Heat diffusion in layered media via the thin-layer method

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SUMMARY

The thin-layer method is a semi-analytical, numerical tool that has been successfully used in the past for wave motion in layered media, and differs from finite elements in that the medium is discretized only along a subset of the problem’s dimensions, i.e. it combines partial discretization with analytical solutions. This paper shows that it can be applied just as well to heat diffusion, and results are given for impulsive as well as distributed point and dipole thermal sources with formulations in Cartesian and cylindrical coordinates. In essence, the method involves the discretization of the governing equations in the direction of layering together with analytical solutions for the remaining directions and for the variation in time. The solutions in response to spatially and temporally impulsive sources can also be used as fundamental solutions in the context of a boundary element formulation. This is illustrated herein by means of an example of a concrete slab overlain by two layers of carbon fiber reinforced polymers (CFRP), where a penny-shaped crack is meant to simulate an area where the CFRP has debonded from the concrete.

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1. INTRODUCTION

Given the routine application of finite elements, finite differences and boundary elements to problems of heat diffusion in engineering, it would seem that most practical problems can be solved effectively simply by recourse to appropriately detailed and sufficiently refined discrete models. Still, there exist large classes of problems for which conventional finite elements are not an optimal or even viable option. Among these are layered systems subjected to spatially distributed thermal sources whose lateral dimensions are large in comparison with the scales of other material

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and field dimensions, such as specimen thickness or the spatial rate of decay of temperatures. In these problems, the size and computational effort that would be entailed by a naive application of a standard numerical model can be very large or even prohibitive. This difficulty is sidestepped herein by means of a hybrid approach that combines the advantages of the finite element method with the power of analytical solutions, a numerical tool that is widely referred to as the thin-layer method, or TLM for short [1, 2]. This method allows finding Green’s functions—the fundamental solutions—for thermal sources applied anywhere within a layered medium, which can later be used as tools for other purposes. For example, a problem of significant interest is the application of infrared thermography to the location of defects within concrete decks and pavements.

In the case of homogeneous solids of regular shape, such as cylinders or spheres, it is possible to find effective solutions to thermal sources anywhere in the medium by the well-known Green’s functions formalism, e.g. Carslaw and Jaeger [3], Feng and Michaelides [4]. Such Green’s functions may also be applied to the problem of homogeneous inclusions of regular shape embedded in full, homogenous spaces exposed to either external or internal heat sources. Alternatively, this technique may also be extended to homogeneous half-spaces, if the surface is modeled by means of boundary elements, e.g. Tadeu et al. [5]. In the case of layered media, one can obtain the requisite Green’s functions by strictly numerical means using the transfer matrix formalism in the frequency–wave number domain, a method that in the wave propagation community is known as the Haskell–Thomson method. Results in space–time are obtained by numerical Fourier transformations over horizontal wave numbers—or Hankel transforms over radial wave numbers—together with inversions from frequency to time via inverse Fourier transforms based on the FFT algorithm, or inverse Laplace transforms using the Stehfest method, e.g. Tadeu et al. [6]. In the case of multiple layers, however, this method loses attractiveness not only because of the large computational effort that it entails, but also because of accuracy issues. By contrast, Haji-Sheik and Beck [7] evaluate the transforms in terms of eigenfunctions, a strategy that parallels some of the features of the TLM described herein, except that they arrive at a transcendental eigenvalue problem that must be solved by numerically intensive search techniques and also runs into difficulties in the presence of complex eigenvalues. All of these difficulties are largely avoided in the TLM [8, 9].

In the ensuing, we first review the basic heat equations, and then proceed to develop the equations for the TLM in Cartesian and cylindrical coordinate systems.

2. HEAT CONDUCTION EQUATIONS

Consider a solid volume of material consisting of locally homogeneous regions within which the thermal properties are constant. Let \( \theta = \theta(x, y, z, t) \) be the absolute temperature in degrees (K) at a point \( x, y, z \) and time \( t \); \( \rho \) be the mass density; \( c \) be the specific heat; \( \rho c \) be the volumetric heat capacity, in \( (\text{JK}^{-1} \text{m}^{-3}) \); \( \lambda_x, \lambda_y, \lambda_z \) = the thermal conductivities along the three coordinate directions in \( (\text{Wm}^{-1}\text{K}^{-1}) \); and \( q^B = q^B(x, y, z, t) \) = a heat source per unit volume within the body which supplies thermal energy, in \( (\text{Wm}^{-3}) \). The heat equilibrium equation at some arbitrary point in the body is then

\[
\rho c \frac{\partial \theta}{\partial t} - \lambda_x \frac{\partial^2 \theta}{\partial x^2} - \lambda_y \frac{\partial^2 \theta}{\partial y^2} - \lambda_z \frac{\partial^2 \theta}{\partial z^2} = q^B
\]
Moreover, at any point on a closed surface $S$ that surrounds an arbitrary portion (i.e., sub-volume) of the body $B$ with unit normal vector pointing in the outward direction $\hat{n}$, the heat flux equation is

\[
\left[ \frac{\lambda_x}{c_x} \frac{\partial}{\partial x} \hat{i} + \frac{\lambda_y}{c_y} \frac{\partial}{\partial y} \hat{j} + \frac{\lambda_z}{c_z} \frac{\partial}{\partial z} \hat{k} \right] \cdot \hat{n} = q^S \tag{2}
\]

in which $q^S$ is the heat flux entering the sub-volume through the surface $S$, in $(Wm^{-2})$, and $\hat{i}, \hat{j}, \hat{k}$ are unit basis vectors for the three coordinate directions. This equation can be used to establish continuity conditions across the interfaces of dissimilar materials, i.e. the heat leaving one region must equal the heat entering the other. Finally, the boundary conditions at external boundaries of the body can be of four types:

- **Adiabatic boundary conditions**, by means of which heat sources are prescribed. These have the form of Equation (2), and are usually specified together with $q^S = 0$.
- **Isothermal boundary conditions**, where the temperature $\theta$ is prescribed.
- **Convection boundary conditions**, where a fluid in contact with external boundaries either removes or supplies heat. Similar in concept to the previous two.
- **Radiation boundary conditions**, where heat lost to the surrounding space through black-body radiation is a function of the body’s absolute surface temperature $\theta$ and that of the surrounding space $\theta_\infty$. This condition is modeled by setting the right-hand side of Equation (2) as $q^S = -\varepsilon \sigma (\theta^3 - \theta_\infty^3)$, where $\varepsilon$ is the emissivity constant ($\varepsilon = 1$ for an ideal black body), and $\sigma = 5.67 \times 10^{-8} (Wm^{-2}K^{-4})$ is the Stefan–Boltzmann constant. The negative sign is to account for a heat loss as opposed to a gain. For moderate changes away from ambient temperature, Stefan’s law can be linearized as $q^S = -4 \varepsilon \sigma \theta_\infty^3 (\theta - \theta_\infty)$, otherwise a non-linear problem will result.

By appropriate modifications and simplifications, Equations (1), (2) can also be expressed in cylindrical coordinates for materials exhibiting cylindrical material symmetry, for spherical systems with spherical material symmetry, and for plane layered systems (i.e., two-dimensional systems) that do not depend on one of the coordinate directions, say coordinate direction $y$, as will be considered in the next section.

3. **TLM FOR A PLANE, HORIZONTALLY LAYERED MEDIUM (2-D)**

Consider a plane, two-dimensional, horizontally layered system containing arbitrarily distributed thermal line sources, which are independent of the third coordinate $y$. The material properties are assumed to be locally homogeneous, but they may vary from layer to layer. To obtain a semi-discrete solution to this problem, proceed to divide the material layers into sub-layers that are thin in the finite element sense. Consider any one of these thin layers as a free body in space, which is bounded by upper and lower plane surfaces $S^U, S^L$, the distance between which is the layer thickness $h$. Without any loss of generality, set temporarily the origin of coordinates on the middle horizon so that the vertical coordinates of the two bounding surfaces are $z^U = (1/2)h$ and $z^L = -(1/2)h$, respectively. Designate the temperatures at these two bounding surfaces as $\theta^U = \theta(x, z^U, t)$ and $\theta^L = \theta(x, z^L, t)$. Using the finite element method, proceed to express the temperature at intermediate points within the layer by means of interpolation functions of order $m$, and ‘nodal’ temperatures.
of the form

\[ \theta(x, z, t) = \mathbf{N} \mathbf{0} \]  

(3)

in which the vector of \( m \)th order interpolation functions is

\[ \mathbf{N} = [N_0 \ldots N_m] \]  

(4)

and the vector of nodal temperatures is

\[ \mathbf{0} = \mathbf{0}(x, t) = \begin{bmatrix} \theta^U \\ \vdots \\ \theta^L \end{bmatrix} \]  

(5)

For example, if the dimensionless vertical coordinate is \( \zeta = z/h \), then the interpolated temperatures within the thin layer for both linear and quadratic expansions are

\[ \theta(x, z, t) = \mathbf{N} \mathbf{0} = \begin{bmatrix} \frac{1}{2} + \zeta & \frac{1}{2} - \zeta \end{bmatrix} \begin{bmatrix} \theta^U \\ \theta^L \end{bmatrix} \]  

(6a)

for a linear expansion, and

\[ \theta(x, z, t) = \mathbf{N} \mathbf{0} = \begin{bmatrix} (2\zeta + 1)\zeta & 1 - 4\zeta^2 & (2\zeta - 1)\zeta \end{bmatrix} \begin{bmatrix} \theta^U \\ \theta^M \\ \theta^L \end{bmatrix} \]  

(6b)

for a quadratic expansion, in which \( \theta^M = \theta(x, z^M, t) \) is the temperature on the middle surface (i.e. an intermediate node).

When the finite element expansion (3) is substituted into the differential equation (1) and boundary conditions (2) (specialized for plane conditions), it will be observed that these equations are not satisfied exactly, but leave instead residuals \( r^B, r^S \) which can be interpreted as spurious external sources:

\[ r^B = q^B - \rho C \frac{\partial \theta}{\partial t} + \lambda_x \frac{\partial^2 \theta}{\partial x^2} + \lambda_z \frac{\partial^2 \theta}{\partial z^2} \]  

(7a)

\[ r^U = q^S U - \lambda_z \frac{\partial \theta}{\partial z} \bigg|_{z=(1/2)h} \]  

(7b)

\[ r^L = q^S L + \lambda_z \frac{\partial \theta}{\partial z} \bigg|_{z=-(1/2)h} \]  

(7c)

These residuals can be disposed off by means of the well-known method of weighted residuals being applied in the ensuing. Let

\[ \delta \mathbf{0} = \mathbf{N} \delta \mathbf{0} = \delta \mathbf{0}^T \mathbf{N}^T \]  

(8)
be an arbitrary virtual temperature field. We then require the weighted residual $\delta \theta r$ of this field, aggregated over the entire volume and exterior surfaces of the thin layer, to be zero in a weak sense, i.e.

$$\int_{-\infty}^{+\infty} \left[ \delta \theta U U^x + \delta \theta h L + \int_{-(1/2)h}^{+(1/2)h} \delta \theta r B d\xi \right] d\xi = 0$$  \hspace{1cm} (9)

Clearly, the above condition will be satisfied if we require the weighted residual aggregated over any arbitrarily thin slice of width $d\xi$ of the layer to vanish, i.e. if

$$\delta \theta U U^x + \delta \theta h L + \int_{-(1/2)h}^{+(1/2)h} \delta \theta r B d\xi = 0$$  \hspace{1cm} (10)

Using Equations (7a)–(7c), the condition (10) is expressed in full as

$$\delta \theta U \left( q^{SU} - \hat{\lambda} \frac{\partial \theta}{\partial z} \right)_{z=(1/2)h} + \delta \theta L \left( q^{SL} + \hat{\lambda} \frac{\partial \theta}{\partial z} \right)_{z=-(1/2)h} + \int_{-(1/2)h}^{+(1/2)h} \delta \theta \left[ q^B - \rho c \frac{\partial \theta}{\partial t} + \hat{\lambda} \frac{\partial^2 \theta}{\partial x^2} + \hat{\lambda} \frac{\partial^2 \theta}{\partial z^2} \right] d\xi = 0$$  \hspace{1cm} (11)

Integrating the last term by parts, we obtain

$$\delta \theta U q^{SU} + \delta \theta L q^{SL} + \int_{-(1/2)h}^{+(1/2)h} \delta \theta \left[ q^B - \rho c \frac{\partial \theta}{\partial t} + \hat{\lambda} \frac{\partial^2 \theta}{\partial x^2} \right] d\xi - \int_{-(1/2)h}^{+(1/2)h} \hat{\lambda} \frac{\partial \theta}{\partial z} \frac{\partial \theta}{\partial z} \xi = 0$$  \hspace{1cm} (12)

Using the finite element expansion (3), (8), the various terms above are

$$\frac{\partial \theta}{\partial t} = \mathbf{N} \dot{\theta}$$  \hspace{1cm} (13a)

$$\frac{\partial^2 \theta}{\partial x^2} = \mathbf{N} \theta''$$  \hspace{1cm} (13b)

$$\frac{\partial \theta}{\partial z} = \frac{1}{h} \frac{\partial \theta}{\partial \zeta} = \frac{1}{h} \frac{\partial \mathbf{N}}{\partial \zeta} \theta$$  \hspace{1cm} (13c)

in which primes denote differentiations with respect to the appropriate variable. Substituting these into Equation (12), expressing the result in matrix form, and requiring the result to be valid for arbitrary variations $\delta \theta^T = [\delta \theta U \ldots \delta \theta L]$, one obtains the TLM equation for a single thin layer as

$$[q^{SU} 0 \ldots q^{SL}]^T + h \int_{-1/2}^{+1/2} \mathbf{N}^T q^B d\zeta - \rho c h \left[ \int_{-1/2}^{+1/2} \mathbf{N}^T \mathbf{N} d\zeta \right] \dot{\theta} + \hat{\lambda} \frac{\lambda}{h} \left[ \int_{-1/2}^{+1/2} (\mathbf{N}^T \mathbf{N}) d\zeta \right] \theta' = 0$$  \hspace{1cm} (14)
Table I. Layer matrices for linear and quadratic expansion.

<table>
<thead>
<tr>
<th></th>
<th>Linear expansion</th>
<th>Quadratic expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>( \frac{\rho c h}{6} \begin{pmatrix} 2 &amp; 1 \ 1 &amp; 2 \end{pmatrix} )</td>
<td>( \frac{\rho c h}{30} \begin{pmatrix} 4 &amp; 2 &amp; -1 \ 2 &amp; 16 &amp; 2 \ -1 &amp; 2 &amp; 4 \end{pmatrix} )</td>
</tr>
<tr>
<td>A</td>
<td>( \frac{\hat{\lambda} h}{6} \begin{pmatrix} 2 &amp; 1 \ 1 &amp; 2 \end{pmatrix} )</td>
<td>( \frac{\hat{\lambda} h}{30} \begin{pmatrix} 4 &amp; 2 &amp; -1 \ 2 &amp; 16 &amp; 2 \ -1 &amp; 2 &amp; 4 \end{pmatrix} )</td>
</tr>
<tr>
<td>G</td>
<td>( \frac{\hat{\sigma} z}{h} \begin{pmatrix} 1 &amp; -1 \ -1 &amp; 1 \end{pmatrix} )</td>
<td>( \frac{\hat{\sigma} z}{3h} \begin{pmatrix} 7 &amp; -8 &amp; 1 \ -8 &amp; 16 &amp; -8 \ 1 &amp; -8 &amp; 7 \end{pmatrix} )</td>
</tr>
</tbody>
</table>

The above can be written in the compact form

\[
q = M \frac{\partial}{\partial t} \theta - A \frac{\partial^2}{\partial x^2} \theta + G \theta
\]  

(15)

with

\[
M = \rho c h \int_{-1/2}^{+1/2} N^T N \, d\xi
\]

(16a)

\[
A = \hat{\lambda} h \int_{-1/2}^{+1/2} N^T N \, d\xi
\]

(16b)

\[
G = \frac{\hat{\sigma} z}{h} \int_{-1/2}^{+1/2} (N')^T N' \, d\xi
\]

(16c)

\[
q = [q^S U \quad 0 \quad \ldots \quad q^S L]^T + h \int_{-1/2}^{+1/2} N^T q^B \, d\xi
\]

(16d)

but in most cases the last formula (16d) is not needed, inasmuch as the sources are specified directly in their nodal form, as will be done in the next section.

Clearly, all three matrices are symmetric, both \( A, M \) are positive definite, and \( G \) is positive semi-definite. In the particular cases of linear and quadratic expansions, these matrices are given in Table I. Having completed the formulation for a single thin layer, we proceed next to consider the complete system of layers, numbering for this purpose the \( N \) layer interfaces (the nodes) from the top down. Clearly, temperature continuity conditions and heat flux conditions across the discrete layer interfaces will be satisfied simultaneously if the layer matrices for all layers are overlapped as shown schematically in Figure 1, a procedure that leads to block-tridiagonal, symmetric system matrices. The global external source vector in this case can be interpreted as the net balance of the heat fluxes entering or leaving any two adjoining layers, which equal the known external sources. If the boundary conditions at the top or bottom external surfaces are isothermal, then the first and/or the last row and column must be discarded. To avoid proliferation of symbols, we proceed to assign exactly the same symbols to the system matrices, so the system equation has exactly the same form as Equation (15), except that it involves a greater number \( N \) of variables. The use of these matrices is demonstrated in the next section.
HEAT DIFFUSION IN LAYERED MEDIA

Figure 1. Overlapping of thin-layer matrices.

4. RESPONSE TO THERMAL LINE SOURCES

Consider a horizontally layered system that is subjected to an impulsive thermal line source, i.e. to a source of the general form \( q = \delta(t) \delta(x) \delta(z - z') \), with \( \delta(\cdot) \) being the Dirac delta function, and \( z' \) is the elevation of the heat source. This problem can be solved by means of the TLM as follows. Subdivide the layered system into an adequate number of thin layers, and assemble the system matrices as described in the previous section. Assuming that \( z' \) coincides with the \( n \)th layer interface, then the source vector in Equation (16d) is of the form

\[
q = \delta(t) \delta(x) e_n
\]

in which \( e_n \) is a vector composed of zeros, except for its \( n \)th element, which is unity. Thus, this vector coincides with the \( n \)th column of an identity matrix of the same size as the global system. The system equation is then of the form

\[
M \frac{\partial}{\partial t} \mathbf{0} - A \frac{\partial^2}{\partial x^2} \mathbf{0} + G \mathbf{0} = \delta(t) \delta(x) e_n
\]

Carrying out a Fourier transform in space \( x \) and time \( t \), we obtain

\[
(i \omega M + k^2 A + G) \mathbf{0} = e_n
\]

To solve this equation, we first solve the homogeneous equation by means of a modal decomposition, i.e. we proceed to solve the eigenvalues problem for a given horizontal wave number \( k \)

\[
\Omega_j M \Phi_j = (k^2 A + G) \Phi_j
\]

Because \( M, A \) are positive-definite matrices and \( G \) is positive semi-definite, and since all three of these matrices are real and symmetric, it follows that all eigenvalues and eigenvectors will be real and non-negative. Indeed, for \( k \neq 0 \), all eigenvalues are positive. Hence, the eigenvalues problem satisfies standard orthogonality conditions, and can be normalized so that

\[
\Phi^\top M \Phi = I \quad (21a)
\]

\[
\Phi^\top (k^2 A + G) \Phi = \Omega \quad (21b)
\]
in which \( \Phi = \{ \phi_j \} \) is the modal matrix and \( \Omega = \text{diag}\{ \Omega_j \} \) is the spectral matrix. Having solved the eigenvalue problem, we proceed to solve for \( \theta \) by modal superposition, i.e. we make an ansatz of the form

\[
\theta = \Phi \gamma
\]  

(22)

Substitution into Equation (19) and multiplication by the transposed modal matrix gives

\[
\Phi^T (i\omega M + k^2 A + G) \Phi \gamma = \Phi^T e_n
\]  

(23)

and in view of the orthogonality conditions, this changes into

\[
(i\omega I + \Omega) \gamma = \Phi^T e_n = \phi_n^T
\]  

(24)

The coefficient matrix on the left is diagonal, so the solution is obtained trivially as

\[
\gamma = \text{diag}\{i\omega + \Omega_j\}^{-1} \phi_n^T = D \phi_n^T
\]  

(25)

Hence,

\[
\theta(k, \omega) = \Phi D \phi_n^T
\]  

(26)

The solution in the time domain, for a given horizontal wave number, follows from an inverse Fourier transform over frequencies, which can be obtained by a contour integration with poles at \( \omega_j = i\Omega_j \). The result is

\[
\theta(k, t) = \Phi E \phi_n^T
\]  

(27)

with

\[
E = \text{diag}\{\exp(-\Omega_j t)\}, \quad t \geq 0
\]  

(28)

Finally, the solution in the spatial domain follows from the inverse Fourier transform

\[
\theta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi E \phi_n^T e^{-ikx} \, dk
\]  

(29)

With \( \Phi = \{ \phi_j \} = \{ \phi_{nj} \} \), the \( m \)th component of the temperature vector due to an impulsive source at the \( n \)th elevation is then

\[
\theta_{mn}(x, t) = \sum_{j=1}^{N} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi_{mj} \phi_{nj} e^{-i\Omega_j t} e^{-ikx} \, dk
\]  

(30)

In the general case, the integrals above can only be evaluated numerically, because both the modal vectors and modal frequencies depend on the horizontal wave number. An important exception is the case of functionally graded materials considered next, which can be integrated analytically, as will be shown.
4.1. Functionally graded material

In the special case of a homogeneous material, or more generally, when the lateral thermal diffusivity $\alpha_x = \lambda_x/\rho c$ is constant throughout the medium, i.e. when the variation with the vertical coordinate of the volumetric heat capacity varies in tandem with the lateral thermal conductivity $\lambda_x = \alpha_x \rho c$, then $\Lambda = \alpha_x M$, in which case the eigenvalues problem reduces to the form

$$ (\Omega_j - k^2 \alpha_x) M \Phi_j \equiv \varepsilon_j M \Phi_j = G \Phi_j $$

(31)

As can be seen, $\varepsilon_j, \Phi_j$ are no longer functions of the horizontal wave number. Hence, the eigenvalue problem needs to be solved just once, after which one proceeds to obtain the wave number-dependent eigenvalues as $\Omega_j = \varepsilon_j + k^2 \alpha_x$. It follows that

$$ \theta_{mn}(x,t) = \int_0^\infty f(x,t) \sum_{j=1}^N \phi_{mj} \phi_{nj} e^{-\varepsilon_j t} $$

(32)

in which (Gradshteyn and Ryzhik [10, p. 338, Equation 3.462-3, or Matlab symbolic integration])

$$ f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\alpha_x t k^2} e^{-ikx} dk = \frac{1}{\sqrt{\pi} \sqrt{4\alpha_x t}} e^{-x^2/4\alpha_x t}, \quad t \geq 0 $$

(33)

Equation (30) or (32) provides Green’s functions for a thermal line source in a layered or in a functionally graded medium. These can be used to obtain the response to more general, spatially distributed thermal sources with arbitrary temporal variations by means of numerical convolutions with the source terms. They can also be used as fundamental solutions in the context of a boundary element formulation, for example, to find solutions when the plate contains a defect such as delamination of finite size. Notice in passing that solutions to heat diffusion problems in general, and Equation (30) or (32) in particular, violate information theory because changes in temperature (even if minute) appear instantly everywhere. Still, heat from a source will appear to spread out gradually despite the fact that the ‘wave front’ moves at infinite speed.

If and when the material properties are constant throughout, the eigenvalue problem can be solved exactly and in closed form for any combination of boundary conditions at the two external bounding surfaces. For example, for $N$ layers with a linear expansion, with adiabatic conditions at the top and isothermal conditions at the bottom, the eigenvalues and eigenvectors can be shown to be given by

$$ \varepsilon_j = \frac{4N^2}{H^2} \frac{\sin^2 \beta_j}{1 - \frac{2}{3} \sin^2 \beta_j} \frac{\lambda_z}{\rho c} $$

(33a)

$$ \phi_{ij} = \sqrt{\frac{2}{\rho c H}} \cos[2(i-1)\beta_j] $$

(33b)

$$ \beta_j = \frac{\pi}{4N} (2j - 1), \quad j = 1, 2, \ldots, N $$

(33c)
In the limit of infinitely many layers, $N \to \infty$, $\beta_j \to 0$, $4N^2 \sin^2 \beta_j \to (\pi/2)^2(2j-1)^2$, in which case
\[
\varepsilon_j = \left( \frac{\pi(2j-1)}{2H} \right)^2 \frac{\lambda_z}{\rho c}
\]
and we recover the exact solution for a continuous, homogeneous plate bounded by an adiabatic boundary at the top and an isothermal boundary at the bottom.

Finally, it should be observed that the starting, spatially homogeneous ambient temperature condition $\theta_\infty$ (i.e. the temperature far away from the thermal source) has no effect on these equations. This is because $\theta_\infty = 1$, so that $\theta_\infty = \theta_\infty^\prime = G \theta_\infty = 0$. The last identity follows because $G$ is only semi-definite, and gives zero when it multiplies a vector of ones. Thus, all that matters is the temperature change and not the absolute temperature.

4.2. Plate of finite width

The previous sections dealt with a plane layered system of infinite width. However, solutions for systems of arbitrary width $L$ can readily be found in several alternative ways. For example, the image source method could be used with a finite number of sources spaced at distances $L$ to simulate any pair of lateral boundary conditions. Alternatively, a system of periodic sources can be simulated by replacing the Fourier integral over the horizontal wave number $k$ by a summation over discrete wave numbers with wave number step $k = 2\pi/L$. Then again, one could also use Green’s functions previously obtained in the context of a boundary element formulation, and proceed to simulate any finite, irregularly shaped region of the layered system.

5. TLM IN CYLINDRICAL COORDINATES

We consider next the TLM formulation of point sources anywhere in a layered medium by recourse to cylindrical coordinates. Inasmuch as this case strongly parallels the plane case, it suffices to present here only the most essential details.

In cylindrical coordinates $r, \varphi, z$, the counterparts to Equations (1), (2) for the heat equilibrium in the body $B$ and on the exterior surface $S$ are
\[
\rho c \frac{\partial \theta}{\partial t} - \lambda_r \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) - \lambda_{\varphi} \frac{1}{r} \frac{\partial^2 \theta}{\partial \varphi^2} - \lambda_z \frac{\partial^2 \theta}{\partial z^2} = q^B
\]
(35)
and
\[
\left( \lambda_r \frac{\partial \theta}{\partial r} e_r + \lambda_{\varphi} \frac{1}{r} \frac{\partial \theta}{\partial \varphi} e_{\varphi} + \lambda_z \frac{\partial \theta}{\partial z} e_z \right) \cdot n = q^S
\]
(36)
in which the thermal conductivities $\lambda_r, \lambda_{\varphi}, \lambda_z$ are constant within each of the horizontal layers and are assumed to be independent of the azimuth (i.e. cylindrical material symmetry). Following the same approach as in the previous sections, we assume a finite element solution for a thin layer by expressing the temperature within this thin layer in terms of the ‘nodal’ temperatures at the bounding surfaces. Also, for expansion of order higher than linear, we must include terms in an
appropriate number of inner surfaces, such as the middle surface. In general, the interpolation functions \( N \) are as before, see Equations (6a), (6b):

\[
\theta(r, \varphi, z, t) = N\theta(r, \varphi, t)
\]  

(compare with Equations (3), (4), (5). Introduction of this expansion into the differential equations (35), (36) and application of the weighted residual method leads to the TLM equation in cylindrical coordinates (compare with (15)):}

\[
q = M \frac{\partial}{\partial t} \theta - A_r \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) - A_\varphi \frac{1}{r^2} \frac{\partial^2 \theta}{\partial \varphi^2} + G \theta
\]  

with

\[
M = \rho c h \int_{-1/2}^{+1/2} N^T N d\zeta
\]  

\[
A_r = \lambda_r h \int_{-1/2}^{+1/2} N^T N d\zeta
\]  

\[
A_\varphi = \lambda_\varphi h \int_{-1/2}^{+1/2} N^T N d\zeta
\]  

\[
G = \frac{\lambda_z}{h} \int_{-1/2}^{+1/2} \left( N' \right)^T N' d\zeta
\]  

As before, the system matrices are obtained by overlapping the element matrices as shown in Figure 1 and, again, we use the same notation for the system as we do for the individual elements, to avoid proliferation of symbols.

6. RESPONSE TO THERMAL POINT SOURCES

In cylindrical coordinates, the system equations for an impulsive point source placed at the \( n \)th thin-layer interface is (compare with (18))

\[
M \frac{\partial}{\partial t} \theta - A_r \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) - A_\varphi \frac{1}{r^2} \frac{\partial^2 \theta}{\partial \varphi^2} + G \theta = \delta(t) \frac{\delta(r)}{2\pi r} e_n
\]  

Indeed, integrating the source term over a small area of arbitrary radius \( a \) enclosing the source, one obtains a unit value, which is the strength of the source.

To solve Equation (40), we proceed to carry out a Fourier–Bessel (i.e. Hankel) transform in the azimuth and in the radial coordinate. For example, the Hankel transform of an arbitrary scalar function \( f(r, \varphi) \) is of the form

\[
\tilde{f}_n(k) = \int_0^\infty r J_n(kr) \int_0^{2\pi} \left( \frac{\cos n\varphi}{\sin n\varphi} \right) f(r, \varphi) \, d\varphi \, dr, \quad n = 0, 1, 2, \ldots
\]  

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However, because the source in Equation (40) does not depend explicitly on the azimuth, only the Fourier terms of order zero survive this transformation. The result is

\[(i\omega \mathbf{M} + k^2 \mathbf{A}_r + \mathbf{G}) \mathbf{\theta} = \mathbf{e}_n\]  

(42)
a result that agrees perfectly with Equation (19). Thus, this equation admits the same spectral decomposition as the plane case, the modes are the same as before, and the response in the wave number–time domain is again given by Equations (27) and (28). To obtain the temperature field in the space–time domain, we carry out an inverse Hankel transform of order zero using Equation (27) as kernel:

\[
\mathbf{\theta}(r, \varphi, t) = \int_0^\infty \mathbf{\Phi} \mathbf{E} \mathbf{\phi}_n^T J_0(kr) k \, dk
\]

(43)
so the \(m\)th component of temperature due to an impulsive thermal source at the \(n\)th elevation is

\[
\theta_{mn}(r, \varphi, t) = \sum_{j=1}^{N} \int_0^{+\infty} \phi_{mj} \phi_{nj} e^{-\Omega_j t} J_0(kr) k \, dk
\]

(44)
In general, the integral transform above must be evaluated numerically, a task that can be accomplished accurately and effectively with the method of Guizar-Sicairos and Gutierrez-Vega [11]. Again, in the case of a functionally graded material, i.e. \(\lambda_r = \lambda_r \rho c\), with \(\lambda_r\) being a constant throughout the medium, then \(\mathbf{A}_r = \lambda_r \mathbf{M}\), the eigenvalue problem is given by Equations (31), (32), changes into

\[
\theta_{mn}(r, \varphi, t) = f(r, t) \sum_{j=1}^{N} \phi_{mj} \phi_{nj} e^{-\varepsilon_j t}
\]

(45a)
with

\[
f(r, t) = \int_0^{+\infty} e^{-\varepsilon_j t} J_0(kr) k \, dk = \frac{1}{2\varepsilon_j t} \exp \left(-\frac{r^2}{4\varepsilon_j t}\right), \quad t \geqslant 0
\]

(45b)
Equations (45a), (45b) provide Green’s function (fundamental solution) for a unit impulsive point source in a functionally layered medium. Resorting to convolution, this solution can also be used to construct additional solutions for sources with arbitrary temporal variation.

7. RESPONSE TO DISTRIBUTED THERMAL SOURCES

We consider next impulsive thermal sources with arbitrary variation in the radial coordinate, i.e. a source of the form

\[
S(r, t) = S(r) \delta(t)
\]

(46)
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Because of the cylindrical symmetry, the Fourier–Bessel transform involves only the components of order zero. Hence, the wave number representation of the spatial component of this source is

\[ S(k) = \int_0^{2\pi} \int_0^{\infty} S(r) J_0(kr) r \, dr \, d\phi \]

\[ = 2\pi \int_0^{\infty} S(r) J_0(kr) r \, dr \]  
(47)

In the particular case of a ring source \( S(r) = S_0 \delta(r - R) \), we obtain from Equation (47)

\[ S(k, R) = 2\pi R S_0 J_0(kR) \]  
(48)

Alternatively, in the case of a disk source of intensity \( S_0 \) that acts within a circular area of radius \( a \), its wave number representation is

\[ S(k, a) = 2\pi S_0 \int_0^{\infty} J_0(kr) r \, dr \]

\[ = 2\pi a S_0 \frac{J_1(ka)}{k} \]  
(49)

which elicits a temperature field

\[ \theta(r, \varphi, z, z', t) = \sum_{j=1}^{N} \int_0^{\infty} [\phi_j(z) \phi_j(z') e^{-\Omega_j t} J_0(kr) S(k, a)] k \, dk \]

\[ = 2\pi a S_0 \sum_{j=1}^{N} \int_0^{\infty} [\phi_j(z) \phi_j(z') e^{-\Omega_j t} J_1(ka) J_0(kr)] \, dk \]  
(50)

in which \( \phi_j(z) = \phi_j(z_m) \equiv \phi_{mj}, \phi_j(z') = \phi_j(z_n) \equiv \phi_{nj} \)

8. THERMAL DIPOLES

Dipoles can be visualized as two thermal point sources of opposite polarity (sign) and in infinitesimally close proximity, whose amplitude is inversely proportional to their distance (the dipole’s arm) while the product of their intensity times the arm is a constant that equals the strength \( S \) of the dipole. Dipoles play a central role in the formulation of cracks via the boundary element method, inasmuch as they allow modeling abrupt discontinuities in temperature across the crack. One can distinguish two basic dipoles depending on the orientation of the dipole’s arm: horizontal and vertical dipoles. Dipoles with other orientations can also be modeled.

8.1. Horizontal dipole

Consider a horizontally layered system within which two point sources of equal and opposite amplitude \( A \) are acting at some common elevation \( z' \) at a short horizontal distance \( 2a \) apart. Also, let \( \theta(r, z, z', t) \) be the cylindrically symmetric temperature field elicited at some point \( r, \varphi, z \) by a
unit point source placed on the axis at elevation $z'$ (i.e. Green’s function for a point source). The two sources forming the dipole can then be represented compactly together as

$$q(r, \varphi, z', t) = A \delta(t) \frac{\delta(r - a)}{r} [\delta(\varphi) - \delta(\varphi - \pi)] \delta(z')$$  \hspace{1cm} (51)

that is, both point sources act at a radial distance $a$ from the axis, the first at zero azimuth, and the other at 180° thereto. It can easily be verified through integration around a small volume enclosing one of the two sources that one recovers the amplitude $A$, as one should. Applying a Fourier–Bessel transform, we obtain the dipole’s wave number representation as

$$\tilde{q}_n(k, z', t) = \int_0^\infty r J_n(kr) \int_0^{2\pi} \left( \frac{\cos n\varphi}{\sin n\varphi} \right) q(r, \varphi, z', t) \, d\varphi \, dr$$

$$= \begin{cases} 0 & \text{even } n \\ 2A \delta(t) \delta(z') \left( 1 \over 0 \right) J_n(ka) & \text{odd } n \end{cases}$$  \hspace{1cm} (52)

For small distances $a$ and to linear approximation,

$$\lim_{a \to 0} J_n(ka) = \begin{cases} \frac{1}{2} ka, & n = 1 \\ 0, & n > 1 \end{cases}$$  \hspace{1cm} (53)

Hence, $\lim_{a \to 0} \frac{1}{2} Ak = \frac{1}{2}k S$, in which $S$ is the strength of the dipole. It follows that

$$\tilde{q}_1(k, z', t) = kS \delta(t) \delta(z'), \quad \tilde{q}_n(k, z', t) = 0, \quad n > 1$$  \hspace{1cm} (54)

is the wave number representation of the dipole, which is non-zero only for an azimuthal Fourier index $n = 1$, and is associated with a variation $\cos \varphi$ in the azimuth. The inverse Fourier–Bessel transform is then

$$\vartheta(r, \varphi, z, z', t) = S \cos \varphi \sum_{j=1}^{N} \int_0^{+\infty} \phi_j(z) \phi_j(z') e^{-\Omega_j t} J_1(kr) k^2 \, dk \, d\varphi$$

$$= -S \cos \varphi \frac{\partial}{\partial r} \sum_{j=1}^{N} \int_0^{+\infty} \phi_j(z) \phi_j(z') e^{-\Omega_j t} J_0(kr) k \, dk$$

$$= -S \cos \varphi \frac{\partial}{\partial r} \vartheta(r, z, z', t)$$  \hspace{1cm} (55)

that is, the response due a unit horizontal dipole is the negative of the derivative with respect to $r$ of the Green’s functions for a point source, and has a variation $\cos \varphi$ in the azimuth. It follows that the dipole’s counterpart to Equation (45b) for a functionally graded material is

$$f(r, t) = \int_0^{+\infty} e^{-z_0 t k^2} J_1(kr) k^2 \, dk = -\frac{\partial}{\partial r} \left[ \frac{1}{2z_0} \exp \left( -\frac{r^2}{4z_0 t} \right) \right]$$

$$= \frac{r}{(2z_0 t)^2} \exp \left( -\frac{r^2}{4z_0 t} \right)$$  \hspace{1cm} (56)
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8.2. Vertical dipole

In this case, the two sources forming the dipole are on the vertical axis at elevations $z' \pm a$. The response is thus

$$\vartheta(r, \varphi, z, z', t) = A[\vartheta(r, \varphi, z, z' + a, t) - \vartheta(r, \varphi, z, z' - a, t)]$$

$$= 2Aa \frac{\partial \vartheta}{\partial z'} = S \frac{\partial \vartheta}{\partial z'}$$  \hspace{1cm} (57)

Now, because of the layered nature of the system (as well as the presence of external surfaces), the derivative with respect to the source location cannot be expressed in terms of the derivative with respect to the receiver location. In addition, in the TLM method being considered herein, the vertical derivatives are discontinuous across each thin layer—at least in the case of a linear expansion. The simplest alternative then is to actually simulate the dipole as two point sources applied on neighboring thin layers, the first immediately above the second, with the positive source at the top (interface $n$) and the negative source underneath (interface $n+1$); observe that the layer indices increase with depth. In that case, the response is simply

$$\vartheta(r, \varphi, z, z', t) = S \frac{\theta_{mn} - \theta_{m,n+1}}{h_n}$$  \hspace{1cm} (58)

in which $\theta_{mn} = \vartheta(r, \varphi, z_m, z'_n, t)$ is the temperature at the receiver elevation $z_m$ caused by a point source placed at elevation $z'_n$, etc. Also, $h_n$ is the thinness (i.e. thickness) of the layer containing the dipole.

8.3. Temperature and heat flux elicited by vertical disk dipole

A vertical disk dipole is obtained from Equation (57) by using the temperature field for a disk source given by Equation (50) in lieu of Equation (44) (or (45a), (45b)). This results in the temperature field

$$\vartheta(r, \varphi, z, z', t) = 2\pi a S \sum_{j=1}^{N} \int_{0}^{\infty} \left[ \phi_j \frac{\partial \phi_j'}{\partial z'} e^{-\Omega_j t} J_1(ka) J_0(kr) \right] dk$$  \hspace{1cm} (59)

The vertical heat flux associated with such a disk dipole is

$$Q_z(r, \varphi, z, z', t) = -\lambda_z \frac{\partial \vartheta}{\partial z}$$

$$= -2\pi a S \lambda_z \sum_{j=1}^{N} \int_{0}^{\infty} \left[ \frac{\partial \phi_j}{\partial z} \frac{\partial \phi_j'}{\partial z'} e^{-\Omega_j t} J_1(ka) J_0(kr) \right] dk$$  \hspace{1cm} (60)

Hence, the average heat flow over an area equal to and concentric with the disk source is

$$\bar{Q}_z = -\frac{1}{\pi a^2} \int_{\text{area}} Q_z r \, dr \, d\varphi = -4\pi S \lambda_z \sum_{j=1}^{N} \int_{0}^{\infty} \left[ \frac{\partial \phi_j}{\partial z} \frac{\partial \phi_j'}{\partial z'} e^{-\Omega_j t} J_1^2(ka) J_0^2(kr) \right] \frac{1}{k} dk$$  \hspace{1cm} (61)
and in particular, at the location of the dipole,

$$\tilde{Q}_z' = -4\pi S \bar{\lambda}z' \sum_{j=1}^{N} \int_{0}^{\infty} \left[ \frac{\partial \phi_j'}{\partial z'} \right]^2 \frac{e^{-\Omega_j J_1^2(ka)}}{k} dk \tag{62}$$

In the particular case of a functionally graded material, the average heat flux over the disk area is

$$\tilde{Q}_z' = -2\pi S f(t) \bar{\lambda}z' \sum_{j=1}^{N} \left( \frac{\partial \phi_j'}{\partial z'} \right)^2 e^{-\varepsilon_j t} \tag{63a}$$

$$f(t) = 2 \int_{0}^{+\infty} e^{-(x_t/a^2)k^2} \frac{J_1^2(k)}{k} dk = 1 - e^{-a^2/2z_t} \left( I_0 \left( \frac{a^2}{2z_t} \right) + I_1 \left( \frac{a^2}{2z_t} \right) \right) \tag{63b}$$

with the $I_n$ being the modified Bessel functions (integral obtained using Matlab). Finally, in the TLM being considered here, the vertical derivatives are replaced by vertical differences at the $n$th layer where the dipole is applied, which results in

$$\tilde{Q}_z' = -\frac{4\pi S \bar{\lambda}z'}{h_n^2} \sum_{j=1}^{N} \int_{0}^{\infty} \left[ (\phi_{nj} - \phi_{n+1,j})^2 e^{-\Omega_j J_1^2(ka)}} {k} \right] dk \tag{64}$$

which is valid for any layered material configuration. Alternatively, for a functionally graded material, we have the much simpler result

$$\tilde{Q}_z' = -\frac{2\pi S \bar{\lambda}z'}{h_n^2} f(t) \sum_{j=1}^{N} (\phi_{nj} - \phi_{n+1,j})^2 e^{-\varepsilon_j t} \tag{65}$$

with $f(t)$ given by Equation (63b). We shall apply Equation (64) in an example of application in the next section.

9. EXAMPLE OF APPLICATION: PENNY-SHAPED CRACK

Consider a test specimen reported by Starnes [12], Starnes et al. [13–15], which consists of a concrete slab 20 mm thick overlain by two layers of carbon fiber reinforced polymers (CFRP) of 0.5 mm thickness each for a total specimen thickness of 21 mm. The fibers within the top layer are parallel to the surface, while those of the bottom layer are perpendicular to it. Milled immediately underneath the CFRP’s is a penny-shaped crack of 25 mm in diameter and 0.2 mm in thickness, which is meant to simulate an area where the CFRP has debonded from the concrete. Starnes reports the thermal properties for these materials listed in Table II.

The specimen is uniformly heated with lamps from above for a time between 0.05 and 3 s with an intensity (heat flux) that ranges from 5 to 100 kW/m$^2$ (square pulses). The outer surfaces of the specimen are assumed by Starnes to be adiabatic.

To model this problem, we proceed to construct a TLM representation where the concrete is divided into 89 thin layers with a finer discretization in the vicinity of the crack, and the CFRP’s are divided uniformly into 20 thin layers each. Considering that Starnes measured the surface temperatures of her specimen in the $x$ direction (along the fibers of the second layer), we make the simplifying approximation in our model that those thermal properties apply to both the $x$ and...
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Table II. Material properties of concrete block overlain with CFRP.

<table>
<thead>
<tr>
<th></th>
<th>CFRP outer layer</th>
<th>CFRP inner layer</th>
<th>Concrete</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho) (kg/m(^3))</td>
<td>1600</td>
<td>1600</td>
<td>2400</td>
</tr>
<tr>
<td>(c) (J/kg/°K)</td>
<td>1200</td>
<td>1200</td>
<td>800</td>
</tr>
<tr>
<td>(\lambda_x) (W/m/°K)</td>
<td>7</td>
<td>0.8</td>
<td>1.5</td>
</tr>
<tr>
<td>(\lambda_y) (W/m/°K)</td>
<td>0.8</td>
<td>0.8</td>
<td>1.5</td>
</tr>
<tr>
<td>(\lambda_z) (W/m/°K)</td>
<td>0.8</td>
<td>7</td>
<td>1.5</td>
</tr>
<tr>
<td>(\alpha_x) (m(^2)/s)</td>
<td>3.6458 \times 10^{-6}</td>
<td>0.4167 \times 10^{-6}</td>
<td>0.7813 \times 10^{-6}</td>
</tr>
<tr>
<td>(\alpha_y) (m(^2)/s)</td>
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</tr>
<tr>
<td>(\alpha_z) (m(^2)/s)</td>
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<td>3.6458 \times 10^{-6}</td>
<td>0.7813 \times 10^{-6}</td>
</tr>
</tbody>
</table>

y directions, i.e. we assume a transversely isotropic material to preserve cylindrical symmetry and thus be able to evaluate the integral transform in \(k\) (an alternative Cartesian TLM formulation would have been possible, but it would have required significantly more computational effort, which is not really necessary). Since this layered system is not functionally graded, we evaluated the inverse Hankel transforms over radial wave numbers using the method of [11] referred to earlier.

**Step 1: Temperature field without crack.** The first step is to find the temperature field \(\theta_{m1}^n\) at any location in the specimen without the crack due to an impulsive heat source at the upper interface, and in particular, the heat flux that passes through the circular area where the crack is located. This is a straightforward task inasmuch as this problem is one-dimensional, but we resort nevertheless to a TLM solution for consistency in the results. This is accomplished by solving Equation (20) with \(k = 0\) (i.e. no variation in horizontal planes), and then substituting the resulting modes and frequencies into the modal superposition Equation (27) with \(n = 1\) (i.e. heat lamp sources at the top layer). The temperature profile produced by the actual, time-varying external source applied at the top of the layered system without the crack is given by the convolution with the source

\[
\theta_{m1}^n(t) = q(t) * \sum_{j=1}^{N} \phi_{mj} \phi_{1j} e^{-\Omega_{jt}}
\]

in which \(q(t)\) is the temporal representation of the source, here a rectangular pulse of amplitude \(q_0\) and duration \(td\). This temperature profile is uniform in horizontal planes. Since the modes do not depend on time, the convolution above is simply the convolution of a rectangular window with an exponential, which gives

\[
\theta_{m1}^n(t) = \begin{cases} 
q_0t \sum_{j=1}^{N} \phi_{mj} \phi_{1j} \left[ \frac{1 - e^{-\Omega_{jt}}}{\Omega_{jt}} \right], & t < td \\
q_0td \sum_{j=1}^{N} \phi_{mj} \phi_{1j} \left[ \frac{e^{-\Omega_{jt}(t-td)} - e^{-\Omega_{jt}t}}{\Omega_{jt}td} \right], & t \geq td 
\end{cases}
\]

The heat flux at the location immediately above the crack (the layer immediately below the \(n\)th interface) is then obtained from the finite difference:

\[
Q_n^* = -\frac{\partial \theta_n^*}{\partial z} = -\frac{1}{h_n} (\theta_{n,1}^* - \theta_{n+1,1}^*)
\]

This is a negative number, indicating that the heat flows down, i.e. in direction opposite to the positive \( z \) direction, which is the direction of falling temperatures.

**Step 2: Crack discontinuity modeled with a disk dipole.** Next, we model the crack by means of a disk dipole with a time-varying intensity yet to be determined. The average heat flux elicited by an *impulsive* disk dipole at the location where it is applied is given by Equation (64). As described in the ensuing, we determine the intensity of the time-varying dipole so that the net heat flux through the crack is zero at all times, which implies that the crack is adiabatic. In other words, the heat flux produced by the disk dipole cancels exactly the heat flux of the external source, but only within the area of the crack. Having found the intensity of the dipole source, we can find the thermal response anywhere else by superposition of the two thermal fields, namely the one due to the external source and that due to the disk dipole. This is, of course, in essence a one-node boundary element solution to this problem.

Let \( S(t) \) be the time-varying strength of the disk dipole source at the location of the crack. The temperature field elicited by such a source follows from the convolution

\[
\theta_{mn} = S(t) \ast \theta_{mn}(t) \tag{69}
\]

A similar expression holds for the average heat flux over the crack:

\[
Q_n(t) = S(t) \ast \bar{Q}_n(t) \tag{70}
\]

To simulate the crack, we establish the condition

\[
Q^*_n + S(t) \ast \bar{Q}_n(t) = 0 \tag{71}
\]

Since we know both \( Q^*_n(t) \) and \( \bar{Q}_n(t) \), we can use the above to find the time-varying amplitudes of the disk source. Once we know \( S(t) \), we can find the temperatures anywhere in the system by superposition of the two fields:

\[
\theta_{mn} = \theta_{mn}^* + S(t) \ast \theta_{mn}(t) \tag{72}
\]

Consider next the above expressions discretized in time, with all arrays being causal, i.e. having implicit zero values for negative times:

\[
Q^*_n(t) = [q^*_0, q^*_1, q^*_2, \ldots], \quad \bar{Q}_n(t) = [q_0, q_1, q_2, \ldots], \quad S(t) = [s_0, s_1, s_2, \ldots] \tag{73}
\]

Hence, the discretized version of \( Q^*_n(t) + S(t) \ast \bar{Q}_n(t) = 0 \) is

\[
q^*_j + \Delta t \sum_{k=0}^j q_{j-k}s_k = 0 \tag{74}
\]

Defining \( \tilde{q}_j = q_j \Delta t \), this can be solved recursively as

\[
q^*_0 + \tilde{q}_0s_0 = 0 \rightarrow s_0 = -q^*_0/\tilde{q}_0 \\quad q^*_1 + \tilde{q}_1s_0 + \tilde{q}_0s_1 = 0 \rightarrow s_1 = -(q^*_1 + \tilde{q}_1s_0)/\tilde{q}_0 \\quad \vdots \\quad q^*_j + \tilde{q}_j s_0 + \tilde{q}_j s_1 + \cdots + \tilde{q}_0s_j = 0 \rightarrow s_j = -(q^*_j + \tilde{q}_j s_0 + \cdots + \tilde{q}_1s_{j-1})/\tilde{q}_0 \tag{75}
\]
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Figure 2. Snapshots of the temperature profiles corresponding to various time instants.

The algorithm described is stable and provides an accurate solution to the example problem being considered.

Finally, Figure 2 shows some snapshots of the resulting temperature profiles within the slab corresponding to various time instants, where the geometry and material properties are as described above. The individual boundaries of the thin layers are also indicated. A uniform heat source with an intensity of 50 kW/m² is applied at the upper surface for a duration of 2 s. Consequently, it can be seen from Figure 2 that the temperature rises from 0 to 2 s, and afterwards smooths out while distributing over the slab. It is noted here that, due to the one-node boundary element approximation, the temperature profile close to the rim of the crack may deviate slightly from the...
theoretically exact solution; however, this could be removed easily by a more refined model of the crack using more elements. In general, the results shown in Figure 2 are excellent and demonstrate the validity and efficiency of the proposed method.

10. MODEL REFINEMENT

The number of layers needed for any given TLM problem follows rules that are similar to those of finite elements. The thickness of the layers depends on the total thickness of the specimen, the spatial variation of the material layers, the spatial distribution of the thermal sources, and the temporal characteristics of the sources i.e. their rate of change with time. Clearly, the layers must be sufficient in number to capture the expected gradients and variations of temperatures throughout the model, and especially so in the vicinity of point sources. The reader is referred to the standard recommendations found in finite element treatises for further insights into this matter.

11. CONCLUSIONS

This paper expounded the application of the thin-layer method (TLM) to heat diffusion problems, and demonstrated its suitability to material configurations where finite elements are not an optimal or even viable option, such as layered systems subjected to spatially distributed thermal sources whose lateral dimensions are large in comparison with the scales of other material and field dimensions. After reviewing the basic heat equations, explicit formulae were derived for impulsive as well as distributed point and dipole sources with formulations in Cartesian and cylindrical coordinates. Furthermore, it was demonstrated how solutions in response to spatially and temporally impulsive sources can also be used as fundamental solutions in the context of a boundary element formulation. The theory presented was illustrated with an example of a concrete slab overlain by two layers of carbon fiber reinforced polymers (CFRP), where a penny-shaped crack is meant to simulate an area where the CFRP has debonded from the concrete. This problem is of significant interest in non-destructive testing, for example, when infrared thermography is used to locate defects within concrete decks and pavements.

REFERENCES


