the vertical). For example, setting this ansatz into the eigenvalue equation (3.20) for linear elements and considering that \( \Omega^2 = \left( \frac{\pi}{N} \right)^2 \xi_s \), \( \xi_s \cos \theta = \xi_s \) and \( \xi_s \sin \theta = \xi_s \), we obtain the (dimensionless) frequencies of the symmetric modes of the plate with mixed boundary conditions as

\[
\Omega_j^2 = \frac{r+1}{2} \left( R^2 \xi_s^2 + S^2 \xi_z^2 \right) \pm \frac{r-1}{2} \sqrt{\left( R^2 \xi_s^2 - S^2 \xi_z^2 \right)^2 + Q^2 S^2 \xi_z^2 \xi_s^2},
\]

\[
\xi_s = \frac{2\pi}{N} \quad j = 0, 2, \ldots, 2N - 1 \quad (+\text{sign} \rightarrow P\text{waves})
\]

\[
j = 1, 2, \ldots, \frac{1}{2} N \quad (-\text{sign} \rightarrow S\text{waves})
\]

(4.2)

where \( R, S, Q \) have the same meaning as in (3.22), but using the given ansatz for the vertical wavenumber. The solution with the plus sign involves only \( P \) waves, while the negative sign involves only \( S \) waves. The case \( j = 0 \) is a \( P \) wave that propagates horizontally and involves only horizontal motion. For \( j = N/2 \), the motion is an \( S \) wave. Thus, we have a total of \( N \) symmetric modes. The associated modal shapes are obtained from Eq. (3.19b), choosing \( R = D = 0 \), and either \( A = 1, C = 0 \) (for the \( P \) modes) or \( A = 0, C = 1 \) (for the \( S \) modes). The ratio of modal components needed to complete the definition of the modes can be obtained from Eqs. (3.20) and (3.21).

For the anti-symmetric mode, we set \( (2j - 1)\lambda_z = H/2 = Nh/2 \), that is, \( \xi_s = (2j - 1)\pi \), which provides odd multiples of a quarter wave from the mid-plane to the surface (a sine for the horizontal, and a cosine for the vertical component). For the linear element, this gives an equation similar to Eq. (4.2), but with the appropriate substitution for \( \xi_s \) given above, and modal indices \( j = 1, 2, \ldots, \frac{1}{2} N \). Thus, we have once more \( N \) anti-symmetric modes. The plate has then a total of \( 2N \) discrete modes.

The evaluation of the modes for quadratic elements follows along lines similar to those for linear elements, except that the results cannot be reduced to simple formulas, but must be evaluated numerically. However, the required computational effort is independent of the number of layers chosen, because the modal solutions (3.24) are of constant order 4. Whether linear or quadratic elements are used, it can be shown that the results converge to the exact formulas for the plate with mixed boundary conditions as the number of layers is increased.

Figs. 10 and 11 compare the discrete frequency-wavenumber spectra for the symmetric and anti-symmetric modes (dashed lines) against the exact analytical solution (solid lines) for Poisson’s ratio \( \nu = 0.31 \). Twelve layers are used for linear elements, and six for quadratic elements, so both have a total of 24 modes, and both are computed with optimal tuning factors. As can be seen,
about a third of all modes are accurate, although the discrepancy is smaller for the real, propagating branches than for the (less important) evanescent branches.

4.4. SV–P waves in a homogeneous stratum or in a Mindlin plate

The solution for either a stratum or a Mindlin plate can be obtained by combining Eqs. (3.19) or (3.24) with the boundary conditions (2.12) or (2.15) for each free surface. However, the ad-hoc approach based on assuming simple solutions for the vertical wavenumber is no longer applicable, because the equations for P and SV waves cannot be uncoupled by considerations of symmetry and anti-symmetry alone, as was done for a plate with mixed boundary conditions. While this increases the effort in finding the frequency spectrum, and the final equations cannot be written in terms of simple formulas—not even for the linear element—the task can readily be accomplished and solutions found for any arbitrary number of layers. However, in as much as the final equations and results are qualitatively similar to those of the plate with mixed boundary conditions shown previously, we omit further details for this case. A full evaluation of Lamb waves in a Mindlin plate obtained with the TLM can be found in [22].

5. Dispersion of guided waves in a half-space

Finally, we consider the problems of guided waves, i.e. Love and Rayleigh waves propagating in a semi-infinite discrete medium. In principle, a model for these problems with discrete layers would entail solving a matrix equation of infinite size. As we shall see, however, it is still possible to find the wave speeds, propagation modes, and dispersion equations for guided waves in such discrete models.

5.1. Love waves

Consider the problem of Love waves in a soft layer underlain by a stiffer half-space. Both the layer and half-space are discretized with thin layers, so each satisfies a solution of the form of Eqs. (3.4) for the linear element, or (3.12) for the quadratic element. Due to space considerations, we present in detail only the linear case. Defining the layer index to be zero at the interface of the stratum with the elastic half-space, we can write the anti-plane interface displacements as

\[ \sqrt{l} \text{Sn} \left( l, c_i \right) + B_l \sin l, \text{Sn} \left( l, c_i \right) \quad \text{for the layer} \]

\[ \sqrt{l} \text{Sn} \left( l, c_H \right) \quad \text{for the half-space} \]

in which \( Z_{l_H} \) must satisfy the boundedness condition, i.e. \( Z_{l_H} \to 0 \) when \( \sqrt{l} \to -\infty \) (waves must decay exponentially with depth). From (2.6), the boundary condition at the surface is

\[ bLv_n + aLv_{n-1} = 0 \]

The continuity of displacements at the half-space interface requires \( A_L = A_H \), while the continuity of stresses demands

\[ \frac{G_L}{h_L} (a_L v_1 + b_L v_0) + \frac{G_H}{h_H} (b_H v_0 + a_H v_{-1}) = 0 \]

in which \( G_L, G_H, h_L, h_H \) are the shear moduli and discrete layer thicknesses of the layer and half-space, respectively. Also, the coefficients \( a_L, b_L \) are given by Eqs. (2.7a,b). They depend only on the horizontal wavenumber and frequency, which are common to both media. However, because they are normalized differently, a sub-index must be attached to each. For example, for the layer, \( \xi_{il} = k_i h_{il}, \) while for the half-
space $\xi_{zL} = k_x h_L$. Also, $\Omega_L = \omega h_L / C_{SL}$ and $\Omega_H = \omega H / C_{SH}$. Combination of Eqs. (5.1) and (5.2) together with the continuity condition yields the eigenvalue problem

\[
\begin{bmatrix}
  b_L \cos N \xi_{zL} + a_L \cos (N-1) \xi_{zL} & b_L \sin N \xi_{zL} + a_L \sin (N-1) \xi_{zL} \\
  a_L \cos N \xi_{zL} + b_L + \kappa (b_H + a_H Z_H^{-1}) & a_L \sin N \xi_{zL}
\end{bmatrix}
\begin{bmatrix} A_L \\ B_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(5.3)

where $\kappa = G_H h_L / G_S h_S$. Also, the vertical wavenumbers in (5.3) must satisfy the dispersion Eq. (3.3), that is,

$$\cos \xi_{zL} = -b_L / a_L$$

and

$$Z_H^{-1} = -b_H / a_H + \sqrt{(b_H / a_H)^2 - 1}.$$ 

What remains is a transcendental eigenvalue in the frequency for any given horizontal wavenumber. Solving this $2 \times 2$ eigenvalue problem via search techniques—a fairly straightforward task—we obtain the Love wave modes for the discrete model.

The solution for quadratic elements follows along similar lines (using Eqs. (3.11), (3.12) and (2.9)), except that the transcendental eigenvalue problem is now of size $4 \times 4$ (for details, see [22]).

We now turn to a numerical example. Consider a soft layer of unit thickness and unit shear velocity ($H = 1$, $C_S = 1$) underlain by a half-space with shear wave velocity $C_H = 2$. Both media have the same unit mass density, $\rho = 1$. The linear model comprises 20 thin layers in the stratum ($N = 20$) (i.e. $h_L = H / N = 0.05$, while the quadratic model contains 10 thin layers. Thus, the discrete model has $N + 1 = 21$ propagation modes. In contrast, the number of Love modes in the continuous

(a) Linear element
(b) Quadratic element

Fig. 12. Dispersion spectra for Love waves (solid lines: exact solution; dashed lines: TLM).

(a) Linear element
(b) Quadratic element

Fig. 13. Numerical dispersion of Rayleigh surface waves. Exact solution is non-dispersive ($= 1$).
problem grows without bound as the frequency increases. Taking into consideration the wave-velocity contrast between the two materials, we choose the thickness of the thin-layers in the half space to be twice as much as for the soft layer, namely \( h_l = 0.10 \). This ensures that the dimensionless frequency \( \Omega = \frac{\omega l}{C0} = \frac{\omega l}{C_S} \) is the same for the layer and the half-space, which means that vertically propagating waves in both media will be represented by the same number of thin layers per wavelength—the latter are proportional to the wave velocity—and thus will exhibit similar discretization errors. Also, this implies that \( \kappa = G_{Hl} h_l / G_t h_l \equiv \rho_l C_{SH} / \rho_t C_{SL} \), i.e. the local stiffness ratio equals the impedance ratio, a desirable property.

Fig. 12(a) and (b) compare the dispersion curves for Love waves in a discrete medium with optimal tuning factors (dashed lines) against the exact analytical solutions (solid lines). As can be seen, some 12 of the 21 modes are accurately represented. Thus, the discrete model represents Love waves properly for the first 2/3 or so of the computable modes, but not for all modes. At first, this may come as a surprise, considering our earlier statements that horizontally propagating body waves exhibit no error. However, Love waves, which also propagate only horizontally, are not body waves, but constitute inhomogeneous waves in the half-space, that is, the vertical wavenumber in the half-space is not zero, but imaginary. Nonetheless, the discrete model performs admirably, even with the few layers considered herein, the quadratic element outperforming the linear element by a substantial margin.

5.2. Rayleigh waves

Finally, we derive the spectrum equation for Rayleigh surface waves in a discrete homogeneous half-space \((l \leq 0)\). This entails combination of the DWE solutions (3.19a) or (3.24) with the boundary conditions (2.12) or (2.15), while disposing of the exponential terms in the solution equations that grow with depth (to satisfy the boundedness condition at \( z = -\infty \)). Thus, for a linear expansion, the displacement ansatz and boundary conditions are of the form

\[
\mathbf{u}_l = \begin{bmatrix} u_l \\ i w_l \end{bmatrix} = A \phi_p Z_p^l + C \phi_s Z_s^l
\]

\[
b_l \mathbf{u}_l + a \mathbf{u}_{-1} = 0
\]

in which \( \phi_p \), \( Z_p \), \( \phi_s \), \( Z_s \) must be admissible solutions of Eq. (3.15). Substituting (5.4a) into (5.4b), we obtain a 2×2 transcendental eigenvalue problem whose solution must be found by search techniques. Rayleigh waves in a discrete layered system modeled with quadratic elements can be solved similarly, except that the eigenvalue problem grows to be of size 4×4.

6. Conclusions

The comprehensive exploration of numerical dispersion artifacts in the TLM presented in this paper has led us to the following conclusions:

- Numerical dispersion can be minimized by the use of appropriate tuning factors \( \mu, \alpha \) to combine consistent and lumped mass and stiffness matrices. Optimal results are obtained using \( \mu = \alpha = 0.55 \) for linear elements, and \( \mu = \alpha = 0.67 \) for quadratic elements. These improve the results obtained with the TLM not only for body waves, but also for guided waves. The improvement is particularly noticeable in the low modes.
- To obtain accurate results, it is necessary to use at least four elements per wavelength for linear elements \((N_z \geq 4)\), and at least two to three elements for quadratic elements \((N_z \geq 2.5)\). The shortest wavelengths in a model can easily be estimated a priori on the basis of the physical wave velocities and the highest frequencies present in the excitation.
- When using the TLM to compute wave fields due to dynamic sources by normal mode superposition, either in the frequency-space domain, or in the time-wavenumber domain, it may be best to restrict the modal summation to the first half or two-thirds of the computable modes, and to discard the highest modes. When following the \( N_z \) rule of thumb, such inaccurate, higher modes will not participate anyway in the response, so nothing is lost by disregarding them. Alternatively, the highest frequency (or wavelength) of the first mode neglected should exceed the highest frequency (or shortest wavelength) in the excitation.
- The numerical dispersion in TLM models is largest when the waves propagate in the direction of layering, and smallest when they propagate parallel to the layers. It was found that guided waves, such as Love and Rayleigh waves, are accurately modeled with the TLM and exhibit little numerical dispersion. The same applies to guided waves in plates, for which closed form expressions were presented. To the writers knowledge, the evaluation of the dispersion char-
acteristics of such guided waves with finite elements may be novel.

- When $P-SV$ waves propagate in a TLM medium (or for that matter, in a conventional finite element grid), the angles of polarization (i.e. directions of particle motions) do not exactly coincide with the direction of propagation and its perpendicular. Closed form formulas were presented for the apparent wave speed and aberration angle as function of the angle of propagation of such waves. It was found that the aberration is generally greatest when the waves propagate within a 30–60° cone. Still, the error is quite small, especially when the rule of thumb on mesh refinement is followed.

- For heterogeneous media, the computational effort associated with solving a wave propagation problem with $N$ linear thin layers is comparable to that of solving a quadratic model containing half as many layers, i.e. $N/2$ (both involve the same number of degrees of freedom, and the penalty for added bandwidth remains quite small). In as much as the accuracy of the quadratic model is generally better than that of the linear model, the outcome of this even for comparable effort conditions, the quadratic element should be preferred. This is particularly true when strains and stresses are needed. (We have also tested cubic elements, and found that they provide little gains in accuracy in comparison to quadratic elements).

References