Wave propagation analysis in inhomogeneous piezo-composite layer by the thin-layer method

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SUMMARY

The thin-layer method (TLM) is used to study the propagation of waves in inhomogeneous piezo-composite layered media caused by mechanical loading and electrical excitation. The element is formulated in the time-wavenumber domain, which drastically reduces the cost of computation compared to the finite element (FE) method. Fourier series are used for the spatial representation of the unknown variables. The material properties are allowed to vary in the depthwise direction only. Both linear and exponential variations of elastic and electrical properties are considered. Several numerical examples are presented, which bring out the characteristics of wave propagation in anisotropic and inhomogeneous layered media. The element is useful for modelling ultrasonic transducers (UT) and one such example is given to show the effect of electric actuation in a composite material and the difference in the responses elicited for various ply-angles. Further, an ultrasonic transducer composed of functionally graded piezoelectric materials (FGPM) is modelled and the effect of gradation on mechanical response is demonstrated. The effect of anisotropy and inhomogeneity is shown in the normal modes for both displacement and electric potential. The element is further utilized to estimate the piezoelectric properties from the measured response using non-linear optimization, a strategy that is referred to as the pulse propagation technique (PPT). Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: thin-layer method; finite elements; wave propagation; piezoelectric composites; inverse problems; normal modes

1. INTRODUCTION

The demand for new materials has led to the development of fibre or particle reinforced composite materials, which are currently being used as the primary structural materials in the aerospace and automobile industries. However, there are several applications of composite materials where they have proved to be inadequate. For example, in high temperature applications,
Metal matrix composite (MMC) materials are used because of their good thermal stability, high thermal conductivity, and their ability to withstand high temperatures. However, a mismatch in the coefficient of thermal expansion (CTE) between the MMC and the base material induces thermal residual stresses, a characteristic problem with MMCs that can lead to failure of the structure. In another example, conventional polymer materials are typically dielectric that is, they are electrically non-conductive and transparent to electro-magnetic radiation. However, these cannot provide protections to electronic equipments from external radiation and do not allow radiation from the component to escape. Thus, there is a need for electrically conductive polymers.

The advent of graded materials solves many of these problems. The MMCs can be graded, for example, to match the CTE of the base materials and thus residual stresses can be minimized. Similarly, conducting polymers can be manufactured from insulating polymers by filling in with conducting particles. Thus, functionally graded materials (FGMs) allow the design of structures with innovative properties and functions that meet spatially varying performance requirements throughout the structure. In addition to reducing thermal stress or preventing peeling of the coating layer, they also arrest micro-crack propagation or provide impact resistance. The recently conceived functionally graded piezoelectric materials (FGPMs) can produce interesting combinations of properties in composites, such as gradients in electrical conductivity across any desired spatial direction. Materials such as these, which are currently being used in UTs and SAW devices, have heterogeneous character on a molecular dimension, but homogeneous at physical scales of the components. Thus, they can be considered to be locally continuous for the purpose of analysing the structure. In as much as applications of these materials can be for structures subjected to impact loading, a detailed understanding of their dynamic behaviour is also necessary.

Impact loadings elicit wave propagation with broad frequency content, that is, they involve a large number of modes. Hence, a numerical solution with finite elements (FEs) using conventional modal superposition may be computationally prohibitive, in which case a direct time integration may remain the only viable alternative. However, for high-frequency loading, the time step typically cannot exceed \( \Delta t_{cr} = 1/20 f_{max} \), where \( f_{max} \) is the maximum frequency content of the load. Similarly, element sizes are restricted by the critical element length \( l_{cr} = \lambda_{min}/20 \), where \( \lambda_{min} \) is the minimum wave length. Another estimation states that there should be approximately eight elements per wavelength [1]. Thus, FE analyses become impractical when large structures, like layered media are being considered. In these cases wave-based frequency domain techniques such as the layer stiffness matrix method (SMM) [2], must be used. An equivalent technique due to Rizzi is the spectral element method (SEM) [3], which should not be confused with a similarly named method due to Faccioli. Both the layer stiffness and spectral element methods are in turn closely related to a method proposed much earlier by Biot [4], which is also described in his book [5]. These techniques are indicated because they obviate the need for fine meshing, as a single element can model a layer of arbitrary thickness and any high (or low) frequency (see Reference [2] for isotropic layer, Reference [6] for composite layer and Reference [7] for inhomogeneous layer). However, the SMM (or SEM) are not suitable for all kinds of inhomogeneities (for which exact solutions are not available) and for piezoelectric materials, where the extra variable (electric potential) has no inertial term. Thus, there is a need for an integral-transform-based technique with the flexibility of FE for discretizing all types of equations while simultaneously preserving the efficiency of the SEM. This alternative method is the thin-layer method (TLM), which combines the best of the both FEM and SEM.
The TLM is a semi-analytical technique used to solve wave propagation problems in partially heterogeneous media. The method is based on an FE discretization of the media in the direction of material property variation, and analytical methods for the remaining co-ordinate directions [8]. Since its inception in the seventies, the TLM has evolved into an efficient technique for the analysis of wave motion in layered soils and other laminated media. Because it involves only a partial discretization of the medium, it entails a relatively small computational effort in comparison to other full-discretization methods, such as the finite difference and the FEM. The TLM was first used by Lysmer [9] to study the propagation of Rayleigh waves in layered earth strata. Since then several authors have contributed to this method and established it as a viable alternative to the FEM for certain classes of problems. An account of this development can be found in Reference [10].

There are two different forms of the TLM available in the literature, the $\omega$-TLM, where variables are transformed into the frequency domain and exact solutions in the space domain are obtained for temporally harmonic loads, and the other one is the $t$-TLM, or the time domain TLM, which allows exact solutions in the time dimension for spatially harmonic motions [11]; in these two alternatives, full solutions in space-time are obtained by numerical integration over either frequencies ($\omega$-TLM) or wavenumbers ($t$-TLM), and both require the solution of an eigenvalue problem involving narrowly banded matrices. Although widely used in the context of horizontally stratified media, the dynamic response of foundations over layered solids and non-destructive testing of laminated composite materials [12], there remain many application areas where further research is warranted. In the light of this need, this paper considers the TLM method formulated in the time domain to determine the propagation of waves in FGM piezoelectric layered media and in FGPM layers.

The literature on the response of inhomogeneous FGMs to dynamic and impact loadings is still limited. Reddy et al. (see References [13–15]) present geometric non-linear analyses of FGM plates with material properties that vary in the thickness direction, using shear deformation plate theories. Gong et al. [16] report on the elastic response of FGM shells subjected to low velocity impact. Ohyoshi et al. [17] consider the response of FGM plates in terms of wave reflection and transmission coefficients. Liu et al. [18–22] using what in effect is the TLM but which they refer to as the strip element method, study FGM plates in which the variation of material properties is approximated by a piecewise linear function. Stress wave propagation in one-dimensional model is considered by Bruck [23], who suggests ways to reduce the stress amplitude and time delay in the occurrence of maximum stress. Liu et al. [24] use confluent hyper-geometric functions to solve analytically the governing equation of wave propagation in FGM. Han et al. [25] use the TLM to analyse waves in a cylinder composed of FGM, and propose a numerical method to analyse transients elicited by impact loads [26]. SH waves in an FGM plate is also investigated by Han and Liu [27] using layer elements in which the material property variation is a quadratic function in the thickness direction. Recently, Santarea et al. [28] used graded finite element and Berezovski et al. [29] used a composite wave propagation algorithm to analyse wave propagation in FGM. Also there exist several applications of SEM to wave propagation analysis in layered FGM. For linear variation, a spectral element (SE) was developed based on approximate wavenumbers [30]. Recently, an exact SE was formed for FGM layer [7], where exponential variation of the material properties was assumed.

FGPM, also known as active FGM, is a relatively new concept. Here the electro-elastic mismatch between constituent material phases is minimized through continuous gradation.
Typical applications lie in the field of ultrasonic transducers (UT). A conventional UT consists of a uniform piezoelectric ceramic plate with a backing bonded to its rear face. However, the ultrasonic pulse waveform of conventional transducers is insufficiently short, because the transducer generates ultrasonic pulse at both surfaces of the ceramic plate upon impulse excitation and these pulses reverberate between the two faces. A few years ago, a functionally graded piezoelectric ceramics plate was created by giving a temperature gradient in the thickness direction \[31\]. It was found that the FGM transducer with a ceramic matched backing and bonded together with an adhesive was able to generate an ultrasonic pulse of short duration. Recently, another UT with FGPM was fabricated \[32\], which was shown capable of generating signals with broader frequency spectra than those of conventional UTs. There are multi-layered actuators \[33\] and PZT/Pt FGPM actuators \[34\], which exemplify the importance of FGPM in practical applications. The process of developments of a typical FGPM is given in References \[35, 36\].

Wave propagation in FGPM was first studied by Liu and Tani \[20, 21\], who considered mainly SH waves. Subsequently, Liu et al. analysed the dispersion of waves in functionally graded piezoelectric plate \[37\]. An exact FE was formulated by Chakraborty et al. \[38\] for a plate in cylindrical bending with graded thermal and electric properties. To the best of the authors’ knowledge, these are the only works on wave propagation in FGPM plates. In addition, some recent works consider the free vibration of functionally graded piezoceramic spheres \[39\] and in rectangular plates \[40\]. Thus, several issues remain to be investigated, such as the propagation of SV-P waves in FGM and FGPM, the effect of anisotropy and inhomogeneity, the response of FGPM layer for electrical pulse excitation, and so forth. Some of these issues are taken up herein, using for this purpose the very efficient TLM formulated in the time domain.

In addition to being able to effectively solve the forward wave propagation problem, an important advantage of the TLM is its ability to solve inverse problems. This advantage emanates from the fact that it takes only a modest computational effort to get the response of a layered structure for specified loading and material properties. There are two classes of inverse problems, namely system identification (given the input and output, determine the system or black box) and source identification (from the measured output and known system, find the input). Material property identification, which is addressed in this paper, falls under the first category. Here, for a given load and responses measured at a number of locations, the material properties of the structure are identified through trial and error simulations that optimize a cost function through least squares, a method that is referred to herein as the ‘mixed numerical/experimental technique (MNET)’. Methods belonging to this class differ from one another in the data chosen for comparison and in the strategy used to update the design variables. For example, Sol et al. \[41\] dealt with the eigenvalues and eigenmodes of the structure for which an error functional was constructed and minimized. Similarly, Kim et al. \[42\] estimated the heat capacity of composites via an error functional expressed as the sum of the squared differences of measured and computed temperature. Reference \[7\] uses non-linear optimization with box-constraints applied at each frequency.

All of the previous are examples of non-linear optimization with constraints, which fall in the category of classical optimization. In this class of methods, the structure is subjected to pulse loading at some point and its response measured at another known location. This response is compared with the response of the numerical model of the structure. At first, the numerical model does not reproduce the experimentally measured response, but in subsequent iterations,
the squared difference in the two responses, which represents the cost function, is minimized by carrying out appropriately small changes in the material properties. The problem of estimating the unknown material parameters becomes then a problem of non-linear optimization in which the cost function is dependent upon the material constants in a non-linear fashion. Hence, the method of non-linear optimization requires repeated computations of the structural response, which entails heavy cost of computation when the structure is modelled by FEs. For this reason, a spectral element model of the structure is most suitable because a single spectral element can replace thousands of FEs. However, this same approach can also be adopted in the TLM, with the advantage that the optimization can now be carried out in either the frequency or time domains, although in the later case a different norm defining the error function (or cost function) is needed. An application of the TLM to FGPM’s taken up in this paper and illustrated by means of examples.

Other works dealing with the estimation of properties of graded material are based on non-classical optimization. Neural networks are used by Han et al. [43] for the material characterization of FGM cylinders, and by Liu et al. [44] for FGM plates. A genetic algorithm (GA)-based method is given in Reference [45]. Nakamura et al. [46] used a Kalman filter technique along with instrumented micro-indentation to estimate the FGM through-thickness compositional variation and a rule-of-mixture parameter that defines the effective properties of FGM.

This paper is organized as follows. First a detailed formulation of the thin layer element is given in Section 2. Several numerical examples of wave propagation in the anisotropic, inhomogeneous and piezoelectric layer are presented in Section 3, including plots of the normal modes of vibration, which reveal the effects of inhomogeneities. In addition, an example is given that illustrates the estimation of material properties. Finally, conclusions are drawn in Section 4.

2. MATHEMATICAL FORMULATION

The governing equations for linear piezoelectric solids are obtained by adding the Maxwell’s equation with the elastodynamic equation for anisotropic materials (see Reference [47]) as

$$\begin{align*}
\sigma_{ij, j} &= \rho \ddot{u}_i, & e_{ijk} E_{j,k} &= -\dot{B}_i, & e_{ijk} H_{j,k} &= \dot{D}_i, & B_{i,i} &= 0, & D_{i,i} &= 0
\end{align*}$$

where conduction current and free charges are assumed absent. The associated constitutive relations for piezoelectric materials are given by the coupled relations

$$\begin{align*}
\sigma_{ij} &= C_{ijkl} S_{kl} - e_{kij} E_k, & D_k &= e_{kij} S_{ij} + e_{ki} E_i
\end{align*}$$

As it is known that the presence of acoustic waves does not cause electromagnetic radiation, there is no coupling between the electric field and the magnetic field in Equation (1). Thus considerable simplification is achieved and the pertinent equations are the first and the last of Equation (1).

We consider a piezoelectric anisotropic medium possessing transversely isotropic symmetry and oriented such that its crystallographical axes coincide with the global reference Cartesian co-ordinate system, which is the $x_1-x_3$ plane (henceforth written as $X-Z$ plane, as is shown in Figure 1). Further, we write $u$ for $u_1$ and $w$ for $u_3$. In this system the non-zero stress, strains,
Figure 1. The thin layer element configuration and the associated d.o.f.s.

The material properties $\tilde{Q}_{ij}$, $e_{ij}$ and $\tilde{e}_{ij}$ all can vary along $Z$ direction. However, they are taken constant in the subsequent steps. Formulations for inhomogeneous materials are given in the next section. Further, the strains and the electric fields are related to the displacement field and electric potential by the relation

$$
\begin{align*}
\sigma_{xx} & = \tilde{Q}_{11} u_{xx} + \tilde{Q}_{13} w_{xz} + \tilde{Q}_{55} (u_{zz} + w_{xz}) + (e_{31} + e_{15})/p_{10} \times e_{xz} \\
\sigma_{zz} & = \tilde{Q}_{13} u_{xx} + \tilde{Q}_{33} w_{zz} + \tilde{Q}_{55} (u_{xx} + w_{zz}) \\
\sigma_{xz} & = 0 \\
D_x & = e_{11} e_{13} e_{15} e_{11} 0 \\
D_z & = e_{31} e_{33} e_{35} 0 e_{33}
\end{align*}
$$

(2)

The governing equations in the $X$–$Z$ plane become

$$
\tilde{Q}_{11} u_{xx} + \tilde{Q}_{13} w_{xz} + \tilde{Q}_{55} (u_{zz} + w_{xz}) + (e_{31} + e_{15}) \phi_{xz} = \rho \ddot{u}
$$

(3)
\[ \tilde{Q}_{55}(u_{xz} + w_{xx}) + \tilde{Q}_{13}u_{xz} + \tilde{Q}_{33}w_{zz} + e_{15}\phi_{xx} + e_{33}\phi_{zz} = \rho \ddot{w} \]  
\[ e_{15}(u_{xz} + w_{xx}) + e_{31}u_{xz} + e_{33}w_{zz} - \varepsilon_{11}\phi_{xx} - \varepsilon_{33}\phi_{zz} = 0 \]

These equations are supplemented by the boundary conditions

\[ t_x = \sigma_{xx}n_x + \sigma_{xz}n_z, \quad t_z = \sigma_{xz}n_x + \sigma_{zz}n_z, \quad D = D_xn_x + D_zn_z \]

The TLM begins by transferring the variables into the wavenumber domain by assuming the solution as

\[
\begin{align*}
\{u(x, z, t)\} = & N \sum_{m=0}^{M-1} \begin{bmatrix} \tilde{u}(z, t) \sin(\eta_m x) \\ \tilde{w}(z, t) \cos(\eta_m x) \\ \tilde{\phi}(z, t) \cos(\eta_m x) \end{bmatrix} \\
\end{align*}
\]

Substituting Equation (6) in Equations (3)–(5), the reduced set of governing equations are

\[ -\eta_m^2 \tilde{Q}_{11}\tilde{u} - \eta_m \tilde{Q}_{13}\tilde{w}_z + \tilde{Q}_{55}(\tilde{u}_{zz} - \eta_m \tilde{w}_z) - \eta_m(e_{31} + e_{15})\tilde{\phi}_z = \rho \ddot{u} \]  
\[ \tilde{Q}_{55}(\eta_m \tilde{u}_z - \eta_m^2 \tilde{w}) + \eta_m \tilde{Q}_{13}\tilde{u}_z + \tilde{Q}_{33}\tilde{w}_{zz} - \eta_m^2 e_{15}\tilde{\phi} + e_{33}\tilde{\phi}_{zz} = \rho \ddot{w} \]  
\[ e_{15}(\eta_m \tilde{u}_z - \eta_m^2 \tilde{w}) + \eta_m e_{31}\tilde{u}_z + e_{33}\tilde{w}_{zz} + \eta_m^2 e_{11}\tilde{\phi} - \varepsilon_{33}\tilde{\phi}_{zz} = 0 \]

2.1. Thin layer element formulation: homogeneous material

To obtain the thin layer element (TLE), the normal procedure of FE formulation is followed with Equations (7)–(9). The unknowns are written as

\[ \tilde{\theta}(z, t) = \sum_{n=1}^{N} \psi_n(z)\tilde{\theta}_n(t) \quad \tilde{\theta} = u, \; w \; \text{or} \; \phi \]

where \( N \) is the number of nodes in an element, \( \psi_n(z) \) is the shape function for \( n \)th node and \( \tilde{\theta}_n \) is the nodal amplitude. To construct a two-noded TLE, \( \psi_n(z) \) are the standard rod shape functions

\[ \psi_1(z) = 1 - z/L, \quad \psi_2(z) = z/L \]

where \( L \) is the thickness of the layer. Similarly, for three-noded TLE, the shape functions for quadratic expansion are

\[ \psi_1(z) = 1 - 3z/L + 2z^2/L^2, \quad \psi_2(z) = 4z/L(1 - z/L), \quad \psi_3(z) = (z/L)(2z/L - 1) \]

where the nodes are at \( z = 0, \; L/2 \; \text{and} \; L, \) respectively. The order of the polynomial can be further increased if a high number of nodes is accommodated in the layer element. However, rather than using the equally spaced grids, it is also expedient to use unequally spaced grids like
that of Chebyshev and Legendre polynomials, which will dramatically increase the convergence rate. However, in the present context, we will restrict ourselves to polynomial shape functions.

The weak form of the Equations (7)–(9), for the \( m \)th wavenumber step can be written as

\[
\delta \mathbf{v}_m^t [ \mathbf{K} ]_m \mathbf{v}_m + \delta \mathbf{v}_m^t [ \mathbf{M} ]_m \ddot{\mathbf{t}}_m = \delta \mathbf{v}_m^t \mathbf{t}_m
\]

(11)

where the \( \mathbf{K}_m \) and \( \mathbf{M}_m \) are the general stiffness and mass matrices. The \( \mathbf{v}_m \) is the vector of unknowns arranged as \( \{ \hat{u}_i \hat{w}_i \hat{\phi}_i \} \), \( i = 1 \ldots N \) and \( \delta \mathbf{v}_m \) is the vector of corresponding virtual unknowns. The \( \mathbf{t}_m \) is the vector of nodal generalized forces, containing both tractions and electric displacements. The \( \mathbf{K}_m \) matrix (of size \( 3N \times 3N \)) is

\[
\mathbf{K}_m = \begin{bmatrix}
\eta_m \ddot{Q}_{11} \mathbf{P}_1 + \dot{Q}_{55} \mathbf{Q}_2 & \eta_m \dot{Q}_{13} \mathbf{P}_2 - \eta_m \ddot{Q}_{55} \mathbf{Q}_1 & \eta_m e_{31} \mathbf{P}_2 - \eta_m e_{15} \mathbf{Q}_1 \\
\eta_m \dot{Q}_{13} \mathbf{Q}_1 - \eta_m \ddot{Q}_{55} \mathbf{P}_2 & \ddot{Q}_{33} \mathbf{Q}_2 + \eta_m \ddot{Q}_{55} \mathbf{P}_1 & e_{33} \mathbf{Q}_2 + \eta_m e_{15} \mathbf{P}_1 \\
\eta_m e_{31} \mathbf{Q}_1 - \eta_m e_{15} \mathbf{P}_2 & e_{33} \mathbf{Q}_2 + \eta_m e_{15} \mathbf{P}_1 & -\eta_m^2 e_{15} \mathbf{P}_1 - e_{33} \mathbf{Q}_2 \\
\end{bmatrix}
\]

(12)

The matrices \( \mathbf{P}_1, \mathbf{P}_2, \mathbf{Q}_1 \) and \( \mathbf{Q}_2 \), each of size \( N \times N \), are defined as

\[
\begin{bmatrix}
\mathbf{P}_1 & \mathbf{P}_2 \\
\mathbf{Q}_1 & \mathbf{Q}_2 \\
\end{bmatrix} = \int_0^L [\mathbf{N} \quad \mathbf{N}']^t [\mathbf{N} \quad \mathbf{N}'] \, dz
\]

where \( \mathbf{N} = \{ \psi_1 \ldots \psi_N \} \), \( \mathbf{N}' = \{ \partial \psi_1 / \partial z \ldots \partial \psi_N / \partial z \} \)

Similarly, the mass matrix \( \mathbf{M}_m \) (\( 3N \times 3N \)) is given as

\[
\mathbf{M}_m = \begin{bmatrix}
\rho \mathbf{P}_1 & 0 & 0 \\
0 & \rho \mathbf{P}_1 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

(13)

Noting that \( \delta \mathbf{v}_m \) is arbitrary, from Equation (11) and grouping the matrices and vectors given in Equations (12) and (13) the governing ODE becomes

\[
\begin{bmatrix}
\mathbf{M}_m & 0 \\
0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\ddot{\mathbf{u}} \\
\ddot{\mathbf{p}} \\
\end{bmatrix} + \begin{bmatrix}
\mathbf{K}_{uu}^m & \mathbf{K}_{u\phi}^m \\
\mathbf{K}_{u\phi}^m & \mathbf{K}_{\phi\phi}^m \\
\end{bmatrix} \begin{bmatrix}
\mathbf{u} \\
\mathbf{p} \\
\end{bmatrix} = \begin{bmatrix}
\mathbf{f} \\
\mathbf{q} \\
\end{bmatrix}
\]

(14)

where \( \mathbf{u} \) (size \( 2N \times 1 \)) is the nodal values of the mechanical displacements and \( \mathbf{p} \) (size \( N \times 1 \)) is the nodal values of the electric potential. Similarly, \( \mathbf{f} \) and \( \mathbf{q} \) are the mechanical and electrical part of the nodal load vector. In \( \mathbf{u} \), the nodal values are arranged as \( \{ u_1, w_1, u_2, w_2, \ldots, u_N, w_N \} \) and similarly in \( \mathbf{f} \) the corresponding nodal tractions are specified.

We wish to solve the ODE in Equation (14) by Newmark’s time integration rule (with parameters 0.5 and 0.25, i.e. constant average acceleration method), which ensures unconditional stability. However, the mass matrix corresponding to \( \mathbf{p} = \{ \phi_1, \ldots, \phi_N \} \) is zero and solutions cannot be obtained simultaneously for \( \mathbf{u} \) and \( \mathbf{p} \). Hence, Equation (14) is written solely in terms of \( \mathbf{u} \) as

\[
\mathbf{M}_m \ddot{\mathbf{u}} + \mathbf{K}_m \mathbf{u} = \mathbf{f}_m
\]
where

\[ \bar{K}_m = K_{uu}^m - K_{u\phi}^m K_{\phi\phi}^{-1} K_{\phi u}^m, \quad \bar{f}_m = f - K_{u\phi}^m K_{\phi\phi}^{-1} q \]

From the solution of \( u \), the electric field is obtained as

\[ p = K_{\phi\phi}^{-1} (q - K_{\phi u}^m u) \]

The above equations are valid for piezoelectric material. However, for anisotropic material without any piezoelectric property, the existing matrices can be used to obtain the mechanical displacement field. The governing equation in this case becomes

\[ M_m \ddot{u} + K_{uu}^m u = f_m \]

For \( N = 2 \), the matrices \( M_m \) and \( K_{uu}^m \) are

\[
M_m = (\rho L t / 6) \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \\
K_{uu}^m = \begin{bmatrix} \hat{Q}_{11} \eta_m L^2 / 3 + \hat{Q}_{55} / L & -\eta_m / 2 (\hat{Q}_{13} - \hat{Q}_{55}) & \hat{Q}_{11} \eta_m L / 6 - \hat{Q}_{55} / L & \eta_m / 2 (\hat{Q}_{13} + \hat{Q}_{55}) \\
\hat{Q}_{33} / L + \eta_m \hat{Q}_{55} / L / 3 & -\eta_m / 2 (\hat{Q}_{13} + \hat{Q}_{55}) & -\hat{Q}_{55} / L + \eta_m \hat{Q}_{55} / L / 3 & \eta_m / 2 (\hat{Q}_{13} - \hat{Q}_{55}) \\
\hat{Q}_{11} \eta_m L / 3 + \hat{Q}_{55} / L & \eta_m / 2 (\hat{Q}_{13} + \hat{Q}_{55}) & \hat{Q}_{33} / L + \eta_m \hat{Q}_{55} / L / 3 & -\eta_m \hat{Q}_{55} / L / 3 \\
\hat{Q}_{11} \eta_m L / 6 - \hat{Q}_{55} / L & \eta_m / 2 (\hat{Q}_{13} - \hat{Q}_{55}) & \hat{Q}_{55} / L / 3 & \hat{Q}_{11} \eta_m L / 6 \end{bmatrix} \]

2.2. Extraction of strains and voltages

At wavenumber \( \eta_m \), the generalized stress \( S_m \) and the generalized strain \( E_m \) are

\[ S_m = \{ \sigma_{xx}, \sigma_{zz}, \sigma_{xz}, D_x, D_z \}_m, \quad E_m = \{ S_{xx}, S_{zz}, 2S_{xz}, E_x, E_z \}_m, \quad S_m = Q E_m \]

where \( Q \) is the matrix in Equation (2). Using the assumed form of the variables given in Equation (6) and the shape functions given in Equation (10) \( E_m \) is related to the unknown variables \( v_m \) as

\[ E_m = B_m v_m, \quad B_m = [ B_1, B_2, \ldots, B_N ]_m, \quad B_i \in \mathbb{R}^{5 \times 3} \]

where each of the \( B_i \) is defined as

\[
B_i = \begin{bmatrix} \eta_m \psi_i & 0 & 0 \\ 0 & \partial \psi_i / \partial z & 0 \\ \partial \psi_i / \partial z & -\eta_m \psi_i & 0 \\ 0 & 0 & \eta_m \psi_i \\ 0 & 0 & -\partial \psi_i / \partial z \end{bmatrix}, \quad i = 1 \ldots N
\]

Once \( E_m \) is obtained, the \( S_m \) is obtained using the constitutive matrix \( Q \). For performing the summations over the wavenumbers, it is to be remembered that the first, second and fifth components are to be multiplied by \( \cos(\eta_m x_s) \) and the third and fourth components are to be multiplied by \( \sin(\eta_m x_s) \), where \( x_s \) is the \( X \) co-ordinate of the point where strain/stress is measured.

### 2.3. Thin layer element formulation: inhomogeneous material

For an inhomogeneous layer, it is assumed that the material properties vary only in \( Z \) direction. The extension of the present formulations to this kind of inhomogeneity is straightforward by the application of the weak form of the governing equation. The general stiffness and mass matrices \( K_m \) and \( M_m \) are obtained by the general rule

\[
K_m = \int_0^L B^T Q(z) B t \, dz, \quad M_m = \int_0^L N^T \rho(z) N t \, dz
\]  

(15)

where

\[
N = [\psi_1, \ldots, \psi_N], \quad \psi_i = \begin{bmatrix} \psi_i & 0 & 0 \\ 0 & \psi_i & 0 \\ 0 & 0 & \psi_i \end{bmatrix}
\]

Further, let us assume that the \( z \) dependency is given by

\[
\tilde{Q}_i = \tilde{Q}_{ij0} f_Q(z), \quad e_{ij} = e_{ij0} f_e(z), \quad \varepsilon_{ij} = \varepsilon_{ij0} f_e(z), \quad \rho = \rho_0 f_\rho(z)
\]

where the forms of \( f_Q, f_e, f_\varepsilon \) and \( f_\rho \) are yet to be decided. Substituting these expressions in Equation (15), the expressions for the \( K_m \) and \( M_m \) are obtained as

\[
K_m = \begin{bmatrix}
\eta_m^2 \tilde{Q}_{110} P_1^Q + \tilde{Q}_{550} Q_2^Q & \eta_m \tilde{Q}_{130} P_2^Q - \eta_m \tilde{Q}_{550} Q_1^Q & \eta_m e_{310} P_2^e - \eta_m e_{150} Q_1^e \\
\eta_m \tilde{Q}_{130} Q_1^Q - \eta_m \tilde{Q}_{550} P_2^Q & \tilde{Q}_{330} Q_2^Q + \eta_m^2 \tilde{Q}_{550} P_1^Q & e_{330} Q_1^e + \eta_m^2 e_{150} P_1^e \\
\eta_m e_{310} Q_1^e - \eta_m e_{150} \bar{P}_2^e & e_{330} Q_2^e + \eta_m^2 e_{150} P_1^e & -\eta_m^2 e_{110} P_1^e - e_{330} Q_2^e
\end{bmatrix}
\]

\[
M_m = \begin{bmatrix}
\rho_0 P_1^p & 0 & 0 \\
0 & \rho_0 P_1^p & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where the matrices involved in the expressions are defined as

\[
\begin{bmatrix} P_1^Q \\ P_2^Q \\ Q_1^Q \\ Q_2^Q \end{bmatrix} = \int_0^L [N N']^t [N N'] f_Q(z) \, dz,
\begin{bmatrix} P_1^e \\ P_2^e \\ Q_1^e \\ Q_2^e \end{bmatrix} = \int_0^L [N N']^t [N N'] f_e(z) \, dz,
\begin{bmatrix} P_1^p \\ P_2^p \\ Q_1^p \\ Q_2^p \end{bmatrix} = \int_0^L [N N']^t [N N'] f_\rho(z) \, dz
\]
The expressions of the functions dictating $z$ dependency can be any integrable function. However, in the present study, they are assumed to be two parameter functions, one linear and another exponential. Together, they cover most of the popular choices of variation used in the production of graded materials. These functions are defined for linear and exponential variations as

$$f_p(z) = p_0 + p_1 a_1 z, \quad f_p(z) = p_0 a_1 \exp[p_1 a_1 z], \quad p = Q, e, \varepsilon, \rho$$

Thus, for linear variation the matrices are

$$P_1 = \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} a_0 + \frac{L^2}{12} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} a_1, \quad P_2 = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} a_0 + \frac{L}{6} \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} a_1$$

$$Q_1 = P_2^T, \quad Q_2 = \left( \frac{a_0}{L} + \frac{a_1}{2} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Similarly, for exponential variation

$$P_1(1, 1) = 2a_0 \alpha/(a_1^3 L^2) - (a_1^2 L^2 + 2 + 2a_1 L)a_0/(a_1^3 L^2)$$
$$P_1(1, 2) = a_0 \alpha(-2 + a_1 L)/(a_1^3 L^2) + (2 + a_1 L)a_0/(a_1^3 L^2)$$
$$P_1(2, 1) = P_1(1, 2)$$
$$P_1(2, 2) = a_0 \alpha(a_1^2 L^2 + 2 - 2a_1 L)/(a_1^3 L^2) - 2a_0/(a_1^3 L^2)$$
$$P_2(1, 1) = -a_0 \alpha/(a_1^2 L^2) + (a_1 L + 1)a_0/(a_1^2 L^2)$$
$$P_2(1, 2) = -P_2(1, 1)$$
$$P_2(2, 1) = -(\alpha(a_1 L - 1)a_0 + a_0)/(a_1^2 L^2)$$
$$P_2(2, 2) = -P_2(2, 1)$$

$$Q_1 = P_2^T, \quad Q_2 = \frac{a_0}{a_1 L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \alpha = \exp[a_1 L]$$

Thus, to retrieve the stiffness and mass matrices of the homogeneous layer, all the $a_0$ and $a_1$ are to be taken to the limit of 1 and 0, respectively.

2.4. Prescription of the boundary conditions

Essential boundary conditions are prescribed in the usual way as is done in FEM, where simply the nodal displacements are arrested or released depending upon the nature of the boundary conditions. The applied tractions are to be prescribed at the nodes. It is assumed that the
forcing function (for symmetric loading) can be written as

\[ F(x, z, t) = \delta(z - z_j) \left( \sum_{m=1}^{M} a_m \cos(\eta_m x) \right) f(t) \]  

(16)

where \( \delta \) denotes the Dirac delta function, \( z_j \) is the \( Z \) co-ordinate of the point where load is applied and the \( z \) dependency is fixed by suitably choosing the node where the load is prescribed. No variation of load along \( Z \) direction is allowed in this analysis. The \( a_m \) are the Fourier series coefficients of the \( x \) dependent part of the load.

With the summation involved in the solution there is one associated window, in the space, \( L_x \). The discrete horizontal wavenumber \( \eta_m \) are related to this window by the number of data \( M \) chosen in the summation, i.e.

\[ \eta_m = 2(m - 1)\pi/L_x = 2(m - 1)\pi/(M\Delta x) \]  

(17)

where \( \Delta x \) is the spatial sampling rate.

3. NUMERICAL EXAMPLES

The developed TLE is validated first to establish its accuracy and efficiency with respect to conventional 2D FE solutions. To this end, both anisotropic and inhomogeneous layers are analysed for high frequency loading and the responses are compared. Subsequently, a piezocomposite layer is analysed and the variation of electric potential is obtained. Models of ultrasonic transducers (UT) are analysed using this method and the effect of ply-stacking on the measured voltage at the piezolayer is obtained. The effect of anisotropy and inhomogeneity on mode shapes of mechanical and electrical d.o.f. is studied in the subsequent example. Finally, the present formulation is utilized to show its advantage in solving inverse problems such as material property identification from the measured response.

3.1. Verification of the TLE: anisotropic layer

Wave propagation in cross-ply composite layer is studied in this section and the results are compared with 2D FE solutions. The material taken is graphite-epoxy (Gr.-Ep.), AS/3501 composites, which has the following material properties: \( E_1 = 144.48 \) GPa, \( E_3 = 9.63 \) GPa, \( G_{13} = 4.13 \) GPa, \( v_{13} = 0.3 \), \( v_{12} = 0.02 \) and \( \rho = 1389 \) kg/m\(^3\). The ply-sequence considered is \([0^0_{10}/90^0_{10}/0^0_{10}]\), where each lamina is 0.01m thick. This large thickness is chosen to differentiate between incident and reflected pulses, although any layer thickness can be analysed. The layered system (shown in Figure 2) is impacted by a high frequency loading. The time history of the load along with its spectrum is shown in Figure 3. As is seen in the figure, the load is of unit magnitude and 50\( \mu \)s duration with an initial padding of 100\( \mu \)s. The load has high frequency content (around 46kHz). This kind of impact load excites many natural modes and the resulting response is normally obtained by a linear combination of these modes. A mode superposition method of analysis for this kind of loading will be computationally expensive and that is where the present element formulation is most useful. For spatial variation, 32 Fourier series coefficients (\( M \) in Equations (6) and (16)) are considered. For concentrated load all the \( a_m \)s are equal to \( 2/L_x \), where \( L_x \) is the window length in \( X \) direction, here taken as 1.0 m, as per the FE model.
The load is applied at the centre of the top layer, first in the Z direction, which generates primarily QP wave and then in the X direction, which generates primarily QSV wave. Response of the structure is measured at several locations along the surface and interfaces. For FE analysis, the layer is modelled with 3600, three-noded plane-strain FEs. In comparison, there are only
30 TLEs in the TLM model. The FE model results in a global system matrix of size $3656 \times 126$, whereas, the TLM model results in a global system matrix (dynamic stiffness matrix) of size $58 \times 3$. For both the cases, Newmark time integration is adopted with a time increment of $1 \mu s$ which means, to get a time history upto $600 \mu s$, the global matrix in FE need to be back-substituted 600 times. For the TLM analysis, however, the system matrix needs to be inverted $600 \times M$ times. Still, this computational requirement is many orders smaller compared to the requirement of the FE analysis. Further, a typical simulation in FE takes $110 \text{s}$ of CPU time, whereas, a SE run takes $14 \text{s}$ in Compaq Alpha Server ES40 with DEC compiler.

For the load applied in the $Z$ direction at point 1, the $Z$-directional velocity $\dot{w}$, is measured at points marked by 2 (see Figure 2). The velocity history of this node is plotted in Figure 4. The response in this case does not start at $100 \mu s$, but at $130 \mu s$. This is the time taken for propagation in the first layer, i.e. $0^\circ$ laminate. Subsequent reflections at around $2.9 \times 10^{-4} \text{s}$ and $3.6 \times 10^{-4} \text{s}$ are due to reflections from the fixed edge ($z = 0.3 \text{ m}$) and free edge ($z = 0.0 \text{ m}$), respectively. Further, the peak at around $5.0 \times 10^{-4} \text{s}$ is the second reflection from the fixed edge. The TLE captures these reflections quite well and except for the last reflection, the response matches satisfactorily with FE response.

Next, the same load is applied at point 1 in the $X$ direction. For this load primarily QSV waves are generated. There will be no wave at the impact point and $X$-directional velocity $\dot{u}$ is measured at the surface point 5 and plotted in Figure 5. The figure shows clearly reflections from the fixed ends. As is seen before, good agreement between FE and TLE responses can be observed.
From these plots, there are few observations that need to be discussed. When a velocity wave encounters a stiffer zone the reflected wave has equal magnitude as the opposite sign of the incident wave. As opposed to that, when the wave encounters a zone of comparatively lower stiffness, the reflected wave has equal amplitude and the same sign as the incident wave. These phenomena are best visible in the reflections from the fixed end (infinite stiffness) and free end (zero stiffness) of a structure. However, reflected waves are also generated at the interfaces of laminates because of a mismatch in stiffness and hence in impedance. In the present model, propagation is considered in the direction of ply-stacking and there is nominal change in stiffness in that direction, due to change in the laminae angle. Hence, the magnitude of the reflected waves from the interface will not be large enough to be visible, in comparison to the boundary-generated waves. Thus whatever reflections are present in the velocity or stress history are solely due to reflections from the boundary.

### 3.2. Verification of the TLE: inhomogeneous layer

Wave propagation in an inhomogeneous layer is studied in this section and the results are again compared with 2-D FE solutions. Three different kinds of materials are taken for this purpose. The first one is steel, the second is alumina and the third is FGM, which smoothly blends the material properties of steel to ceramic. The metal, ceramic and FGM layer are all of thickness 0.1 m. The material properties of the layers are as follows. For steel, the Young's modulus $E$ is 210 GPa, Poisson's ratio $v = 0.3$ and $\rho = 7800$ kg/m$^3$. Similarly, for ceramic, $E = 390$ GPa, $v = 0.3$ and $\rho = 3950$ kg/m$^3$. It is assumed that the gradation is exponential. Thus, for these
material properties $Q_{a_0}$ and $\rho_{a_0}$ are 1 and $Q_{a_1}$ and $\rho_{a_1}$ are 6.1904 and $-6.8041$, respectively.

The layered system is shown in Figure 2, where the materials are written in parentheses.

The structure is impacted at node 1, both in the $X$ and $Z$ directions and the bottom edge is fixed. The load history is as before. Again, the load is applied at the centre of the top layer first in the $Z$ direction and then in the $X$ direction. The response of the structure is measured at several locations along the surface and the interfaces. For FE analysis, the layer is modelled with 3900, three-noded plane-strain FEs. To model inhomogeneity, material properties are evaluated first at the nodes depending upon the specified variation type (here, exponential). Next, within each element the properties are interpolated using the shape function of the displacement field. Accordingly, the element mass and stiffness matrices are modified. In comparison, there are only 30 TLEs in the spectral model. The FE and TLM models generate global system matrices of the same order as in the previous example. Again, the Newmark time integration method is adopted with a time increment of $1\mu s$.

For the load applied in $Z$ direction at node 1, $Z$-directional velocity $\dot{w}$, is measured at point marked by 2 (see Figure 2). The velocity history of this node is plotted in Figure 6. Again, the response in this case does not start at 100 $\mu s$, but at 110 $\mu s$. This is the time needed for propagation in the first layer, i.e. ceramic layer. Subsequent reflections at around $1.68 \times 10^{-4}$ s and $2.5 \times 10^{-4}$ s are due to the reflections from the fixed edge ($z = 0.3$ m) and free edge ($z = 0.0$ m), respectively.

Next, the same load is applied at point 1 in the $X$ direction. For this load primarily SV waves are generated. There will be no wave at the impact point and $Z$-directional velocity $\dot{w}$ is measured at the surface point 4 (see Figure 2) and plotted in Figure 7. Several reflections

Figure 6. $P$-wave propagation at the interface (point 2), solid line—TLM, dashed line—2D FEM.
from the fixed ends are visible. As before, good agreement between FE and TLE responses can be observed. These responses establish the developed TLE in terms of accuracy, efficiency and low cost of computation.

3.3. Verification of the TLE: homogeneous piezoelectric layer

The last two subsections establish the TLE as an efficient tool for wave propagation analysis in anisotropic and inhomogeneous layer. In this subsection, the TLE is used to verify its effectiveness in analysing piezo-composite structures. For verification purposes, a layer of 0.1 m depth and 1.0 m width is considered, which is made up of PZT-4 material with the following properties, (see Reference [48]).

\[
C_{11} = 132 \times 10^9, \quad C_{13} = 73 \times 10^9, \quad C_{33} = 115 \times 10^9, \quad C_{55} = 26 \times 10^9 \\
e_{31} = -4.1, \quad e_{33} = 14.1, \quad e_{15} = 10.5, \quad \rho = 7500 \\
e_{11} = 584.1 \times 10^{-11}, \quad e_{33} = 712.4 \times 10^{-11}
\]

where \( C_{ij} \)s are in N/m\(^2\), \( e_{ij} \)s are in C/m\(^2\), \( e_{ij} \)s are in F/m and \( \rho \) is in kg/m\(^3\).

The TLE model of the layer is made up of 30 TLEs, which yields a system matrix of order 90 \( \times \) 5, as each node now has 3 d.o.f. The corresponding FE model has 500 elements, 10 in depth direction and 50 in width direction. This FE model is not adequate for analysing the problem exactly. However, analysis for piezoelectric material requires condensation of the
electric degrees of freedom and that involves inversion of square matrices. Thus, for the present model, there are total 1526 active d.o.f., out of which 508 are electrical d.o.f. to be condensed. Thus a square matrix of order 508 must be inverted at the beginning, which requires large memory space. Typically, using LAPACK subroutine xGETRF and xGETRI, the matrix is inverted in 0.306 s of CPU time, the final matrices $M_m$ and $K_{uu}$ are formed in 1.45 s and the system is solved for 500 steps in 211 s of CPU time, in a Compaq Alpha Server ES40. This computation time and memory requirements increase drastically with the increase in element numbers, and the TLE model is computationally far superior than the FE model for the present layered media analysis.

The same load of previous examples is applied at the mid-point of the top surface, where the bottom of the layer is fixed. The electric potential $\phi_1[\tau\text{m}]$ is measured at two points on the surface. Figure 8 shows the variation of $\phi$ at the impact point along with the FE result. As the figure suggests, the history resembles SV wave type seen before. Thus multiple reflections coming from the fixed end is visible. The comparison with FE result reveals that the element has accurately captured the variation at the initial part of the time history. The mismatch at the later part of the history suggests inadequacy of the FE mesh for the given mechanical loading.

When the potential is measured 0.1 m away from the impact point (on the surface) the history looks like that which is given in Figure 9. The important issues to be noted are the severe reduction of the amplitude (1/10) from the previous case, slow rise of the waveform and the stronger reflections (compared to the incident pulse) from the boundary. Thus, like thermal wave, electric waves are also localized in space. In comparison, at a distance of 0.1 m, the elastic wave velocity is reduced by half of the original (as Figure 5 suggests). Figure 10 shows the variation of the Z velocity for this model at the impact site. The figure also suggests strong agreement between the FE and TLE models and thus again ascertains its accuracy.
3.4. TLE model of ultrasonic transducers

The analysed piezolayer is rather unrealistic, because of its large depth. In practical applications, typically the piezolayer has 100 μm thickness, e.g. when used in ultrasonic transducer. In an effort to model this kind of structures, two different piezo-composite layered arrangements are considered as is shown in Figure 11. In the first model, there is a piezolayer (PZT-4) of 0.01 m thickness (d) grown on a substrate of 0.09 m thickness (ℓ) and made up of Gr.-Ep. (AS/3501) material. All the material properties are as taken before. There are total 20 TLEs, where 2 elements are used to model the piezolayer. The composite is taken devoid of any piezoelectric effect. To avoid singularity in the $K_{ij\phi\phi}$, a nominal value of $\varepsilon_{ij}$ is assigned for this material and all the $\phi$ d.o.f. (for the nodes starting from 3 to 21) are restrained. The model is impacted first at the surface with the same load and voltage generated at the top layer and at the piezo-composite interface is measured and plotted in Figure 12. As the figure suggests, the voltage measured at the top surface is less than that at the interface. Also it is clear that the waveform subsides quickly, which shows the temporal localization of the electric wave for this thin piezolayer.

Next, a unit voltage, with a time dependency as shown in Figure 3, is applied at the piezolayer and the $Z$ velocity is measured at the top surface, at the interface and mid-layer of the composite. These histories are shown in Figure 13. As is suggested by the figure, there is a steady decline in amplitude as one goes away from the surface. Further, several reflections are visible which are generated at the fixed end and the interface due to mismatch in material properties of PZT and Gr.-Ep.
Next, we consider a \( p/\theta_80/p \) sequence of layers (Figure 11), where \( \theta \) is varied between 0\(^\circ\) and 90\(^\circ\). Here, each lamina is 1 mm thick \((\ell = 0.08)\) and made up of Gr.-Ep. material with the same material properties as before. Each piezolayer is 0.01 m thick \((d)\) and the first one is used as actuator and the second one as sensor. First, a unit voltage is applied at the actuator layer and the same is measured at the sensor layer. The voltage has the same time dependency as before. For \( \theta = 0^\circ \) and 90\(^\circ\), the measured voltage is plotted in Figure 14. It is evident that the signatures are markedly different in these two cases, where 0\(^\circ\) ply-stacking renders faster propagation speed. Further, there are phase and amplitude mismatches, which grow over time. Thus, identification of layer properties is possible if suitably standardized. In comparison, however, for an applied normal loading at the surface (same time history as applied before)
the voltage histories measured at the second piezolayer (Figure 15) do not show any marked difference and thus mechanical loading is not suitable for identifying layer properties over a small propagating length.

As a final example, we consider the use of FGPM in UT. Typically, the piezolayer thickness \((d)\) varies from 0.5 to 2.0 mm, whereas the backing material thickness \((\ell)\) will be \((18 + d)\) mm [32]. However, in this example we take a comparatively larger dimension of the piezolayer to magnify the effect of gradation. The \(d\) is taken as 0.025 mm, whereas \(\ell = 0.075\) m (see Figure 11(b)). It is assumed that the material properties are linearly varying within the FGPM, where two different gradations are considered. In the first set only the piezoelectric properties are varying in such a way that they vanish at the piezolayer-backing material interface, i.e. \(p_{a0} = 1\) \((p = Q, e, \rho, \varepsilon)\), \(Q_{a1} = 0, \rho_{a1} = 0, e_{a1} = -40.0, \varepsilon_{a1} = -40.0\). In the second set, the elastic and inertial properties are also varying, somewhat arbitrarily, as: \(Q_{a1} = 40, \rho_{a1} = 3\) and all the other parameters are same as in the first set. The same voltage history of the previous example is applied in the piezolayer and the resulting elastic motion is measured at the mid-layer of the structure, i.e. at \(z = 0.05\) m. The measured \(Z\) velocity histories at this point are plotted in Figure 16. The response due to the actuation of a homogeneous piezolayer is also plotted in the same figure (shown in dashed line).

As the figure suggests, gradation in piezoelectric properties influences mostly the speed of wave propagation. In this case, the group speed is decreased compared to the homogeneous case, as can be seen from the late appearances of the peaks. However, the peak amplitudes are marginally affected. On the other hand, if the elastic and inertial gradations are introduced, there is considerable change in the peak amplitude as well as in the wave speed, which is evident from the increasing phase mismatch.

3.5. Natural modes of vibration

In this section, the basic components of the waves, i.e. the natural modes of vibration are investigated. These modes superpose with suitable weight factors to produce the waves whose time domain representation we have seen so far. The same layer discussed in Section 3.1 is taken for analysis. Three cases are considered. First, a uni-directional laminate, \([0\overline{3}0]\), second, a cross-ply laminate, \([0_{10}/90_{10}/0_{10}]\) and finally a uni-directional laminate with graded material, \([0_{10}/f0_{10}/0_{10}]\), where \(f\) denotes the FGM. All the layers are fixed at the bottom, \(z = 0.3\) m. The mode shapes for fully elastic analysis are obtained by the normal procedure of quadratic eigenvalue problem (QEP) solving over the stiffness and mass matrices, \(M_m\) and \(K_m^{uu}\), where \(m\) denotes the horizontal \((X)\) mode number. In this example, all the modes are obtained for \(m = 10\), i.e. \(\eta = 56.55\) m\(^{-1}\). The first four mode shapes for each of the cases are plotted in Figures 17–19. In both the cases of composite laminate, the \(Z\) component of the displacement assumes higher value than the \(X\) component. The effect of asymmetry is pronounced only in \(u\) mode, which is shown in Figure 18. Further, asymmetry introduces discontinuity in the mode shapes. However, for FGM, the characteristics of the mode shapes are completely changed. The FGM layer, which ranges from \(z = 0.1\) to \(0.3\) m, has the maximum amplitude for first three modes, where the effect of gradation is comparatively less at the fourth mode.

For the modes of electric potential, the same layer is taken with the material properties of PZT-4, as given before. The potential mode is extracted from the displacement modes, for which the eigenvalue is solved for \(K_m\) and \(M_m\). The potential is extracted from the displacement modes \(u_m\) by the relation \(\phi_m = -K_m^{m-1}K_m^{u}u_m\). Two sets of materials are considered, first one,
Figure 14. Voltage measured at the second piezolayer (solid line—$p/0^\circ/p$, dashed line—$p/90^\circ/p$).

Figure 15. Voltage measured at the second piezolayer (solid line—$p/0^\circ/p$, dashed line—$p/90^\circ/p$).
homogeneous PZT-4, and the second one, linearly graded PZT-4 with previously considered inhomogeneous parameters. The mode shapes are shown in Figures 20 and 21, for homogeneous and FGPM material, respectively. It is seen that for homogeneous material, the distribution closely resembles the \( w \) distribution in Figure 17, however, the situation changes completely for FGPM, where, most of the non-zero values are clustered around \( z = 0.1 \) m. With increase in mode number, length of the non-zero potential region increases. Thus for all the cases, effect of inhomogeneity is predominant for lower modes.

3.6. Estimation of the piezoelectric properties

Estimation of the electric/elastic properties through PPT requires a set-up of non-linear least square formulation where the statement of the problem is

\[
\min_{q} \sum_{t} e_t(q)^2, \quad q \in \mathbb{R}^N
\]

i.e. a function \( e : \mathbb{R}^N \rightarrow \mathbb{R}^M \) is to be minimized in the least square sense. The function is defined as the difference between the experimentally measured response (e.g. velocity or voltage) and the numerically obtained response.

In the present case, \( N \) is the number of design variables for which estimations will be obtained. The \( M \) is the number of sensor points where the response of the structures is

![Figure 16. Z velocity at \( z = 0.05 \) m (solid line—FGPM and dashed line—homogeneous piezolayer).](image-url)
recorded. The recorded signal in the time domain is partitioned into several time intervals and optimization is performed in each interval independently. The error in the $k$th partition $T_k$, where $T_k = [t_{k-1}, t_k]$ is defined as
\[ e = \frac{|f_s(t_k) - f^e(t_k)|}{\|f^e(t)\|_\infty}, \quad t \in T_k \]
where $f_s$ and $f^e$ are the simulated and experimental data, respectively. Thus, the error is based on the deviation at the end of the interval, which is largest in the whole interval. Thus, the numerator in the error definition is essentially $\|f_s - f^e\|_\infty$. Although several other norms could be chosen for this purpose, the currently defined norm is found most sensitive to a change in the design variable $q$.

The same layered structure (Figure 2) is taken, where now all the layers are made up of PZT-4 material, whose material properties are as given before. The same load as applied before (Figure 3) is considered for testing of the above method. The response ($\phi$) is measured at node 1 and this response is used as the experimental data ($f^e$) for all the parametric estimation, thus, $M = 1$ in this case. Hence, from the previous discussion, only one variable can be estimated at a time. In this study, the elements of the piezoelectric stress tensor $e_{13}, e_{15}$ and $e_{33}$ are estimated. It is found that the measured response is not so susceptible to small changes in the dielectric permittivity $\varepsilon$ and hence, its estimation cannot be performed with sufficient accuracy. The optimization is performed with box-constraints, i.e. lower and upper bounds are specified for all the variables. The variables are estimated at six intervals equally spaced between 120

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Figure 17. Depthwise displacement distribution, uni-directional laminate $[0_{30}]\eta = 56.55$ (abscissa indicates modal amplitude (m)).
Figure 18. Depthwise displacement distribution, asymmetric laminate $[0_{10}/90_{10}/0_{10}]$, $\eta = 56.55$ (abscissa indicates modal amplitude (m)).

Figure 19. Depthwise displacement distribution, functionally graded composite layer $[0_{10}/f0_{10}/0_{10}]$, $\eta = 56.55$ (abscissa indicates modal amplitude (m)).
Figure 20. Electric potential distribution, homogeneous PZT-4, $\eta = 56.55$ (abscissa indicates modal potential amplitude (Vm)).

Figure 21. Electric potential distribution, FGPM, PZT-4, $\eta = 56.55$ (abscissa indicates modal potential amplitude (Vm)).
Figure 22. Estimation of $\varepsilon_{31}$.

Figure 23. Estimation of $\varepsilon_{33}$.
and 320 $\mu$s. The optimization is performed by the MATLAB® function LSQNONLIN, where the Levenberg–Merquardt algorithm is used. The tolerances in residue and function values are set at $1.0 \times 10^{-09}$ for all estimations.

Figure 22 shows the estimation of $e_{31}$ whose actual value is $-4.1 \text{C/m}^2$. It can be seen that in all the intervals, the estimated values are quite close to the exact value. These values are shown at the end of each interval. The plot with circle denotes the experimental data $f^e$, which the simulation is trying to achieve. At each iteration the simulated history $f^s$, comes closer to the experimental data. The firm lines show these iterations. The initial guess is given as $-1$, with a lower bound of $-10$ and upper bound of $0$. As the figure suggests, there is not much change for different iterations as the response is quite the same for different values of $e_{31}$. At each interval 10 iterations are required to estimate the value correctly.

Similarly, estimation of $e_{33}$ is shown in Figure 23 and as the figure suggests the same accuracy of previous estimation is attained in this case also. The exact value is $14.1 \text{C/m}^2$, which is achieved exactly at four intervals. The iterated histories can be seen quite clearly in this case (the firm lines), where convergence to $f^e$ occurs from both above and below the experimental data. The optimization takes 12 iterations in each case.

However, the same accuracy is lost in the estimation of $e_{15}$, which is shown in Figure 24. As the figure reveals, there are two instances (interval 3 and 5), when the estimated value is quite away from the desired value (10.5). Other than these cases, three times the exact value is captured and once the estimation is off by 14%. The iterations and monotonic convergence are visible in the plots of $f^s$ (again shown by the firm lines). It takes 14 iterations in each interval for this convergence.
As mentioned before, only one variable is estimated at a time, i.e. $M = 1$ in this case. Next, efforts can be directed towards estimating more than one variable at a time with more number of sensor points. However, measuring only voltages at several points may prove to be insufficient for this kind of estimation and mechanical response data like, velocity or acceleration measurements need to be used in conjunction. This will also be useful in mixed estimation where elastic properties or density of the material can also be identified.

Overall, it can be concluded that the piezoelectric parameters can be estimated quite efficiently by the PPT in conjunction with the TLM.

4. CONCLUSION

The present TLE is an efficient tool for modelling anisotropic and inhomogeneous piezoelectric layers and analysing wave propagation phenomena. It is shown that the proposed element captures the essential wave propagation behaviour of anisotropic and inhomogeneous layers quite effectively. Two piezo-composite layered structures are analysed to show the ease of the present formulation in possible applications, like UT modelling. This element is also suitable for modelling FGPM. It is shown that the signatures from laminates are more sensitive to electrical actuation in comparison to mechanical loading. Thus dual actuator-sensor-based UT is suitable for material identification and can be extended to flaw or delamination detection. Further, wave propagation in FGPM is shown in an example, which reveals considerable effect of gradation on mechanical response. It is shown that in absence of elastic and inertial gradations, FGPM affects only the propagating speed. The distributions of normal modes reveal interesting effects of anisotropy and inhomogeneity, which are shown pronounced in low-frequency modes. The element is used in the PPT to extract piezoelectric material properties from the measured experimental response.

NOMENCLATURE

- $\sigma_{ij}$: elements of the stress tensor
- $S_{ij}$: elements of the strain tensor
- $C_{ijkl}$: elements of the elastic stiffness tensor
- $e_{kij}$: elements of the piezoelectric stress tensor
- $\varepsilon_{ki}$: elements of the dielectric permittivity tensor
- $D_i$: elements of the electric displacement tensor
- $E_i$: elements of the electric field tensor
- $u_i$: elements of the mechanical displacement tensor
- $\phi$: electric potential
- $x_i$: elements of the co-ordinate tensor
- $n_i$: elements of the surface normal tensor
- $t_i$: elements of the surface traction tensor
- $\rho$: density
- $t$: thickness of a layer
- $\eta_m$: $m$th horizontal wavenumber
- $\{ \}$, $i$ differentiation with respect to $x_i$
- $\{ \}$ differentiation with respect to time
REFERENCES


