BOUNDARY INTEGRAL METHOD FOR STRATIFIED SOILS

by

EDUARDO KAUSEL
RALF PEEK

June 1982

Sponsored by the National Science Foundation
Division of Problem-Focused Research
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Abstract

This report presents the derivation of a Boundary Integral Method for stratified soils. The theory presented makes use of a discrete algorithm for the Green’s Functions, published in an earlier report in this series. The method presented can be applied to a variety of problems in geomechanics, such as the scattering of waves by rigid inclusions, soil-structure interaction, etc.
1. Introduction

Application of the Boundary Integral Method (BIM) to a heterogeneous medium is generally not practicable, because the Fundamental Solutions (or Green's Functions) for such a material are not known. For horizontally layered soils, only integral solutions, not suitable for use with the Boundary Integral Method, were available in the past for these Green's Functions. More recently, however, these difficulties have been overcome - at least for soil deposits of finite depth - with the development of an explicit solution for the Green's Functions corresponding to dynamic loads in the interior of a layered stratum (Kausel, 1981). These solutions are used in this report to formulate Geomechanics problems involving layered soils, with the Boundary Integral Method. To avoid duplications of effort, we shall rely heavily on the above cited work, to which the reader is referred for definitions and notation. While the required mathematical developments might appear at first complicated and abstruse, the method is highly efficient, and not difficult to implement in computer codes.

2. Boundary Integral Method

a) Continuous case:

Consider a layered stratum of finite width, with a cavity or excavation as shown in Fig. 1. From the dynamic equilibrium equation

$$\sigma_{ij} \dot{u}_j + b_i - \gamma \ddot{u}_i = 0$$

we obtain multiplying by the virtual displacements $u_i^*$, and integrating over the volume by parts

$$0 = \iiint_V (\sigma_{ij} \dot{u}_j + b_i - \gamma \ddot{u}_i) u_i^* \, dV =$$

$$= \int_S t_i u_i^* \, dS - \iiint_V \sigma_{ij} \varepsilon_{ij}^* \, dV + \iiint_V (b_i - \gamma \ddot{u}_i) u_i^* \, dV$$

(2)

in which $t_i = \alpha_{ij} \sigma_{ij}$ = tractions on the boundary $S = S_1 + S_2 + S_3 + S_4$. 
Fig. 1
Interchanging the roles of \( u_i \) and \( u_i^* \), we can write

\[
0 = \iint t_i^* u_i \, dS - \iiint \sigma_{ij}^* \varepsilon_{ij} \, dV + \iiint (b_i^* - \gamma i_i) u_i \, dV \tag{3}
\]

For harmonic loads, with frequency \( \omega \), \( u_i = -\omega^2 u_i \), \( u_i^* = -\omega^2 u_i^* \), so that \( u_i u_i^* = u_i^* u_i = -\omega^2 u_i u_i^* \). Also, the symmetry of the material properties tensor implies that \( \sigma_{ij}^* \varepsilon_{ij}^* = \sigma_{ij} \varepsilon_{ij} \). Subtracting then 3) from 2) and considering the above relations, we obtain

\[
\iiint t_i u_i^* \, dS + \iiint b_i u_i^* \, dV = \iiint t_i^* u_i \, dS + \iiint b_i^* u_i \, dV \tag{4}
\]

Choosing as virtual state the problem of dynamic loads within the body without excavation (Fig. 2), and considering the boundary conditions

\begin{align*}
\ell_j - \ell_j^* &= 0 \quad \text{on} \quad S_2 \quad \text{(free surface)} \\
u_i &= u_i^* = 0 \quad \text{on} \quad S_3 \quad \text{(lateral boundaries)} \quad \text{and} \quad S_4 \quad \text{(bedrock)} \\
b_i &= 0 \quad \text{(no real body forces)}
\end{align*}

we obtain

\[
\iiint t_i u_i^* \, dS - \iiint t_i^* u_i \, dS + \iiint b_i^* u_i \, dV \tag{5}
\]

At this point, we can take the lateral boundaries to infinity. Equation (5) is valid as written, with \( t_i^*, u_i^* \) representing the Green's functions for body loads \( b_i^* \) acting on the laterally unbounded stratum. If more than one virtual state \( b_i^*, t_i^*, u_i^* \) is considered, say \( b_{ij}^*, t_{ij}^*, u_{ij}^* \), with \( j \) identifying the virtual state, equation (5) would transform into

\[
\iiint t_i u_{ij} \, dS = \iiint t_{ij}^* u_i \, dS + \iiint b_{ij}^* u_i \, dV \tag{6}
\]

Considering now point loads in the \( j^{th} \) coordinate direction applied at \( \vec{r} = \vec{r}_0 \).
\[ b_{ij}^* = \delta_{ij} \delta(\vec{r} - \vec{r}_0) \]  

(7)

with \( \delta_{ij} \) = Kronecker's delta, \( \delta(\vec{r} - \vec{r}_0) \) = Dirac's delta function.

we obtain

\[ \iiint V b_{ij}^* u_i \, dV = u_j(\vec{r}_0) \]  

(8)

so that

\[ u_j = \iiint_{S_1} t_i u_{ij}^* \, dS - \iiint_{S_1} t_{ij}^* \, dS \]  

(9)

which gives the actual displacement in the \( j^{th} \) coordinate direction at point \( \vec{r} = \vec{r}_0 \) in \( V \) in terms of the actual and virtual states at the boundary \( S_1 \). When the point of application of the virtual load \( \vec{r}_0 \) is taken on the surface \( S_1 \), the integrals in (9) will present singularities that can be evaluated by isolating small volumes around the point considered.

b) Discrete case

The Green's functions for layered soils that will be used in this work are not the true functions for the continuous solid, but approximations obtained by discretization of the displacement field in the directions of layering. Thus, a counterpart to formula (9) must be developed that accounts for this effect; while arbitrary excavation slopes could be considered, the expressions to be developed in this work will be restricted to cavities that have vertical and horizontal walls only. Thus, horizontal planes must coincide with the elementary interfaces; the vertical walls, however, may have any arbitrary generatrix (Fig. 2).

Consider the discrete layered system shown in Fig. 2. We isolate a single layer of small thickness \( h \) and infinite lateral dimensions, and having a perforation of arbitrary shape (Fig. 3). Considering then the principle of virtual displacements, which is satisfied by the fundamental discrete solution, we can write

\[ \iiint_V \delta v_{ij} \sigma_{ij} \, dV + \iiint_V \delta u_i (\vec{r} \ddot{u}_i - b_i) \, dV - \iiint_S \delta u_i t_i \, dS = 0 \]  

(10)
Substituting $\epsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji})$ and integrating by parts,

$$
\iint_S \delta u_i \ (\alpha_{ij} \sigma_{ij} - t_i) dS - \iiint_V \delta u_i \ (\sigma_{ij} + b_i - \gamma \ddot{u}_i) dV = 0 \quad (11)
$$

We define now consistent interface edge load vectors (forces per unit length at the edges of the perforation; $m$ and $m+1$ refer to the interfaces)

$$
Q = \begin{bmatrix}
Q_m \\
Q_{m+1}
\end{bmatrix}, \quad 
\eta_m = \begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix} = \begin{bmatrix}
q_i
\end{bmatrix}_m \quad (12)
$$

as well as the interface displacement vectors

$$
V = \begin{bmatrix}
U_m \\
U_{m+1}
\end{bmatrix}, \quad 
U_m = \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix} = \begin{bmatrix}
u_i
\end{bmatrix}_m \quad (13)
$$

such that for arbitrary displacements $\delta V$, the following relation is satisfied:

$$
\delta V^T Q = \int_0^h \delta u_i \ t_i \ dz - \int_0^h \delta u_i \ \sigma_{ij} \ dz \quad (14)
$$

The integrals are evaluated on the vertical walls. This implies choosing edge loads satisfying

$$
\iint_{S_v} \delta u_i \ (t_i - \alpha_{ij} \sigma_{ij}) \ dS = 0 \quad \text{where } S_v = \text{vertical surface} \quad (15)
$$

It follows from equations (11) and (15) that

$$
\iiint_{S_h} \delta u_i \ t_i \ dS = \iiint_{S'_h + S''_h} \delta u_i \ \sigma_{ij} \ dS - \iiint_V \delta u_i \ (\sigma_{ij} + b_i - \gamma \ddot{u}_i) dV \quad (16)
$$
We define also the consistent interface tractions vector

\[ T = \begin{pmatrix} T_m \\ T_{m+1} \end{pmatrix}, \quad T_m = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \quad (17) \]

and stress vector (the minus sign follows from the fact that \( \alpha_{\nu j} \) in the lower horizon is the negative of \( \alpha_{\nu j} \) on the upper horizon \( \nu + 3 \))

\[ S^{(3)} = \begin{pmatrix} S_{m}^{(3)} \\ S_{m+1}^{(3)} \end{pmatrix}, \quad S_m^{(3)} = \begin{pmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{pmatrix} \quad (18) \]

such that for arbitrary displacements \( \delta U \),

\[ \delta V^T T = \delta V^T S - \int_0^h \delta U \left( \sigma_{11} \frac{\partial u_1}{\partial x} + b_i - \gamma \ddot{u}_i \right) \, dz \quad (19) \]

The term in parenthesis in the integrand represents the \( i^{th} \) wave equation; while it is zero at every point for the true solution, it is not for the discretized solution considered here. Hence, the consistent interface tractions \( T \) are not equal to the internal stresses \( S \). Equations (14) and (19) are valid in any coordinate system, provided that the field variables (and wave equation) are appropriately defined. In cartesian coordinates (1\( \rightarrow x \), 2\( \rightarrow y \), 3\( \rightarrow z \)), the covariant derivative is equal to the partial derivative \( (\sigma_{ij} \frac{\partial}{\partial x} = \sigma_{ij,j}) \); for cylindrical coordinates (1\( \rightarrow \rho \), 2\( \rightarrow \phi \), 3\( \rightarrow z \), on the other hand, the expansion of the covariant derivative is more involved.

Equations (14) and (19) are used to compute the consistent edge tractions and interface tractions at the position \( x,y \), where they are required.

The integrals are computed by expanding the displacements \( \delta u_i \), \( u_i \) within the layer in terms of the interface displacements \( \delta U \), \( U \) using linear interpolation. For arbitrary variations of the interface displacements \( \delta U \), one obtains then the consistent forces and tractions \( Q, T \). The computation of these integrals is presented later in this work.

In terms of the consistent edge loads \( Q_m \) and interface tractions \( T_m \) defined by eqs. (14) and (19), we may write equation (10) as
\[
\begin{align*}
&\int_C^* u_m^T Q_m^* \, ds + \int_C^* u_{m+1}^T Q_{m+1}^* \, ds + \int_{S_h^1}^* U_m^T T_m^* \, dS + \int_{S_h^2}^* U_{m+1}^T T_{m+1}^* \, dS + \\
&+ \int_{C''} \int_{C''} u_i b_i^* \, dV = \int_{C''} \int_{C''} \varepsilon_i \sigma_i \, dV + \int_{C''} \int_{C''} \gamma u_i \ddot{u}_i \, dV \tag{20}
\end{align*}
\]

with the real displacement field playing the role of virtual displacements. Interchanging then the real displacement field and the fundamental solution (*) we obtain

\[
\begin{align*}
&\int_C^* u_m^T Q_m^* \, ds + \int_C^* u_{m+1}^T Q_{m+1}^* \, ds + \int_{S_h^1}^* U_m^T T_m^* \, dS + \int_{S_h^2}^* U_{m+1}^T T_{m+1}^* \, dS + \\
&+ \int_{C''} \int_{C''} u_i^* b_i \, dV = \int_{C''} \int_{C''} \varepsilon_i^* \sigma_i \, dV + \int_{C''} \int_{C''} \gamma u_i^* \ddot{u}_i \, dV \tag{21}
\end{align*}
\]

Subtracting (21) from (20), with \( u_i^* \ddot{u}_i = u_i^* \ddot{u}_i = -2 u_i^* u_i^* \), \( \varepsilon_i^* \sigma_i \) = \( \varepsilon_i^{*}\sigma_i^{*} \), \( U_m^T Q_m^* = Q_m^* U_m^* \) (etc), and \( b_i = 0 \) we obtain

\[
\begin{align*}
&\int_C^* q_m^* T_m^* \, ds + \int_C^* q_{m+1}^* T_{m+1}^* \, ds + \int_{S_h^1}^* T_m^* U_m^* \, dS + \int_{S_h^2}^* T_{m+1}^* U_{m+1}^* \, dS + \\
&+ \int_{C''} \int_{C''} b_i^* u_i \, dV = \int_C^* q_m^* T_m^* \, ds + \int_C^* q_{m+1}^* T_{m+1}^* \, ds + \\
&+ \int_{S_h^1}^* U_m^T T_m^* \, dS + \int_{S_h^2}^* U_{m+1}^T T_{m+1}^* \, dS \tag{22}
\end{align*}
\]

We consider now a stratum with an excavation (Fig. 4). Since such a system is in essence an assembly of perforated layers, each satisfying an equation of the form (22), it suffices to overlap the contributions of the various layers to form the corresponding equations for the system of layers. On the other hand, the consistent tractions \( T \) and \( T^* \) are continuous at the layer interfaces; hence, their contribution to the integrals cancel upon overlapping, except in those regions of the horizontal planes that remain exposed due to the excavation.
The equations for the system are then

\[
\sum_{\text{Exposed rings}} \int_C q_{ij} \, u_i \, ds + \sum_{\text{hor. surfaces}} \int_S t_{ij} \, u_i \, ds + u_j(\hat{r}_o) =
\]

\[
\sum_{\text{Exposed rings}} \int_C u_{ij}^* \, q_i \, ds + \sum_{\text{hor. surfaces}} \int_S u_{ij}^* \, t_i \, ds
\]

(23)

The term in \( u_j \) arises from considering a point load at \( r = \hat{r}_o \), as in equation (7). If the virtual point load is within the excavated soil, then this term must be dropped. The additional index \( j \) identifies the various fundamental states (2 horizontal point loads and 1 vertical point load). Evaluation of equation (23) can be achieved by discretization of the rings and exposed surfaces into "elements" over which the displacements and tractions can be assumed to be uniform. The above integrals will have singularities whenever the point loads are taken on the exposed surfaces. These integrals are evaluated in the sections that follow.

3. Green's functions

The integrals implied by equation (23) may be evaluated with the assumption that the real boundary displacements and tractions are piecewise uniform in horizontal planes, and vary linearly with the vertical coordinate. Expressions are also needed for the stresses in terms of the displacements, and for the Green's functions associated with the problem of dynamic point loads in the interior of a layered stratum. Some of these equations are given in ref. 1 (Kausel, 1981). which we shall use extensively in the following; for the sake of brevity, this reference will be denoted simply as R81.13. Also, equations from this report will be identified with a prefix E; for example, equation (E6a), refers to equation (6a) in R81.13 (i.e., in Kausel, 1981).
We define first the following coefficients: (refer to eqs. E6a,b)

<table>
<thead>
<tr>
<th>loading case</th>
<th>μ</th>
<th>c</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>horizontal x load</td>
<td>1</td>
<td>\cos θ</td>
<td>-\sin θ</td>
</tr>
<tr>
<td>horizontal y load</td>
<td>1</td>
<td>\sin θ</td>
<td>\cos θ</td>
</tr>
<tr>
<td>vertical load</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
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(24a, 24b, 24c)

which shall be used to specify the variation of the displacements (and stresses) with the azimuth.

3.1 Green's functions for displacements

3.1.1 Horizontal point loads.

From E88 a, b, c, the Green's functions for the displacements due to a horizontal point load are: (Fig. 4)

\[
u_\rho = \frac{p}{4\pi} \left[ \sum_{l=1}^{2N} \phi^{m}_x \phi^{n}_x \frac{d}{d\rho} H_1^{(2)}(k_{l\rho})/k_{l\rho} + \frac{1}{\rho} \sum_{l=1}^{N} \phi^{m}_y \phi^{n}_y H_1^{(2)}(k_{l\rho})/k_{l\rho} \right] c \tag{25a}
\]

\[
u_\theta = \frac{p}{4\pi} \left[ \sum_{l=1}^{2N} \phi^{m}_x \phi^{n}_x H_1^{(2)}(k_{l\rho})/k_{l\rho} + \sum_{l=1}^{N} \phi^{m}_y \phi^{n}_y \frac{d}{d\rho} H_1^{(2)}(k_{l\rho})/k_{l\rho} \right] s \tag{25b}
\]

\[
u_z = -\frac{p}{4\pi} \left[ \sum_{l=1}^{2N} \phi^{m}_x \phi^{n}_x H_1^{(2)}(k_{l\rho}) \right] c \tag{25c}
\]

in which \( p \) = value of the load, \( i = \sqrt{-1} \); \( H_1^{(2)}(k_{l\rho}) \) are second Hankel functions of the \( j \)th order; and \( \phi^{m}_x, \phi^{m}_y, \phi^{n}_y \) are the components of the natural modes of wave propagation in the stratum with wavenumbers \( k_{l\rho} \). R and L refer to the Rayleigh and Love modes (E14 through E22b, E50, E171 through E196). \( N \) is the total number of sublayers. The index \( m \) refers to the location (interface number) at which the displacements are observed, while \( n \) identifies the elevation at which the point load (at \( \rho = 0 \)) is applied. Also, the coefficients \( c, s \) are given by (24a) or 24b).
In matrix notation (rows \( \rightarrow m \), columns \( \rightarrow n \)), these functions may be written as

\[
U_\rho = \frac{\rho}{4i} (\phi_x \frac{d}{d\rho} H^R L K^{-1} R \phi_x^T + \frac{1}{\rho} \phi_y H^L L K^{-1} L \phi_y^T) c \quad (26a)
\]

\[
U_\theta = \frac{\nu}{4i} (\frac{1}{\rho} \phi_x H^R L K^{-1} R \phi_x^T + \phi_y \frac{d}{d\rho} H^L L K^{-1} L \phi_y^T) s \quad (26b)
\]

\[
U_z = -\frac{\rho}{4i} (\phi_z H^R L \phi_x^T) c \quad (26c)
\]

with

\[
H^R_j = \text{diag}(H^{(2)}_j(k_{R0})) \quad , \quad \ell = 1, \ldots , 2N \quad (21a)
\]

\[
H^L_j = \text{diag}(H^{(2)}_j(k_{L0})) \quad , \quad \ell = 1, \ldots , N \quad (27b)
\]

being diagonal matrices assembled with second Hankel functions of \( j \)-th order (\( j=1 \) in eqs. 26). They satisfy the recurrence relation (omitting the indices \( R, L \))

\[
\frac{d}{d\rho} H^R_j = H^R_{j-1} K - \frac{j}{\rho} H^R_j \quad (28)
\]

The diagonal matrices \( K^R \), \( K^L \) contain the Rayleigh and Love wavenumbers \( k^R_{\ell} \), \( k^L_{\ell} \).

Using the recurrence relations, equations (26) may also be written as

\[
U_\rho = \frac{\rho}{4i} \left[ (\phi_x \frac{d}{d\rho} H^R L K^{-1} R \phi_x^T - \phi_y \frac{d}{d\rho} H^L L K^{-1} L \phi_y^T) + \phi_y H^L L \phi_y^T \right] c \quad (29a)
\]

\[
= \frac{\rho}{4i} \left[ \phi_x H^R L \phi_x^T - \frac{1}{\rho} (\phi_x H^R L K^{-1} R \phi_x^T - \phi_y H^L L K^{-1} L \phi_y^T) \right] c \quad (29b)
\]
\[ U_\theta = \frac{p}{4i} \left[ \phi_x H_0^L \phi_x^T - (\phi_x \frac{d}{d\phi} H_1^R K_1^{-1} \phi_x^T - \phi_y \frac{d}{d\phi} H_1^L K_1^{-1} \phi_y^T) \right] s \quad (29c) \]

\[ = \frac{p}{4i} \left[ \phi_y H_0^L \phi_y^T + \frac{1}{\mu} (\phi_x H_1^R K_1^{-1} \phi_x^T - \phi_y H_1^L K_1^{-1} \phi_y^T) \right] s \quad (29d) \]

\[ U_z = \frac{p}{4i} (\phi_x \frac{d}{d\phi} H_0^R K_1^{-1} \phi_x^T) c \quad (29e) \]

The behavior of these functions for small argument (i.e., near the point load) can be examined by considering the ascending series of the Hankel functions:

\[ H_0 = - \frac{2i}{\pi} I + (1 - \frac{2i}{\pi} \gamma) I + \frac{i}{2\pi} \frac{\rho^2 K^2 L}{4} [1 - \frac{2i}{\pi} (\gamma - 1)] K^2 + ... + o(\rho^4 \ln \rho) \quad (30a) \]

\[ H_1 = \frac{1}{\rho} \frac{2i}{\pi} K^{-1} - \frac{i}{\pi} \frac{\rho}{K L} + \frac{\rho}{2} [1 - \frac{i}{\pi} (2\gamma - 1)] K + ... + o(\rho^3 \ln \rho) \quad (30b) \]

in which \( I \) is the identity matrix; \( \gamma = 0.5772156649 \) (Euler's constant); and

\[ L = \text{diag} \left\{ \ln \left( \frac{1}{2} k_{xy} \rho \right) \right\} \quad (31) \]

which may be split into

\[ L = \ln \left( \frac{\rho}{\rho_0} \right) I + \text{diag} \left\{ \ln \left( \frac{1}{2} k_{xy} \rho_0 \right) \right\} \quad (32) \]

with \( \rho_0 \) being an arbitrary reference radius.

Substituting eqs. (30a,b) into eqs. (29), and considering the orthogonality conditions specified by (E196) as well as observing the identity \( C_x \equiv C_y \) (see Appendix 1 and RI0.13), we obtain

\[ U_\rho = - \frac{p}{4} \left\{ \ln \left( \frac{\rho}{\rho_0} \right) (A_x^{-1} + A_y^{-1}) + (\phi_x L R \phi_x^T + \phi_y L L \phi_y^T) + \frac{1}{2} (2\gamma + 1 + i\pi) A_x^{-1} \right. \]

\[ + \left. \frac{1}{2} (2\gamma - 1 + i\pi) A_y^{-1} \right\} c + ... + o(\rho^2 \ln \rho) \quad (33a) \]
\[
U_\theta = -\frac{p}{4\pi} \left( \ln \left( \frac{\rho}{\rho_0} \right) (A_x^{-1} + A_y^{-1}) + (\phi_x L_R \phi_x^T + \phi_y L_L \phi_y^T) + \frac{1}{2} (2\gamma-1+i\pi)A_x^{-1} + \frac{1}{2} (2\gamma+1+i\pi)A_y^{-1} \right) s + \ldots 0 \ (\rho^2 \ln \rho) \tag{33b} \]

\[
U_z = -\frac{p}{4\pi} \rho \left( \ln \left( \frac{\rho}{\rho_0} \right) A_z^{-1} B_{zx} A_x^{-1} - \phi_z L_R \phi_z^T + \frac{1}{2} (2\gamma-1+i\pi)A_z^{-1} B_{zx} A_x^{-1} \right) c + \ldots 0 \ (\rho^3 \ln \rho) \tag{33c} \]

with the notation

\[
L_R = \text{diag} \left\{ \ln \left( \frac{1}{2} k^R_{\xi \phi} \rho \right) \right\} \tag{34a} \\
L_L = \text{diag} \left\{ \ln \left( \frac{1}{2} k^L_{\xi \phi} \rho \right) \right\} \tag{34b} \\

\]

Since \( \rho_0 \) is a constant, the above matrices are also constant. Equations (33) show that the singularity at the origin (\( \rho=0 \)) is logarithmic, and that the displacements in the neighborhood of the axis are independent of frequency. Also, since \( A_x^{-1} \), etc. are not diagonal, the singularity extends over the entire vertical axis. Notice that \( U_\rho \approx U_\theta \), \( U_z \to 0 \) (since \( \lim \rho \ln \rho = 0 \)).

3.1.2 Vertical point load

From equations (30a,b,c) the Green's functions for vertical point loads are

\[
u_\rho = \frac{p}{4\pi} \sum_{\lambda=1}^{2N} \phi_m \phi_n H_1^{(2)} \left( k_{\lambda \rho} \right) \tag{35a} \\
u_\theta = 0 \tag{35b} \\
u_z = \frac{p}{4\pi} \sum_{\lambda=1}^{2N} \phi_m \phi_n H_0^{(2)} \left( k_{\lambda \rho} \right) \tag{35c} \\

which in matrix notation reads
\[ U_\rho = \frac{p}{4i} \phi_x H^R_{\rho} \phi_z^T \]  
\[ U_\theta = 0 \]  
\[ U_z = \frac{p}{4i} \phi_z H^R_{\theta} \phi_z^T \]  

For points close to the axis, one obtains using eqs. (30a,b):

\[ U_\rho = \frac{p}{4\pi \rho} \left\{ \ln(\frac{\rho}{\rho_0}) (A^{-1}_x B_{xz} A^{-1}_z) - \phi_x L_R K_R \phi_z^T + \right. \]
\[ + \frac{1}{2} (2\gamma - 1 + i\pi) A^{-1}_x B_{xz} A^{-1}_z \} \right\} + \ldots \Omega^3 (\ln \rho) \]  
\[ U_\theta = 0 \]  
\[ U_z = -\frac{p}{2\pi} \left\{ \ln(\frac{\rho}{\rho_0}) A^{-1}_x + \phi_z L_R \phi_z^T + \frac{1}{2} (2\gamma + i\pi) A^{-1}_z \right\} + \right\} \right\} \right\} + \ldots \Omega^2 (\ln \rho) \]  

Again, the singularity is logarithmic, and independent of frequency. Also, \( U_\rho \to 0 \).

3.2 Green's functions for the internal stresses

3.2.1 Edge tractions in vertical planes

With reference to Fig. 5, the stresses in tangential (circumferential) and meridional planes can be written as:

\[
\begin{bmatrix}
\sigma_\rho \\
\tau_{\rho \theta} \\
\tau_{\rho z}
\end{bmatrix} = \begin{bmatrix}
\lambda + 2G & \cdot & \cdot \\
\cdot & G & \frac{\partial}{\partial \rho} + \cdot & \cdot & \cdot & \lambda \\
\cdot & G & \cdot & \frac{\partial}{\partial z} + G & \cdot & \cdot & \cdot & \frac{1}{\rho} \frac{\partial}{\partial \theta} + \\
\lambda & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{bmatrix} \begin{bmatrix}
\frac{1}{p} \\
\frac{1}{p} \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{bmatrix} \begin{bmatrix}
u_\rho \\
u_\theta \\
u_z
\end{bmatrix}
\]  

(38)
\[
\begin{align*}
\begin{bmatrix}
\tau & \theta \\
\sigma & \theta \\
\tau & Z & B
\end{bmatrix} &= 
\begin{bmatrix}
\lambda & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \rho} + \tau & \cdot & \cdot \\
\cdot & \lambda & \cdot \\
\cdot & \cdot & \lambda + 2G
\end{bmatrix}
\begin{bmatrix}
\rho \\
\cdot \\
\cdot
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \rho} + \tau & \cdot & \cdot \\
\cdot & \lambda & \cdot \\
\cdot & \cdot & \lambda + 2G
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \theta} & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \rho} & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\end{align*}
\]

\[\begin{align*}
\frac{\partial}{\partial \rho} + \tau & \cdot & \cdot \\
\cdot & \lambda & \cdot \\
\cdot & \cdot & \lambda + 2G
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\end{align*}
\]

On the other hand, the displacement vector in cylindrical coordinates is of the form

\[
U = T \bar{U} = 
\begin{bmatrix}
\mu \\
c \\
s
\end{bmatrix}
\begin{bmatrix}
\bar{u}_\rho \\
\bar{u}_\theta \\
\bar{u}_z
\end{bmatrix}
\]

in which \(c, s\) are given by (24) (compare with E6a,b). The superscript bar indicates that the variation with the azimuth has been separated. Defining also \(\mu = \text{Fourier index}\)

\[
\tilde{S}^\rho = 
\begin{bmatrix}
\sigma_p \\
\tau_p \theta \\
\tau_p Z
\end{bmatrix}
\begin{bmatrix}
\mu \\
c \\
s
\end{bmatrix}
\begin{bmatrix}
\bar{u}_\rho \\
\bar{u}_\theta \\
\bar{u}_z
\end{bmatrix}
\]

we obtain, substituting (40) and (41) into (38)

\[
\tilde{S}^\rho = 
\begin{bmatrix}
\sigma_p \\
\tau_p \theta \\
\tau_p Z
\end{bmatrix}
\begin{bmatrix}
\lambda + 2G & \cdot & \cdot \\
\cdot & \lambda & \cdot \\
\cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \rho} + \tau & \cdot & \cdot \\
\cdot & \lambda & \cdot \\
\cdot & \cdot & \lambda + 2G
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \theta} & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\]

\[\begin{align*}
\frac{\partial}{\partial \rho} + \tau & \cdot & \cdot \\
\cdot & \lambda & \cdot \\
\cdot & \cdot & \lambda + 2G
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\end{align*}
\]
On the other hand, substituting (40) into (39), we obtain (after factoring out the coefficients $c$, $s$):

$$
\begin{align*}
\mathbf{S}^\theta &= \begin{bmatrix}
\tau^\theta \\
\sigma^\theta \\
\tau^\theta \\
\tau^\theta
\end{bmatrix} = \begin{bmatrix}
G & \partial \rho \\
\lambda & \partial \rho \\
\partial \rho & \partial \rho \\
G & \partial \rho
\end{bmatrix} + \begin{bmatrix}
\mu G & -G \\
\lambda + 2G & -\mu (\lambda + 2G) \\
\partial \rho & \partial \rho \\
\mu G & \partial \rho
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_\rho \\
\mathbf{u}_\theta \\
\mathbf{u}_z
\end{bmatrix}
\end{align*}
$$

(43)

which is based on the definitions

$$
\mathbf{S}^\theta = \begin{bmatrix}
\sigma^\theta \\
\tau^\theta \\
\tau^\theta \\
\tau^\theta
\end{bmatrix} = \begin{bmatrix}
\mathbf{s} \\
\mathbf{c} \\
\mathbf{s} \\
\mathbf{s}
\end{bmatrix} \begin{bmatrix}
\tau^\theta \\
\tau^\theta \\
\tau^\theta \\
\tau^\theta
\end{bmatrix} = \mathbf{T}_\mu \mathbf{S}^\theta
$$

(44)

$$
\mathbf{T}_\mu = \text{diag} (s, c, s)
$$

(45)

(Compare $\mathbf{T}_\mu$ in (45) with $\mathbf{T}_\mu$ in (40)!)  

Within a sublayer, both the real and virtual displacements vary linearly with the vertical coordinate, that is

$$
\begin{align*}
\delta \mathbf{U} &= \xi \mathbf{U}_m + (1-\xi) \mathbf{U}_{m+1} \\
\mathbf{U} &= \xi \mathbf{U}_m + (1-\xi) \mathbf{U}_{m+1}
\end{align*}
$$

(46)

which also holds for the amplitudes $\hat{\mathbf{U}}$.

Considering now equation (14) in cylindrical coordinates, we can substitute (46) in conjunction with either (42) or (43) and obtain the consistent edge tractions (forces per unit length).
a) Circumferential edge tractions

In this case, \( \alpha_{\nu \rho} = 1, \alpha_{\nu \theta} = \alpha_{\nu z} = 0 \) \((\nu = \rho)\). Integrating (14) with \(dz = \kappa d\zeta\), in combination with (42) and (46), and requiring the result to be valid for arbitrary virtual displacements \(\delta V\), we obtain the amplitudes of the circumferential edge tractions:

\[
Q^\rho = \begin{pmatrix}
O^\rho_m \\
O^\rho_{m+1}
\end{pmatrix} = \left( A_m \frac{\partial \nu}{\partial \rho} + D_m + \frac{1}{\rho} E_m \right) \begin{pmatrix}
U_m \\
U_{m+1}
\end{pmatrix}
\]

(47)

with matrices \(A_m, D_m, E_m\) given in tables 1 and 2. (This equation is the same as E131). The actual edge tractions are

\[
Q^\rho = \begin{pmatrix}
T_{\mu} \\
T_{\mu + 1}
\end{pmatrix} \begin{pmatrix}
\tilde{O}^\rho_m \\
\tilde{O}^\rho_{m+1}
\end{pmatrix}
\]

(48)

which follows from consideration of eq. (41).

b) Meridional edge tractions

The direction cosines are now \( \alpha_{\nu \rho} = \alpha_{\nu z} = 0, \alpha_{\nu \theta} = 1 \) \((\nu = \theta)\). Integrating (14) with \(dz = \kappa d\zeta\) in combination with (43) and (46), and requiring the result to be valid for arbitrary virtual displacements \(\delta V\), we obtain the amplitudes of the meridional edge tractions:

\[
\tilde{Q}^\theta = \begin{pmatrix}
\tilde{O}^\theta_m \\
\tilde{O}^\theta_{m+1}
\end{pmatrix} = \left( \tilde{A}_m \frac{\partial \nu}{\partial \rho} + \tilde{D}_m + \frac{1}{\rho} \tilde{E}_m \right) \begin{pmatrix}
\tilde{U}_m \\
\tilde{U}_{m+1}
\end{pmatrix}
\]

(49)

The matrices \(\tilde{A}_m, \tilde{D}_m, \tilde{E}_m\) are given in table 3. Considering also eq.(44), we obtain the actual edge tractions as

\[
Q^\theta = \begin{pmatrix}
\tilde{T}_{\mu} \\
\tilde{T}_{\mu + 1}
\end{pmatrix} \begin{pmatrix}
\tilde{O}^\theta_m \\
\tilde{O}^\theta_{m+1}
\end{pmatrix}
\]

(50)

c) Edge tractions in arbitrary vertical plane

Consider a vertical plane not passing through the origin; the normal
to this plane forms an angle $\nu$ with the radial direction, so that
$\alpha_{\nu_0} = \cos \nu, \alpha_{\nu_\theta} = \sin \nu, \alpha_{\nu_z} = 0$ (Fig. 6). Expanding eq. (14) in terms of the circumferential and meridional stresses, and substituting the equivalents (41) and (44), we obtain

$$\delta V^T Q^\nu = \int_0^h \left\{ \delta u_0 \left( c \cos \nu \tau_{\nu_0} + s \sin \nu \tau_{\nu_\theta} \right) + \delta u_\theta \left( s \cos \nu \tau_{\nu_\theta} + c \sin \nu \tau_{\nu_0} \right) \\
+ \delta u_z \left( c \cos \nu \tau_{\nu z} + s \sin \nu \tau_{\nu \theta} \right) \right\} \, dz$$

(51)

With the displacement expansions (46) and the requirement that the result be valid for arbitrary virtual displacements $\delta V$, we obtain the edge tractions in the $\nu$-plane as (note the absence of superscript bar!):

$$Q^\nu = \cos \nu \, Q^\nu + \sin \nu \, Q^\theta$$

(52)

$$Q^\nu = \begin{bmatrix} Q^\nu_m \\ Q^\nu_{m+1} \end{bmatrix}, \quad Q^\theta = \begin{bmatrix} q_{\rho \nu} \\ q_{\theta \nu} \end{bmatrix}$$

(53)

3.2.2 Green's functions for the edge tractions in vertical planes

The consistent edge tractions (forces per unit length) in cylindrical coordinates associated with the fundamental solutions corresponding to the point loads in the interior of the stratum are simply obtained by replacing equations (25) (without the coefficients $c, s$!) and/or (35) into equations (47) and (49). For results see tables 4 and 5.

3.2.3 Tractions in horizontal planes

The consistent tractions in horizontal planes follow from equation (19), which can be written as

$$(\delta U^T_m \delta U^T_{m+1}) \begin{bmatrix} \tau^m_m \\ -\tau^m_{m+1} \end{bmatrix} = (\delta U^T_m \delta U^T_{m+1}) \begin{bmatrix} \bar{S}^m_m \\ -\bar{S}^m_{m+1} \end{bmatrix} - \int_0^h \delta U^T \bar{W} \, dz$$

(54)

in which the wave equation matrix $\bar{W}$ (in cylindrical coordinates) is


\[
\bar{W} = \begin{bmatrix}
\lambda + 2G & \cdots & \cdots \\
G & \cdots & \cdots \\
\cdots & \cdots & \lambda + 2G \\
\end{bmatrix} \frac{\partial}{\partial \rho} \bar{U} + \begin{bmatrix}
G & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \lambda + 2G \\
\end{bmatrix} \frac{\partial}{\partial z} \bar{U} + \begin{bmatrix}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\lambda + G & \cdots \\
\end{bmatrix} \frac{\partial^2}{\partial \rho \partial z} \bar{U}
\]

\[
+ \frac{1}{\rho} \begin{bmatrix}
\lambda + 2G & -\mu (\lambda + G) & \cdots \\
\mu (\lambda + G) & G & \cdots \\
\cdots & \cdots & G \\
\end{bmatrix} \frac{\partial}{\partial \rho} \bar{U} + \frac{1}{\rho} \begin{bmatrix}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \mu (\lambda + G) \\
\end{bmatrix} \frac{\partial}{\partial z} \bar{U}
\]

\[
- \frac{1}{\rho^2} \begin{bmatrix}
\lambda + 2G + \mu^2 G & -\mu (\lambda + 3G) & \cdots \\
-\mu (\lambda + 3G) & \mu^2 (\lambda + 2G) + G & \cdots \\
\cdots & \cdots & \mu^2 G \\
\end{bmatrix} \bar{U} + \gamma \omega^2 \bar{U}
\]  \tag{55}

and the stress vector \( \bar{S} \) is

\[
\bar{S} = \begin{bmatrix}
\bar{\tau}_{\rho z} \\
\bar{\tau}_{\theta z} \\
\bar{\sigma}_{z}
\end{bmatrix} = \begin{bmatrix}
\cdots & \cdots & G \\
\cdots & \cdots & \cdots \\
\lambda & \cdots & \cdots \\
\end{bmatrix} \frac{\partial}{\partial \rho} \bar{U} + \begin{bmatrix}
G & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \lambda + 2G \\
\end{bmatrix} \frac{\partial}{\partial z} \bar{U} +
\]

\[
\frac{1}{\rho} \begin{bmatrix}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \mu G \\
\lambda & -\mu \lambda & \cdots \\
\end{bmatrix} \bar{U}
\]  \tag{56}

The superscript bar indicates again separation of the variation with the azimuth, i.e.

\[
\begin{align*}
T_m &= T_{\mu} T_m \\
S_m &= T_{\mu} S_m \\
\bar{W} &= T_{\mu} \bar{W}
\end{align*}
\]  \tag{57}
and a similar relationship for $T_{m+1}, S_{m+1}$. (The diagonal matrix $T_{m}$ should not be confused with the tractions vector $T_{m}$.) To evaluate (54) for the fundamental states, we rewrite equations (25) and (35) in the form

$$U = T_{m} U$$

$$\bar{U} = \frac{1}{N} \sum_{\ell=1}^{2N} H_{\ell} \bar{H}_{\ell}$$

in which

$$H_{\ell} = \begin{pmatrix}
\frac{d}{d\phi} H^{(2)}_{\mu}(k_{\phi}) & H^{(2)}_{\mu}(k_{\phi}) \\
H^{(2)}_{\mu}(k_{\phi}) & \frac{d}{d\phi} H^{(2)}_{\mu}(k_{\phi}) \\
\ddots & \ddots & \ddots
\end{pmatrix}$$

$$F_{\ell} = \begin{pmatrix}
f_{\phi_{\ell}}^{\rho} \\
f_{\phi_{\ell}}^{\theta} \\
f_{\phi_{\ell}}^{Z}
\end{pmatrix} = (1-\xi) F_{m}^{\ell} + \xi F_{m+1}^{\ell}, \quad 0 \leq \xi \leq 1$$

For horizontal point loads, $\mu=1$, and

$$F_{\ell}^{m} = p \begin{pmatrix}
\phi_{x}^{m} & n_{x}^{m} / k_{x}^{R} \\
0 & 0 \\
\phi_{z}^{m} & n_{z}^{m} / k_{z}^{R}
\end{pmatrix}, \quad k_{x}^{R}, \quad 1 \leq \ell \leq 2N$$

(Rayleigh modes)

$$F_{\ell}^{m} = p \begin{pmatrix}
0 \\
\phi_{y}^{m} & n_{y}^{m} / k_{y}^{L} \\
0
\end{pmatrix}, \quad k_{y}^{L}, \quad 2N+1 \leq \ell \leq 3N$$

(Love modes)

Note that the index $\ell$ for the Love modes has been shifted by $2N$, in order to be able to write $\bar{U}$ as a single summation.
For vertical point loads, on the other hand, \( \mu = 0 \) and

\[
F_{m}^m = - \rho \left\{ \begin{array}{c}
\frac{m}{m} \\
\frac{n}{n} \\
\frac{R}{R} \\
0 \\
\frac{n}{n} \\
\frac{R}{R}
\end{array} \right\} , \quad k_{m} = k_{m}^R , \quad 1 \leq \ell \leq 2N \quad (63a)
\]

\[
F_{m}^m = 0 , \quad 2N+1 < \ell < 2N \quad (63b)
\]

Substituting (59) into (55) and (56), we obtain after considerable algebra (which requires also consideration of the Bessel differential equation):

\[
W = \frac{1}{4} \sum_{\ell = 1}^{3N} H_{m}^\ell \left[ -k_{m}^2 \left\{ 1 \begin{array}{ccc}
(\lambda+2G) & \cdots \\
G & \cdots \\
0 & \cdots 
\end{array} \right\} - k_{m} \left\{ 1 \begin{array}{ccc}
\cdots \\
-1 & \cdots \\
\cdots & \cdots 
\end{array} \right\} \frac{\partial}{\partial z} + 
\begin{array}{ccc}
G & \cdots \\
G & \cdots \\
0 & \cdots 
\end{array} \right] \frac{\partial}{\partial z} + \gamma_{m}^2 \left\{ 1 \begin{array}{ccc}
1 \\
1 \\
1 
\end{array} \right\} F_{m}^m \quad (64)
\]

and

\[
S = \frac{1}{4} \sum_{\ell = 1}^{3N} H_{m}^\ell \left[ k_{m} \left\{ \begin{array}{ccc}
\cdots & -G \\
\cdots & \cdots \\
\cdots & \cdots 
\end{array} \right\} + \left\{ 1 \begin{array}{ccc}
G \\
G \\
0 & \cdots 
\end{array} \right\} \frac{\partial}{\partial z} \left\{ 1 \begin{array}{ccc}
\lambda+2G \\
\lambda & \cdots \\
\lambda & \cdots 
\end{array} \right\} F_{m}^m \quad (65)
\]

Substituting (64) and (65) into (54), and considering virtual displacements \( \delta U = \delta U_{m} + (1-\xi)\delta U_{m+1} \), we obtain for arbitrary interface virtual displacements (see also table 1):
\[
\begin{pmatrix}
\bar{T}_m \\
-\bar{T}_{m+1}
\end{pmatrix} = \frac{1}{4i} \sum_{\ell=1}^{3N} \begin{pmatrix}
H^l_{\ell} & \ldots & H^l_{\ell}
\end{pmatrix} \begin{pmatrix}
A_m k^2 + B_m k^2 + C_m - \omega^2 m
\end{pmatrix} \begin{pmatrix}
F^m \\
F^{m+1}
\end{pmatrix}
\]

or defining the modal stresses (see also E155 in R13.13)

\[
\begin{pmatrix}
S_{m,\ell} \\
-S_{m+1, \ell}
\end{pmatrix} = K_{m,\ell} \begin{pmatrix}
F^m_{\ell} \\
F^{m+1}_{\ell}
\end{pmatrix}
\]

we obtain finally

\[
\bar{T}_m = \frac{1}{4i} \sum_{\ell=1}^{3N} H^l_{\ell} S_{m,\ell}
\]

(68a)

\[
\bar{T}_{m+1} = \frac{1}{4i} \sum_{\ell=1}^{3N} H^l_{\ell} S_{m+1,\ell}
\]

(68b)

These equations give the interface tractions for \( \phi > 0 \). On the other hand, at the point of application of the point load (\( \phi = \gamma \)), a singularity develops which needs to be considered separately (see below). The evaluation of equations 68a,b is presented in table 9.

The above expressions for the consistent tractions are identical to those obtained by evaluation of the Hankel transforms of the modal stresses in the wavenumber domain. In R81.13, the Green's functions for the tractions corresponding to horizontal and vertical disk loads were evaluated in this manner. To demonstrate the agreement with the procedure employed here, consider a disk load of intensity \( qR = \frac{p}{\pi R} \) (i.e., \( q\pi R^2 = p \)) and examine the limits of the functions \( I_3/\pi R \) and \( -I_1/\pi Rk \) as \( R \to 0 \) (see R81.13, tables 2 and 3, i.e., as the disk load becomes a point load: \( \lim_{R \to 0} \frac{I_3}{\pi R} = 0, \lim_{R \to 0} \frac{-I_1}{\pi Rk} = 0 \)).
\[
\lim_{R \to 0} \left( \frac{I_3}{\pi R} \right) = \frac{1}{4i k} \left\{ H_1^{(2)}(k \rho) - \frac{2i}{\pi k} \right\}
\]
\[
\text{def} \quad = \frac{1}{k} H_1
\]

\[
\lim_{R \to 0} \left( \frac{-I}{k \pi R} \right) = -\frac{1}{4i k} H_0^{(2)}(k \rho)
\]
\[
\text{def} \quad = -\frac{1}{k} H_0
\]

where the $H_0^{(2)}$ and $H_1^{(2)}$ are second Hankel functions, and $H_0, H_1$ are modified Hankel functions as defined above (identical to those in appendix 1).

The equation at the top of page 56 in R81.13 then becomes (for a horizontal disk load):

\[
\lim_{R \to 0} \frac{1}{\pi R} \left\{ \frac{d}{d \rho} I_3 \rho I_3 \rho I_3 \rho \ldots \right\} = \frac{1}{k_{\xi}} \left\{ \frac{d}{d \rho} H_1 \frac{1}{\rho} H_1 \frac{1}{\rho} H_1 \ldots \right\}
\]

\[
= \frac{1}{4i} \frac{1}{k_{\xi}} \begin{bmatrix}
\frac{d}{d \rho} H_1^{(2)} & \frac{1}{\rho} H_1^{(2)} & \ldots \\
\frac{1}{\rho} H_1^{(2)} & \frac{d}{d \rho} H_1^{(2)} & \ldots \\
\vdots & \vdots & \ddots
\end{bmatrix} + \frac{2i}{k_{\xi} \rho} 2 \pi \begin{bmatrix}
1 & -1 & \ldots \\
-1 & 1 & \ldots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\]

\[
= \frac{1}{4i} \frac{1}{k_{\xi}} (H_{\xi} + H_{\xi})
\]

\[
= \frac{1}{k_{\xi}} \begin{bmatrix}
\frac{d}{d \rho} H_1 & \frac{1}{\rho} H_1 & \ldots \\
\frac{1}{\rho} H_1 & \frac{d}{d \rho} H_1 & \ldots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\]

\[
\text{def} \quad = \frac{1}{k_{\xi}} H_{\xi} \text{ (note transposition of bar!)}
\]
where $H_{1 \lambda}$ is as in equation (60), and $|H_{1 \lambda}|$ is assembled as equation 60, but with modified Hankel functions (69a,b) instead. Alternatively, for a vertical disk load, we obtain

$$\lim_{R \to 0} \frac{1}{k_{\lambda} \pi R} \begin{pmatrix}
\frac{d}{d\rho} I_{1 \lambda} & \frac{1}{\rho} I_{1 \lambda} \\
\frac{1}{\rho} I_{1 \lambda} & \frac{d}{d\rho} I_{1 \lambda} \\
\cdot & \cdot & -k_{\lambda} I_{1 \lambda}
\end{pmatrix} = -\frac{1}{k_{\lambda}} \frac{1}{4i} H_{1 \lambda}$$

where $H_{1 \lambda}$ and $k_{\lambda}$ are defined.

(70b)

We have then

$$|H_{1 \lambda}| = \frac{1}{4i} (H_{1 \lambda}^* + L_{1 \lambda})$$

$$L_{1 \lambda} = \frac{2i}{\pi \rho k_{\lambda}} \begin{pmatrix}
1 & -1 & \cdot \\
-1 & 1 & \cdot \\
\cdot & \cdot & k_{\lambda} \rho
\end{pmatrix}$$

for a horizontal load

(71a)

$$H_{1 \lambda} = \frac{1}{4i} H_{1 \lambda}^*$$

$$L_{1 \lambda} = 0$$

for a vertical load

(71b)

Substituting these expressions into Equation E157, we obtain (omitting the azimuthal factor $T_{m}$, hence the superscript bar!).

$$\bar{T}_m = \frac{1}{4i} \sum_{\lambda=1}^{3N} \left( (H_{1 \lambda}^* + L_{1 \lambda}) S_{m \lambda} + L_{1 \lambda} \tilde{S}_{m \lambda} \right)$$

(72)

in which $S_{m \lambda}$ is defined by equation 67, $\tilde{S}_{m \lambda}$ is the truncated modal stress defined as (with $C_m = G_m - \omega^2 M_m$, see Table 1)
\[
\begin{align*}
\begin{cases}
\tilde{S}_{m,\ell} \\
\tilde{S}_{m+1,\ell}
\end{cases}
= \left( A_m \cdot k_\ell^2 + B_m \cdot k_\ell \right) \begin{cases}
F_m^\ell \\
F_{m+1}^\ell
\end{cases}

= (\kappa_m^\ell - C_m^\ell) \begin{cases}
F_m^\ell \\
F_{m+1}^\ell
\end{cases}

= \begin{cases}
S_m^\ell \\
S_{m+1,\ell}
\end{cases} - \begin{cases}
S_m^\ell \\
S_{m+1,\ell}
\end{cases}
\end{align*}
\]

(73)

and \( L_\ell \) is given for a horizontal load by (see also R81.13, page 59)

\[
L_\ell = \frac{2i}{\pi R^2 k_\ell} \begin{cases}
1 & 1 & . \\
1 & 1 & . \\
. & . & -k_\ell \rho
\end{cases}, \quad 0 \leq \rho < R
\]

(74a)

\[
= \frac{2i}{\pi \rho^2 k_\ell} \begin{cases}
-1 & 1 & . \\
1 & -1 & . \\
. & . & -k_\ell \rho
\end{cases} = - L_\ell, \quad R \leq \rho
\]

(74b)

while for a vertical load, from R81.13, E170 and table 3: (Note: the negative sign in the entry for \( g \) in table 2, R81.13 is in error; it should be positive):

\[
L_\ell = \frac{4i}{\pi R^2 k_\ell} \begin{cases}
. & . & . \\
. & . & . \\
. & . & -1
\end{cases}, \quad \rho < R
\]

(75a)

\[
= 0 \quad (= - L_\ell), \quad \rho > R
\]

(75b)
Consider first the case $p > R$. For both horizontal and vertical loads, the stresses are

$$T_m = \frac{1}{4 \pi} \sum_{\ell=1}^{3N} \left( (H_\ell + iL_\ell) S_{m,\ell} - iL_\ell \left( S_{m,\ell} - \tilde{S}_{m,\ell} \right) \right)$$

$$= \frac{1}{4 \pi} \sum_{\ell=1}^{3N} \left( H_{m,\ell} + iL_{m,\ell} \right)$$

and $L_\ell$ is non-zero only for horizontal loads. However, in this latter case, the second summation can be expanded to

$$\sum_{\ell=1}^{3N} \begin{pmatrix} iL_{m,\ell} & \tilde{S}_{m,\ell} - \tilde{S}_{m+1,\ell} \\ -iL_{m,\ell} & \tilde{S}_{m+1,\ell} \end{pmatrix} = \sum_{\ell=1}^{3N} \begin{pmatrix} iL_{m,\ell} \\ -iL_{m,\ell} \end{pmatrix} \begin{pmatrix} C_{m,\ell} \\ F_{m,\ell} \end{pmatrix} \begin{pmatrix} \tilde{S}_{m,\ell} - \tilde{S}_{m+1,\ell} \\ \tilde{S}_{m+1,\ell} \end{pmatrix}$$

and considering the special structure of the matrices $L_\ell$ and $C_m$ (table 1), we can permute the first product, and obtain

$$\sum_{\ell=1}^{3N} \begin{pmatrix} iL_{m,\ell} & \tilde{S}_{m,\ell} - \tilde{S}_{m+1,\ell} \\ -iL_{m,\ell} & \tilde{S}_{m+1,\ell} \end{pmatrix} = C_{m,\ell} \sum_{\ell=1}^{3N} \begin{pmatrix} iL_{m,\ell} F_{m,\ell} \\ -iL_{m,\ell} F_{m+1,\ell} \end{pmatrix}$$

Now

$$\sum_{\ell=1}^{3N} iL_{m,\ell} F_{m,\ell} = \frac{2ip}{\pi \rho} \sum_{\ell=1}^{3N} \begin{pmatrix} 1 & -1 & \vdots \\ -1 & 1 & \vdots \\ \vdots & \ddots & k_{\ell,\ell} \end{pmatrix} \begin{pmatrix} m_{\ell} \\ \phi_x \phi_y \\ \phi_{x^2} \phi_{y^2} \end{pmatrix} \begin{pmatrix} \phi_{n_x} \phi_{n_y} \end{pmatrix}$$

(from (62a, h), (71a) and combining Rayleigh and Love modes such that $\phi_{n_x} = 0$ for $\ell > 2N$, $\phi_{n_y} = 0$ for $\ell \leq 2N$).

Now,

$$\sum_{\ell=1}^{3N} \phi_{x_{\ell}} n_{\ell,\ell} / k_{\ell,\ell} = -c_{m,n}$$

$$\sum_{\ell=1}^{3N} \phi_{y_{\ell}} n_{\ell,\ell} / k_{\ell,\ell} = -c_{n,m}$$

(80a)
where $c_{-x}^{-1}, c_{-y}^{-1}$ are symbolic representations of the elements of the inverses

$$c_x^{-1} = \begin{bmatrix} c_{mn}^{-1} \end{bmatrix} = -\phi_x K_R^{-2} \phi_x^T$$  \hspace{1cm} (81a)

$$c_y^{-1} = \begin{bmatrix} c_{mn}^{-1} \end{bmatrix} = -\phi_y K_L^{-2} \phi_y^T$$  \hspace{1cm} (81b)

Since $c_x = c_y$, then $c_{mn}^{-x} = c_{mn}^{-y}$

Also

$$\sum_{\lambda=1}^{3N} \phi_{\lambda} \phi_{\lambda}^m n_\lambda / k_\lambda = 0$$  \hspace{1cm} (because of orthogonality)

and the cross products in $\phi_{\lambda} \phi_{\lambda}^m n_\lambda$ are zero because Rayleigh and Love modes do not overlap in $\lambda$. Hence, we obtain

$$\sum_{\lambda=1}^{3N} h_{\lambda} f_{m}^{\lambda} = \frac{-21 \omega^2}{2 \pi} \begin{bmatrix} c_{mn}^{-x} - c_{mn}^{-y} \\ \lambda = 1 \sum_{\lambda} \lambda \tilde{S}_{m\lambda} \end{bmatrix} = \mathbb{0}$$  \hspace{1cm} (82)

so that

$$\sum_{\lambda=1}^{3N} \sum_{\lambda} \lambda \tilde{S}_{m\lambda} = 0$$  \hspace{1cm} (83)

Thus, the interface stresses are

$$T_m = \frac{1}{4\pi} \sum_{\lambda} h_{\lambda} \tilde{S}_{m\lambda}$$  \hspace{1cm} (84)

which agrees with equation (68a)). Hence, the formulation in this work and in R81.13 agree. It remains to evaluate the singularity at the origin. We return for this purpose to equation (72), which we rewrite as

$$T_m = \sum_{\lambda=1}^{3N} h_{\lambda} s_{m\lambda} + \frac{1}{4\pi} \sum_{\lambda=1}^{3N} L_{\lambda} \tilde{S}_{m\lambda}$$  \hspace{1cm} (85)

Let's consider first the second summation, which we expand to (see equation 73)
\[ \frac{1}{4\pi} \sum_{\ell = 1}^{3N} \left\{ \begin{array}{c} L_{\ell} \\ L_{\ell} \\ -S_{m+1, \ell} \end{array} \right\} \left\{ \begin{array}{c} \tilde{S}_{m\ell} \\ \tilde{S}_{m\ell} \\ -S_{m+1, \ell} \end{array} \right\} = \frac{1}{4\pi} \sum_{\ell = 1}^{3N} \left\{ \begin{array}{c} L_{\ell} \\ L_{\ell} \\ -S_{m+1, \ell} \end{array} \right\} \left\{ A_{m\ell} k^2_{\ell} + B_{m\ell} k_{\ell} \right\} \left\{ \begin{array}{c} F_{m\ell} \\ F_{m+1\ell} \end{array} \right\} \] (86)

But from equations (62), (63), (74), (75), and table 1, we obtain for the first three rows of (86) in the case of a horizontal load: (again, combining Rayleigh and Love modes):

First row:

\[ \frac{p}{2\pi R^2} \sum_{\ell = 1}^{3N} \left( \frac{h}{6} \left( \lambda + 2\mu \right) \left( 2\phi^m_{x} + \phi^m_{y} + \phi^{m+1}_{x} + \phi^{m+1}_{y} \right) + G \left( 2\phi^m_{y} + \phi^{m+1}_{y} \right) \right) \left( \phi^m_{x} + \phi^m_{y} \right) \] (87a)

Second row: same as first row.

Third row:

\[ \frac{p}{2\pi R^2} \rho \sum_{\ell = 1}^{3N} \left( \frac{h}{6} G \left( 2\phi^m_{z} + \phi^{m+1}_{z} \right) k_{\ell} + \frac{1}{2} \left( \lambda - G \right) \phi^m_{x} + \frac{1}{2} \left( \lambda + G \right) \phi^m_{y} \right) \left( \phi^m_{x} + \phi^m_{y} \right) \] (87b)

But

\[ \sum_{\ell = 1}^{3N} \phi^m_{x} \left( \phi_{x} + \phi_{y} \right) = \sum_{\ell = 1}^{2N} \phi^m_{x} \phi_{x} = a^{-x}_{mn} \]

\[ \sum_{\ell = 1}^{3N} \phi^m_{y} \left( \phi_{x} + \phi_{y} \right) = \sum_{\ell = 1}^{2N} \phi^m_{y} \phi_{y} = a^{-y}_{mn} \] (88)

\[ \sum_{\ell = 1}^{3N} \phi^m_{z} \left( \phi_{x} + \phi_{y} \right) k_{\ell} = \sum_{\ell = 1}^{2N} \phi^m_{z} \phi_{x} = 0 \]

where \( a^{-x}_{mn}, a^{-y}_{mn} \) are symbolic representations of the elements of the inverses

\[ A^{-1}_{x} = \left\{ a^{-x}_{mn} \right\} = \phi_{x} \phi_{x}^T \] (89)

\[ A^{-1}_{y} = \left\{ a^{-y}_{mn} \right\} = \phi_{y} \phi_{y}^T \]
Also, since ρ = 0, the last row need not be considered. Hence, in the case of a horizontal point (small disk) load

\[
\frac{1}{4T} \sum_{\ell=1}^{3N} \left\{ \frac{L}{2} \frac{S_m}{S_{m+1, \ell}} \right\} = \frac{p}{2\pi R^2} \left\{ \begin{array}{ll}
\frac{h}{\lambda + 2G} (2a^{-x}_{mn} + a^{-x}_{m+1,n}) + \frac{h}{\rho} G(2a^{-y}_{mn} + a^{-y}_{m+1,n}) \\
\text{same as above} \\
0 \\
\frac{h}{\lambda + 2G} (a^{-x}_{mn} + 2a^{-x}_{m+1,n}) + \frac{h}{\rho} G(a^{-y}_{mn} + 2a^{-y}_{m+1,n}) \\
\text{same as above} \\
0
\end{array} \right. 
\]  

(90)

For our applications, we are interested in the integrals

\[
\Delta_{xx}^F = \int_0^R \int_{\theta_1}^{\theta_2} t_{xx} \rho \, d\rho \, d\theta 
\]

(91a)

\[
\Delta_{yx}^F = \int_0^R \int_{\theta_1}^{\theta_2} t_{yx} \rho \, d\rho \, d\theta 
\]

(91b)

\[
\Delta_{zx}^F = \int_0^R \int_{\theta_1}^{\theta_2} t_{zx} \rho \, d\rho \, d\theta 
\]

(91c)

For a horizontal -x load, the arguments of the above integrals are

\[
t_{xx} = t_{zp} \cos^2 \theta + t_{z0} \sin^2 \theta 
\]

(92a)

\[
t_{yx} = (t_{zp} - t_{z0}) \sin \theta \cos \theta 
\]

(92b)

\[
t_{zx} = t_{zz} \cos \theta 
\]

(92c)
Considering the first three rows of equations (90) together with (91) and (92), we obtain after integration, with \( \alpha = \theta_2 - \theta_1 \)

\[
\Delta f_{xx}^m = \frac{\alpha}{4\pi} \rho \left[ \frac{h}{6} (\lambda + 2G) (2a_m^{x-n} + a_m^{-y_{m+1,n}}) + \frac{h}{6} G (2a_m^{x-y} + a_m^{-y_{m+1,n}}) \right]
\]

(93a)

\[
\Delta f_{yx}^m = 0 \quad \text{(x load!)}
\]

(93b)

\[
\Delta f_{zx}^m = 0
\]

(93c)

In particular, for a full disk \( \theta_1 = 0, \theta_2 = 2\pi \), so that \( \alpha = 2\pi \). Similar expressions can be written also for \( \Delta f_{xx}^{m+1}, \Delta f_{yy}^{m+1}, \Delta f_{zz}^{m+1} \), considering for this purpose the last three rows of eq. 90. The combined results for the upper and lower interfaces of the \( m^{th} \) layer are then

\[
\begin{pmatrix}
\Delta f_{xx}^m \\
\Delta f_{xx}^{m+1}
\end{pmatrix} = \frac{\alpha \rho}{4\pi} \begin{pmatrix}
\frac{h}{6} (\lambda + 2G) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_m^{-x} \\ a_{m+1,n}^{-x} \end{bmatrix} + \frac{h}{6} G \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_m^{-y} \\ a_{m+1,n}^{-y} \end{bmatrix}
\end{pmatrix}
\]

(94)

The \( y \) and \( z \) resultants are all zero. For a horizontal-y load, on the other hand, the resultants are obtained by permutation of the sub-indices \( x \) and \( y \). (Note that if Poisson's ratio is constant throughout the stratum, then \((\lambda + 2G) a_m^{-x} = G a_m^{-y} \) and \((\lambda + 2G) a_m^{-x} = G a_m^{-y} \).

On the other hand, if the contributions of the various layers are overlapped, then equation (94) transforms (for \( \alpha = 2\pi \)) into

\[
\Delta F = \frac{p}{2} \left[ A_x A_x^{-1}(n) + A_y A_y^{-1}(n) \right]
\]

(95)

where \( A_x^{-1}(n), A_y^{-1}(n) \) represent the \( n^{th} \) columns of the inverses of \( A_x \), \( A_y \). Clearly, this reduces to \( \Delta F = p \{ E_n \} \), where \( E_n = \{ \delta_{mn} \} \) is a column vector whose only non-zero entry \( (\delta_{nn}, = 1) \) corresponds to the \( n^{th} \) component. Thus, the singular term considered here balances exactly the external load applied at the \( n^{th} \) interface. A similar demonstration can
be used for a horizontal y-load. Equations (93) give then the fraction of the external force carried by the mth layer.

On the other hand, for a vertical load, equation (86) results in

First two rows = 0 (because of 75a)

Third row:

\[
\frac{p}{\pi R^2} \sum_{\ell=1}^{2N} \left\{ \frac{h}{6} G (2 \phi_m \phi_z + \phi_{m+1} \psi \ell) + \frac{1}{2 (\lambda - G)} \phi_x \psi_k + \frac{1}{2 (\lambda + G)} \phi_x \psi_{k+1}, \frac{n}{k} \right\} \phi_z \psi \ell \]
\]

As in eqs. (83), we define

\[
A_{-z}^{-1} = \begin{bmatrix} a_{-z}^{-1} \\ a^{-1} \end{bmatrix} = \tilde{\phi}_z \phi_z \tilde{T} \]

Hence

\[
\frac{1}{4T} \sum_{\ell=1}^{3N} \begin{bmatrix} \tilde{I}_\ell & \tilde{S}_m \psi \ell \\ -\tilde{I}_\ell & -\tilde{S}_{m+1} \psi \ell \end{bmatrix} = \frac{p}{\pi R^2} \begin{bmatrix} 0 \\ \frac{h}{6} G (2 a_{mn}^{-z} + a_{m+1, n}^{-z}) \\ 0 \\ 0 \end{bmatrix} \]

For a vertical load, (\mu = 0), we are interested in the integral

\[
\int_0^R \int_0^\theta \int_0^1 t_{zz} \rho \ d\rho \ d\theta = \alpha R \int_0^R \int_0^\infty \tilde{t}_{zz} \rho \ d\rho
\]
Substituting (99) into (100) and integrating, we obtain

\[
\begin{bmatrix}
\Delta^m_{zz} \\
\Delta^{m+1}_{zz}
\end{bmatrix} = \frac{\rho \alpha}{2\pi} \frac{h_0}{6} \begin{bmatrix} 2 & 1 \\
1 & 2 \end{bmatrix} \begin{bmatrix} a^{-z}_{m} \\
a^{-z}_{m+1,n}\end{bmatrix}
\]  
(101)

which gives the fraction of the external vertical load carried by the \(m^{th}\) layer. As before, if the contributions of all layers are overlapped, and \(\alpha = 2\pi\), we obtain

\[
\Delta F = \rho \ A_z^{-1}(n) = \rho \ E_n
\]  
(102)

where again \(A_z^{-1}(n)\) denotes the \(n^{th}\) column of the inverse of \(A_z\), and

\[
E_n = \{e_{mn}\}
\]

is a column vector whose only non-zero element \((e_{nn} = 1)\) is the \(n^{th}\). Thus, the singularity is balanced entirely by the external load.

To complete the formulation, we must examine now the contribution of the first term in equation (85) to the load carried by the \(m^{th}\) layer. The first term of (85), written in full, is

\[
\sum_{\nu=1}^{3N} \sum_{\nu=1}^{3N} \left\{ \frac{d}{d\rho} H_{\mu} \right\} = \frac{\mu}{\rho} H_{\mu} \right\} 
\]  
(103)

where \(H_{\mu}\) are the modified Hankel functions defined by equations (69) and in appendix 1. The derivatives of these functions are:

\[
\frac{d}{d\rho} H_0 = k H_0 - \frac{1}{\rho} H_1 = 0 \quad (\ln \rho)
\]  
(104a)

\[
\frac{d}{d\rho} H_0 = - (k H_1 + \frac{1}{2 \pi \rho}) = - \frac{1}{2 \pi \rho} + 0 \quad (\rho \ln \rho)
\]  
(104b)
For a horizontal point load, the terms in $H_1$ and $dH_1/d\rho$ are of order $\rho \ln \rho$ and $\ln \rho$, respectively. Thus, their integrals in the neighborhood of the origin vanish, i.e.,

$$\lim_{R \to 0} \int_0^R H_1 \, d\rho = 0 \quad (105a)$$

$$\lim_{R \to 0} \int_0^R \frac{d}{d\rho} H_1 \, d\rho = 0 \quad (105b)$$

Hence, equation (103) does not contribute terms to the singular integral in the horizontal load case.

In the vertical load case, on the other hand, $t_{zz}$ is logarithmic (since it is associated with the functions $H_0$) and $t_{z\theta} = 0$ (since the Love modes do not participate). The remaining term $t_{z\rho}$ contains the singular term in $1/\rho$ (from the derivative of $H_0$), which written in full is

$$t_{z\rho}^m = \frac{p}{2\pi \rho} \sum_{\ell=1}^{2n} \left\{ \frac{1}{h} \left( \lambda + 2G \right) \left( 2\phi_x^m \phi_x^{m+1,\ell} + \phi_x^{m+1,\ell} \phi_x^m \right) k_\ell + \frac{1}{h} \left( \lambda - G \right) \phi_z^{m,\ell} - \left( \lambda + G \right) \phi_z^{m+1,\ell} \right\} \phi_z^{n,\ell} \quad (106)$$

The above expression will frequently vanish as a result of the orthogonality conditions $A_x \phi_x K_x \phi_x^T + B_{xz} \phi_x \phi_z^T = 0$ and $\phi_x K_{xz} \phi_z^T = 0$. While the latter condition guarantees the vanishing of the term in square brackets in (106), the former appears only incompletely in the other terms, so that the cancellation cannot be assured. However, when considering a full disk (bottom or ceiling of excavation + $\alpha = 2\pi$), the integration over the disk vanishes, since $t_{z\rho}$ is radially symmetric. Furthermore, in the case of a circular sector ($\alpha = 2\pi$), as may be found at the intersection of horizontal and vertical walls, the singularity contributed by (106) cancels with a corresponding term in the integral over the vertical walls (see section 4.2.1.1). Thus, the first term in equation (85) (i.e., equation 103) need not be considered in the evaluation of the
singularity at the origin; its regular contribution, on the other hand, will be taken up in later sections (4.2.1.2).

It follows from the preceding that equations (94) and (101) capture completely the singularity at the origin. These equations are summarized in table 15.
4. **Evaluation of Boundary Integrals**

Equation 23 requires the evaluation of two distinct types of integrals: On the left-hand side, we have the integrals of the (virtual stresses) $\mathbf{x}$ (real displacements), while on the right-hand side, we have the integrals of the (virtual displacements) $\mathbf{x}$ (real stresses). These will be calculated in the following.

4.1 **Point load not on cell**

For integration cells not coinciding with the location of the virtual point loads, the most expedite method is to take the integrals over the cell as simply the area of the cell times the integrand evaluated at the center of the cell. This is based on the assumption that the real displacements and stresses are piecewise uniform in horizontal planes (constant over each cell), and that the virtual displacements and stresses vary monotonically and smoothly so that their averages over the cell equal approximately the value at the center.

With reference to Fig. 7, we have

$$
\left\{ \int_a q_{ij}^* \, u_i \, ds = \left\{ \int_a q_{ij}^* \, ds \right\} \, u_i = a \, q_{ij}^* \, u_i \right\}
$$

$$
\left\{ \int_A t_{ij}^* \, u_i \, dS = \left\{ \int_A t_{ij}^* \, dS \right\} \, u_i = A \, t_{ij}^* \, u_i \right\}
$$

where $q_{ij}^*$ is the average virtual edge traction over the edge segment of length $a$, and $t_{ij}^*$ is the average traction over the cell of surface $A$. Since the virtual load is not on the edge or cell under consideration, the tractions increase/decrease monotonically over the integration length/area; hence, the averages can be approximated by the values of the virtual traction at the center of the segment or area. If the edge is not smooth at a point, we can evaluate $q_{ij}^*$ on the projection of the edge onto the tangential plane. From Eq. 52, namely, we see that the edge tractions are of the form

$$Q^v = Q^0 \cos \nu + Q^0 \sin \nu$$

(108)
For two edges of length \(a/2\) each, and forming angles \(\alpha, \beta\) with the radius (Fig. 7), the integral (for uniform tractions) is

\[
\mathcal{Q}^\nu = \frac{a}{2} (Q^\alpha + Q^\beta) = \frac{a}{2} \left[ Q^0 (\cos \alpha + \cos \beta) + Q^0 (\sin \alpha + \sin \beta) \right] \quad (109)
\]

in which \(\mathcal{L}\) is the equivalent edge length, and \(Q^\nu\) is the equivalent tractions vector. From (108) and (109) it follows that

\[
\begin{align*}
\mathcal{L} \cos \nu & = \frac{a}{2} (\cos \alpha + \cos \beta) \\
\mathcal{L} \sin \nu & = \frac{a}{2} (\sin \alpha + \sin \beta)
\end{align*}
\]

(110)

which can be satisfied if

\[
\begin{align*}
\mathcal{L} & = a \cos \frac{\alpha - \beta}{2} \\
\nu & = \frac{\alpha + \beta}{2}
\end{align*}
\]

(111)

The above equations define the projection of the edge onto the tangential plane.

Equations similar to (107) can also be written for the virtual displacements - real traction integrals:

\[
\begin{align*}
\int_a u^*_{ij} q_i \, dS &= a u^*_{ij} q_i \\
\int_A u^*_{ij} t_i \, dS &= A u^*_{ij} t_i
\end{align*}
\]

(112)

in which \(u^\star_{ij}\) is the average virtual displacement which may be taken as the displacement at the center of the cell.
4.2 **Point load on integration cell**

When the integration path contains a virtual point load, the integrand exhibits a singularity that must be computed by isolating a small volume in the neighborhood of the load. For the discrete solution presented here, special consideration is also required for integration points on the same vertical line as the point load (i.e., having the same \( x, y \) coordinates). In the following sections, we shall evaluate these integrals for the various cases of interest.

4.2.1 **Virtual stresses x real displacements**

4.2.1.1 **Integration over vertical walls**

Consider the cylindrical sector of radius \( a \) shown in Fig. 8. It is bounded by two vertical walls forming an angle \( \alpha \), a cylindrical wall, and two horizontal circular sectors. The vertical walls define either an edge of the excavation, or in the particular case of a flat angle \( (\alpha = \pi) \), a wall of the excavation. A virtual point load is applied somewhere on the axis of the cylindrical sector. We are interested in the integrals over the vertical walls. To avoid the singularity on the axis, we cut out a cylindrical sector of small radius \( \varepsilon \), and consider the integral over the vertical surfaces of the resulting body in the limit when \( \varepsilon \) tends to zero (Fig. 9). We seek then the limit of the integral

\[
\lim_{\varepsilon \to 0} \mathcal{C} T \mathcal{T}^* \mathcal{S} \mathcal{S}_3 \mathcal{S}_3
\]

(113)

in which \( \mathcal{U} \) is the vector of real displacements, and \( \mathcal{T}^* \) is the vector of tractions associated with the fundamental solution(s). The real displacements vary linearly between layers, and are uniform over horizontal surfaces of (small) radius \( a \). Isolating a small volume defined by two neighboring layer interfaces (Fig. 9), we can apply the principle of virtual displacements to this body, but reversing the roles of the virtual and real fields:

\[
\mathcal{C} \mathcal{T} \mathcal{T}^* \mathcal{U} \mathcal{V} = \mathcal{C} \mathcal{T} \mathcal{T}^* \mathcal{S} \mathcal{S}_3 \mathcal{S}_3
\]

(114)

but \( \mathcal{S} = \mathcal{S}_1 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_3 \). Hence
Figure 7
\[ \int_{S_1'} \int_{S_2'} \int_{S_3'} U^T T^* dS - \int_{S_2''} \int_{S_1''} U^T T^* dS - \int_{S_1''} U^T \varepsilon^* dV + \int_{S_1''} Y^T U^* dV \]

(115)

We have now an equivalent expression for the integral sought above. The first integral on the right-hand side (whose result we may call \( J \)) need not be evaluated, because its contribution to the total integral is equal and opposite to the contribution of the corresponding integrals for the neighboring layers (continuity of tractions!). Hence, when the integrals for the various layers (including the bottom of the excavation!) are overlapped, this term cancels; also, at the free surface (for an open excavation) the tractions for the fundamental solution are zero \((T^* = 0)\). It remains then to evaluate the other integrals.

a) Integration over \( S_1'' \):

The area element is in this case \( dS = a d\theta dz \), with \( dz = h d\psi \). The integration over \( z \) is not necessary, since it has already been carried out in section 3 when evaluating the consistent edge tractions. For either the upper or lower edge, we have from Eqs. 47,48:

\[ \Omega_\mu = \begin{pmatrix} \Sigma_{\phi \phi}^* \\ \Sigma_{\phi \theta}^* \\ \Sigma_{\phi z}^* \end{pmatrix}, \quad Q^* = T_\mu \Omega^* \]  

(116)

With reference to Fig. 10, we have then for the differential virtual forces \( dF^* = Q^* d\theta \)

\[ dF^* = d \begin{pmatrix} f_{\phi}^* \\ f_{\theta}^* \\ f_z^* \end{pmatrix} = \begin{pmatrix} c & s & -q_{\phi \rho}^* \\ s & c & -q_{\theta \rho}^* \\ -q_{\phi z}^* & -q_{\theta z}^* & c \end{pmatrix} \rho d\theta \quad (\rho = \text{a!}) \]  

(117)

and the differential forces in Cartesian coordinates (\( \Rightarrow \) tilde!)
\[
\begin{align*}
\frac{dF^*}{d\gamma} &= \begin{bmatrix}
 f_x^* \\
 f_y^* \\
 f_z^*
\end{bmatrix} = \begin{bmatrix}
 \cos \theta & -\sin \theta \\
 \sin \theta & \cos \theta \\
 0 & 1
\end{bmatrix} \begin{bmatrix}
 f_x^* \\
 f_y^* \\
 f_z^*
\end{bmatrix} \\
\end{align*}
\]

so that

\[
\begin{align*}
\frac{dF^*}{d\gamma} &= \begin{bmatrix}
 f_x^* \\
 f_y^* \\
 f_z^*
\end{bmatrix} = \begin{bmatrix}
 c \cos \theta & -s \sin \theta \\
 c \sin \theta & s \cos \theta \\
 0 & 1
\end{bmatrix} \frac{\theta_2^*}{\theta_2^* - \theta_1^*} d\theta
\end{align*}
\]

Integration of the above expression over \(\theta\) between limits \(\theta_1\) and \(\theta_2\), with the definition

\[
\frac{\theta}{\theta} = \frac{\theta_2 + \theta_1}{2}
\]

\[
\alpha = \theta_2 - \theta_1
\]

and considering the values of \(c, s\) specified by Eq. 24, we obtain

\[
F^* = Z \frac{\theta_2^*}{\theta_2^* - \theta_1^*}
\]

in which

\[
Z = \begin{cases}
Z_x & \text{for a horizontal } x \text{ load} \\
Z_y & \text{for a horizontal } y \text{ load} \\
Z_z & \text{for a vertical } z \text{ load}
\end{cases}
\]

\[
Z_x = \begin{bmatrix}
\frac{1}{2}(\alpha + \sin \alpha \cos 2\theta) & \frac{1}{2}(\alpha - \sin \alpha \cos 2\theta) \\
\frac{1}{2} \sin \alpha \sin 2\theta & -\frac{1}{2} \sin \alpha \sin 2\theta \\
\frac{1}{2} \sin \alpha \sin 2\theta & 2 \sin \frac{\alpha}{2} \cos \theta
\end{bmatrix}
\]

(123a)
\[
Z_y = \begin{pmatrix}
\frac{1}{2} \sin \alpha \sin 2\bar{\theta} & -\frac{1}{2} \sin \alpha \sin 2\bar{\theta} \\
\frac{1}{2} (\alpha - \sin \alpha \cos 2\bar{\theta}) & \frac{1}{2} (\alpha + \sin \alpha \cos 2\bar{\theta}) \\
2 \sin \frac{\alpha}{2} \sin \bar{\theta} &
\end{pmatrix}
\]

(123b)

\[
Z_z = \begin{pmatrix}
2 \sin \frac{\alpha}{2} \cos \bar{\theta} & . & . \\
2 \sin \frac{\alpha}{2} \sin \bar{\theta} & . & . \\
. & . & \alpha
\end{pmatrix}
\]

(123c)

Considering now equation (121) for each of the two interfaces,

\[
\begin{align*}
\left\{ \begin{array}{c}
F^*_{m} \\
F^*_{m+1}
\end{array} \right\} &= \rho \left\{ \begin{array}{c}
Z \ Q^*_m \\
Z \ Q^*_{m+1}
\end{array} \right\} = \rho \left\{ \begin{array}{c}
Z \\
Z
\end{array} \right\} \left\{ \begin{array}{c}
\overline{Q}^*_m \\
\overline{Q}^*_{m+1}
\end{array} \right\}
\end{align*}
\]

(124)

and in view of equation 47, it follows

\[
\begin{align*}
\left\{ \begin{array}{c}
F^*_{m} \\
F^*_{m+1}
\end{array} \right\} &= \rho \left\{ \begin{array}{c}
Z \\
Z
\end{array} \right\} \left( A_m \rho \frac{\partial}{\partial \rho} + \rho D_m + \xi_m \right) \left\{ \begin{array}{c}
U^*_m \\
U^*_{m+1}
\end{array} \right\}
\end{align*}
\]

(125)

This equation gives the virtual forces (in Cartesian coordinates) contributed by \( S_1 \). Hence

\[
\iint_{S_1} \left[ \begin{array}{c}
U^T_m \\
U^T_{m+1}
\end{array} \right] \left[ \begin{array}{c}
F^*_{m} \\
F^*_{m+1}
\end{array} \right] dS = (U^T_m U^T_{m+1}) \left\{ \begin{array}{c}
F^*_{m} \\
F^*_{m+1}
\end{array} \right\}
\]

(126)

in which \( U_m \), \( U_{m+1} \) are the (real) displacement vectors in Cartesian coordinates. The detailed expression for the consistent virtual forces will be presented later.
b) Volume integral for strain energy:

From the stress-strain relation (constitutive equation):

$$\sigma = C\varepsilon$$  \hspace{1cm} (127)

\[ C = \begin{bmatrix}
\lambda + 2G & \lambda & \lambda & \cdots & \cdots & \\
\lambda & \lambda + 2G & \lambda & \cdots & \cdots & \\
\lambda & \lambda & \lambda + 2G & \cdots & \cdots & \\
\cdots & \cdots & \cdots & G & \cdots & \\
\cdots & \cdots & \cdots & \cdots & G & \\
\cdots & \cdots & \cdots & \cdots & \cdots & G
\end{bmatrix} \hspace{1cm} (128)\]

and the strain displacement relation

\[ \varepsilon = \begin{bmatrix}
\frac{\partial}{\partial \rho} & \cdots & \\
\frac{1}{\rho} \frac{\partial}{\partial \theta} & \cdots & \frac{\partial}{\partial z} \\
\cdots & \cdots & \cdots \\
\frac{\partial}{\partial z} & \cdots & \frac{\partial}{\partial \rho} \\
\frac{1}{\rho} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \rho} \left(\frac{1}{\rho}\right) & \cdots \\
\cdots & \frac{\partial}{\partial z} & \frac{1}{\rho} \frac{\partial}{\partial \theta}
\end{bmatrix} \hspace{1cm} \varepsilon, \quad U = \begin{bmatrix} u_\rho \\ u_\theta \\ u_z \end{bmatrix} \hspace{1cm} (129)\]

as well as the Fourier decomposition of the strains (with c, s being given again by eqs. 24):
\[ \begin{align*}
\dot{\varepsilon} &= \begin{pmatrix}
\epsilon_{\rho \rho} \\
\epsilon_{\theta \theta} \\
\epsilon_{z z} \\
\gamma_{\rho z} \\
\gamma_{\theta \rho} \\
\gamma_{\theta z}
\end{pmatrix} - \begin{pmatrix}
c^{-1} \epsilon_{\rho \rho} \\
c^{-1} \epsilon_{\theta \theta} \\
c^{-1} \epsilon_{z z} \\
c^{-1} \gamma_{\rho z} \\
c^{-1} \gamma_{\theta \rho} \\
c^{-1} \gamma_{\theta z}
\end{pmatrix} \begin{pmatrix}
c \mathbf{I}_4 \\
\mathbf{I}_4 \\
s \mathbf{I}_2
\end{pmatrix} = \bar{T}_\mu \bar{\varepsilon} \\
\text{in which } \bar{T}_\mu &= \begin{pmatrix}
c \mathbf{I}_4 \\
\mathbf{I}_4 \\
s \mathbf{I}_2
\end{pmatrix}
\end{align*} \] (130)

we can compute the "real" and "virtual" strains and stresses within the body.

For the virtual displacements, we obtain

\[ \ddot{\varepsilon}^* = \begin{pmatrix}
1 & . & . \\
. & . & . \\
. & . & 1 \\
. & 1 & . \\
. & . & . \\
. & . & .
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial \rho} \ddot{U}^* + \frac{1}{\rho} \dot{U}^* \\
\ddot{U}^* + \frac{1}{\rho} \dot{U}^* \\
\frac{\partial}{\partial z} \ddot{U}^*
\end{pmatrix} = \begin{pmatrix}
1 & -\mu & . \\
. & . & . \\
. & . & 1
\end{pmatrix} \ddot{U}^* + \begin{pmatrix}
1 & . & . \\
. & \mu & . \\
. & . & 1
\end{pmatrix} \frac{\partial}{\partial z} \ddot{U}^* \] (132)

On the other hand,

\[ \varepsilon \mathbf{T} \sigma^* = \varepsilon^T \mathbf{C} \varepsilon^* = \varepsilon^T \bar{T}_\mu \varepsilon^* = \bar{T}_\mu \mathbf{C} \bar{\varepsilon}^* \] (133)

\[ = (\bar{T}_\mu \varepsilon)^T \mathbf{C} \bar{\varepsilon}^* \]
We need then an expression for $\mathbf{\bar{T}}_{\mu} \varepsilon$. For the real displacement field

$$
U = \begin{bmatrix}
    u_\rho \\
    u_\theta \\
    u_z
\end{bmatrix} = \begin{bmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    u_x \\
    u_y \\
    u_z
\end{bmatrix}
$$

(134)

The Cartesian displacements are constant in the interval $\varepsilon < \rho < a$, and change linearly with $z$; hence

$$
\frac{\partial}{\partial \rho} U = 0
$$

(135)

$$
\frac{1}{\rho} \frac{\partial}{\partial \theta} U = -\frac{1}{\rho} \begin{bmatrix}
    \sin \theta & -\cos \theta & . \\
    \cos \theta & \sin \theta & . \\
    . & . & .
\end{bmatrix} \begin{bmatrix}
    u_x \\
    u_y \\
    u_z
\end{bmatrix}
$$

(136)

The strains are then (expanding 129 and dropping the term in $\frac{\partial}{\partial \rho}$)

$$
\varepsilon = \frac{1}{\rho} \begin{bmatrix}
    1 & . & . \\
    . & 1 & . \\
    . & . & 1
\end{bmatrix} U + \begin{bmatrix}
    . & . & . \\
    . & 1 & . \\
    . & . & 1
\end{bmatrix} \frac{\partial}{\partial \rho} U + \frac{1}{\rho} \begin{bmatrix}
    1 & . & . \\
    . & 1 & . \\
    . & . & 1
\end{bmatrix} \frac{\partial}{\partial \theta} U + \frac{2}{\rho} \begin{bmatrix}
    1 & . & . \\
    . & 1 & . \\
    . & . & 1
\end{bmatrix} \frac{\partial^2}{\partial \rho^2} U
$$

(137)

In the computation of $\mathbf{\bar{T}}_{\mu} \varepsilon$, using the above expressions, we obtain the following terms;
\[ T_U \frac{1}{\rho} \begin{bmatrix} 1^{st \ matrix} \end{bmatrix} U = \frac{1}{\rho} \begin{bmatrix} 1^{st \ matrix} \end{bmatrix} \begin{bmatrix} c \cos \theta & c \sin \theta \\ -s \sin \theta & s \cos \theta \\ c \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \] (138)

\[ T_U \begin{bmatrix} 2^{nd \ matrix} \end{bmatrix} \frac{\partial}{\partial z} U = \begin{bmatrix} 2^{nd \ matrix} \end{bmatrix} \begin{bmatrix} c \cos \theta & c \sin \theta \\ -s \sin \theta & s \cos \theta \\ c \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \] (139)

\[ T_U \begin{bmatrix} 3^{rd \ matrix} \end{bmatrix} \frac{1}{\rho} \frac{\partial}{\partial \theta} U = \begin{bmatrix} 3^{rd \ matrix} \end{bmatrix} \frac{1}{\rho} \begin{bmatrix} -s \sin \theta & s \cos \theta \\ -c \cos \theta & -c \sin \theta \\ . & . & . \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \]

\[ = \begin{bmatrix} 3^{rd \ matrix} \end{bmatrix} \frac{1}{\rho} \begin{bmatrix} . & 1 & . \\ . & -1 & . \\ . & . & . \end{bmatrix} \begin{bmatrix} c \cos \theta & c \sin \theta \\ -s \sin \theta & s \cos \theta \\ c \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \]

\[ = \frac{1}{\rho} \begin{bmatrix} . & . & . \\ . & . & . \\ -1 & . & . \\ . & . & . \\ . & . & 1 \end{bmatrix} \begin{bmatrix} c \cos \theta & c \sin \theta \\ -s \sin \theta & s \cos \theta \\ c \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \] (140)

Since the first (eq. 138) and third term (eq. 140) are equal and opposite, they will cancel; only the term in \( \partial/\partial z \) survives. Hence,
\[
\begin{align*}
\bar{T} = & \begin{pmatrix}
. & . & . \\
. & . & 1 \\
1 & . & . \\
. & . & 1 \\
. & . & .
\end{pmatrix} \begin{pmatrix}
c \cos \theta & c \sin \theta & . \\
-s \sin \theta & s \cos \theta & . \\
. & . & c
\end{pmatrix} \frac{\partial}{\partial z} \bar{U} \\
& (141)
\end{align*}
\]

with \( \bar{U} \) being the vector of displacements in Cartesian coordinates. We notice that the matrix in \( c \cos \theta \) etc. is the same as in equation (119); when integrated over \( \theta \), it will again give rise to the matrix \( Z \) (eq.122). It is then convenient to perform the integration over \( \theta \) at this time, and to replace the above matrix by \( Z \) in what follows.

Combining (128), (132) and (141), we obtain

\[
\int_{0}^{2 \pi} (\bar{T}_U \epsilon)^T \tilde{C} \tilde{Z}^* d\theta =
\]

\[
\frac{\partial}{\partial z} \bar{U}^T \bar{Z}^T \begin{pmatrix}
. & . & . & G \\
. & . & \lambda & . \\
. & \lambda & . & . \\
G & \lambda & . & \lambda +2G
\end{pmatrix} \frac{\partial}{\partial \rho} \bar{U}^* + \frac{1}{\rho} \begin{pmatrix}
. & . & \mu G \\
. & -\mu \lambda & . \\
\mu \lambda & . & . \\
G & \lambda & . & \lambda +2G
\end{pmatrix} \bar{U}^* =
\]

\[
(142)
\]

Integration over \( z \), with \( dz = h d\xi \) and

\[
\bar{U}^* = \xi \bar{U}^*_m + (1-\xi) \bar{U}^*_m + 1 \]

\[
\bar{U} = \xi \bar{U}_m + (1-\xi) \bar{U}_{m+1}
\]

we obtain

\[
\int_{\xi}^{\xi+1} (\bar{U}^T \bar{Z}) d\nu = (\bar{U}^T \bar{Z}) \begin{pmatrix}
\bar{F}^*_m \\
\bar{F}^*_{m+1}
\end{pmatrix}
\]

\[
(144)
\]
with

\[
\begin{pmatrix}
Z \\
\text{F}^*_{m} \\
\Gamma^*_{m+1}
\end{pmatrix} =
\begin{pmatrix}
Z \\
\text{F}^*_{m} \\
\Gamma^*_{m+1}
\end{pmatrix} \int_{\varepsilon} a \left( D^T_{m} \rho + \frac{\partial}{\partial \rho} N_{m} + \rho G \right) \begin{pmatrix}
\text{U}^*_{m} \\
\text{U}^*_{m+1}
\end{pmatrix} d\rho
\]

(145)

in which the matrices \( G_{m}, D_{m} \) are given in tables 1 and 2. Also,

\[
N_{m} = \frac{1}{2} \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \mu G & \cdot & \cdot \\
\lambda & -\mu \lambda & \cdot & \lambda & +\mu \lambda \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\mu G & \cdot & \cdot & -\mu G \\
-\lambda & \mu \lambda & \cdot & -\lambda & \mu \lambda & \cdot
\end{pmatrix}
\]

(146)

Integrating eq. 145 by parts, we obtain

\[
\begin{pmatrix}
\text{F}^A_{m} \\
\text{F}^*_{m} \\
\text{F}^*_{m+1}
\end{pmatrix} =
\begin{pmatrix}
Z \\
\text{F}^*_{m} \\
\Gamma^*_{m+1}
\end{pmatrix} \int_{\varepsilon} a \left( D^T_{m} \rho \text{V}^* + (N-D^T) \right) \left( \text{V}^* \right) \left( \rho \text{V}^* \right) d\rho + G \int_{\varepsilon} a \left( \rho \text{V}^* \right) d\rho
\]

(147)

with the shorthand

\[
\text{V}^* = \begin{pmatrix}
\text{U}^*_{m} \\
\text{U}^*_{m+1}
\end{pmatrix}
\]

(148)

Since the Green's functions are logarithmic when \( \varepsilon \to 0 \), the term \( D^T_{\rho} \text{V}^* \) will vanish in the lower limit, i.e.,

\[
D^T_{\rho} \text{V}^* \int_{\varepsilon} a = D^T_{\rho} \text{V}^* \int_{a}
\]

(149)
c) Volume integral for inertial term:

By manipulations that parallel those in the previous section, the following can be established:

\[
\iint V U^T \dot{U} dV = -\omega^2 \begin{pmatrix} Z \\ Z \end{pmatrix} M_m \int_\epsilon^a \rho V^* d\rho
\]

with \( M_m \) being given in table 1. Hence, the virtual forces due to inertia are:

\[
\begin{pmatrix} F^*_m \\ F^*_{-m+1} \end{pmatrix} = -\omega^2 \begin{pmatrix} Z \\ Z \end{pmatrix} M_m \int_\epsilon^a \rho V^* d\rho
\]

(151)

d) Combination of the contributions of all integrals:

Combining the results of the preceding sections a, b, c, i.e., equations (125), (147), (151), we obtain finally

\[
J + \iiint U^T \dot{U}^* dS = \int_{S_3'} \int_{S_3''} \begin{pmatrix} F^*_m \\ F^*_{-m+1} \end{pmatrix}
\]

(152)

The term in \( J \) represents the contribution of the horizontal surfaces at the intersection with the vertical walls. The total virtual forces are then

\[
\begin{pmatrix} F^*_m \\ F^*_{-m+1} \end{pmatrix} = \begin{pmatrix} Z \\ Z \end{pmatrix} \left[ -A_m \frac{\partial}{\partial\rho} + \rho (D_m^T - D_m) - E_m \right] \begin{pmatrix} V^* \end{pmatrix}_a +
\]

+ \( (N_m - D_m^T) \int_\epsilon^a \rho V^* d\rho + (E_m - \omega^2 M_m) \int_\epsilon^a \rho V^* d\rho \)

(153)

It remains only to evaluate the integrals. Using the asymptotic expansions for the Green's functions (eqs. 33, 37) and the orthogonality conditions (E196), one obtains after lengthy algebra the results presented in tables 10, 11 and 12. The consistent virtual forces derived from eq. 153 are given in table 13.
4.2.1.2 **Integration over horizontal planes**

Consider an equivalent circle (cell) of radius \( a \) on the horizontal plane of interest, and having the center at the intersection of the vertical line passing through the point of application of the point load, and the horizontal plane (Fig. 11). The circle has the same area as the true cell, that is, \( \pi a^2 - b^2 \). The integral over this equivalent area should closely approximate the integral over the cell with more complicated geometry.

To evaluate the integrals over the horizontal cell, we can use the results already available for the vertical walls. For this purpose, we erect a cylinder over the cell at elevation \( m+1 \) with material properties that are identical to those of the \( m^{\text{th}} \) layer (Fig. 12), and cut a narrow slit as shown. The slit defines two vertical walls \( S_3^1, S_3^2 \) forming an angle \( \delta = 2\pi - \alpha \). From formula (152), we have

\[
\iiint_{S_3^1 + S_3^2} \mathbf{U}^T \mathbf{T}^* \, d\mathbf{S} = -J + \mathbf{V}^T \begin{bmatrix} \mathbf{F}^*_{-m} \\ \mathbf{F}^*_{-m+1} \end{bmatrix} \tag{154}
\]

where \( J \) represents the contribution of the two lids of the cylinder. In the limit when \( \alpha \to 2\pi \), \( \delta \to 0 \), and \( S_3^1 \) and \( S_3^2 \) coalesce. Since they have opposing normals, their contributions to the integral are equal and opposite in sign, and the lefthand side vanishes. Hence

\[
J = \mathbf{V}^T \begin{bmatrix} \mathbf{F}^*_{-m} \\ \mathbf{F}^*_{-m+1} \end{bmatrix} = (\mathbf{U}^T_{-m} \mathbf{U}^T_{-m+1}) \begin{bmatrix} \mathbf{F}^*_{-m} \\ \mathbf{F}^*_{-m+1} \end{bmatrix} \tag{155}
\]

Alternatively, the left-hand side can be written as

\[
J = \int_{S_2^1} \mathbf{U}^T_{-m} \mathbf{T}^*_{-m} \, d\mathbf{S} - \int_{S_2^2} \mathbf{U}^T_{-m+1} \mathbf{T}^*_{-m+1} \, d\mathbf{S} \tag{156}
\]

(ie the minus sign in the second integral is the result of integrating over the bottom lid with negative normal). Requiring (153) and (155) to be equal for arbitrary (real displacements \( \mathbf{U}_{-m}, \mathbf{U}_{-m+1} \)) (assumed to be uniform over \( S_2^1, S_2^2 \) such that they can be factored out of the integrals), we obtain finally the virtual forces:
Bottom of Excavation

Figure 11
\[
\begin{align*}
\int_{S_2}^* T_m^* dS &= F_m^* \\
\int_{S_2}^* T_{m+1}^* dS &= -F_{m+1}^*
\end{align*}
\tag{157a}
\]

It should be noted that equations (157) imply that the virtual load is not within the \( m \)th layer; loads placed at the \( m, m+1 \) interfaces are eluded as shown in Fig. 12. On the other hand, when \( \alpha = 2\pi \), we have from eqs. (123):

\[
Z_x = \pi \begin{bmatrix} 1 & 1 & \cdots \\ \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots \end{bmatrix}
\tag{150a}
\]

\[
Z_y = \pi \begin{bmatrix} \vdots & \vdots & \cdots \\ 1 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}
\tag{158b}
\]

\[
Z_z = \pi \begin{bmatrix} \vdots & \vdots & \cdots \\ \vdots & \vdots & 2 \end{bmatrix}
\tag{158c}
\]

To obtain then the consistent virtual tractions, we evaluate eq.(153) with the \( Z \) matrices above, using for this purpose the formulae in Table 13. The results of this operation are given in Table 14. It should be noted that the point load is not in the excavated soil, but above or below it, respectively. Alternatively, the consistent virtual forces contributed by the horizontal cells can be obtained directly from the expressions for the consistent horizontal tractions derived earlier (Equations 68a,b and Table 9).
For example, for a horizontal \( x \) load,

\[
\begin{align*}
t_z &= \bar{t}_z \cos \theta \\
t_z &= \bar{t}_z \sin \theta \\
t_x &= \bar{t}_z \cos^2 \theta + \bar{t}_z \sin^2 \theta
\end{align*}
\]

\[
f_x = \int_0^a t_x \rho \, d\rho \, d\theta = \pi \int_0^a (\bar{t}_z + \bar{t}_z) \rho \, d\rho
\]

and from table 9, we can substitute the expressions for \( \bar{t}_z \) and \( \bar{t}_z \) required for the above integral. Inspection of the resulting expression (see table 9) reveals the following integrals: (refer also to tables 7, 8, 11):

\[
\int_0^a \bar{u}_j \rho \, d\rho = \bar{y}_j
\]

\[
\int_0^a (\bar{v}_j \rho + \bar{w}_j) d\rho = \int_0^a \frac{d}{d\rho} (\rho \bar{w}_j) d\rho = \rho \bar{w}_j \bigg|_0^a = \rho \bar{w}_j \bigg|_{\rho=a}, \quad j = \rho, \theta
\]

\[
= -a \left[ \frac{d}{d\rho} \bar{u}_\rho + \frac{1}{\rho} (\bar{u}_\rho - \bar{u}_\theta) \right] \bigg|_{\rho=a} \quad \text{for } j = \rho
\]

\[
= a \left[ \frac{d}{d\rho} \bar{u}_\theta + \frac{1}{\rho} (\bar{u}_\rho - \bar{u}_\theta) \right] \bigg|_{\rho=a} \quad \text{for } j = \theta
\]

\[
\int_0^a (\rho \bar{w}_z - \bar{u}_z) d\phi = \int_0^a (\rho \frac{d\bar{u}_z}{d\rho} - \bar{u}_z) d\phi
\]

\[
= -\rho \bar{u}_z \bigg|_0^a = -\rho \bar{u}_z \bigg|_{\rho=a}
\]

Considering the above integrals and the equations in table 4, one obtains for \( f_x \) the same result as in table 14.
At the intersection of a vertical and horizontal wall, the contribution of the horizontal cell adjacent to the edge should not be considered, because it has been accounted for in the integration over the vertical wall (included in the J term in equation (152)). Hence, the consistent virtual forces in table 14 are only computed for the nodes not adjacent to the walls.

4.2.2 Virtual displacements x real stresses

On the one hand, these integrals are easier to evaluate than those in section 4.2.1, since the virtual displacements are better behaved ("less singular") than the virtual stresses; on the other hand, a complicating factor is the fact that the (unknown) real tractions are not defined at a point, but have an associated surface with given direction. If the boundary is continuous at the node under consideration, no difficulties arise, while if it is discontinuous, some additional manipulations are required.

4.2.2.1 Integrals over vertical planes:

With reference to Fig. 13, consider the integral over the shaded region (cell) on the exposed vertical wall for the horizontal x direction:

$$\int \left[ u_x^* t_x \right] \mathrm{dz} \mathrm{ds} = \int \left[ u_x^* \xi + u_x^* (1-\xi) \right] \left[ \xi \frac{t_x^m}{t_x} + (1-\xi) t_x^{m+1} \right] \mathrm{hd}\xi \mathrm{ds}$$

$$= \int \left\{ \begin{array}{cc}
\left[ u_x^* \right] & \xi^2 \\
& \xi(1-\xi)
\end{array} \right\} \left\{ \begin{array}{cc}
\left[ u_x^* \right] & t_x^m \\
& t_x^{m+1}
\end{array} \right\} \mathrm{hd}\xi \mathrm{ds}$$

$$= \int \left[ u_x^* \right] \frac{h}{6} \left[ \begin{array}{cc}
2 & 1 \\
1 & 2
\end{array} \right] \left[ t_x^m \right] \mathrm{ds} \quad (159)$$

But

$$u_x^* = \frac{1}{2a} \int u_x^* \mathrm{ds} = \text{average horizontal displacement} \quad (160)$$

and
Figure 13
\[
\begin{align*}
\begin{bmatrix} Q_x^m \\ Q_x^{m+1} \end{bmatrix} &= 2a \begin{bmatrix} q_x^m \\ q_x^{m+1} \end{bmatrix} = 2a \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} t_x^m \\ t_x^{m+1} \end{bmatrix} \\
\end{align*}
\]

(161)

- consistent nodal forces.

Hence

\[
\int u_x^* \ t_x \ dz \ ds = \begin{bmatrix} u_x^m & u_x^{m+1} \end{bmatrix} \begin{bmatrix} Q_x^m \\ Q_x^{m+1} \end{bmatrix}
\]

(162)

a) For a horizontal x-load:

\[
u_x^* = \cos^2 \theta \ u^*_\rho + \sin^2 \theta \ u^*_\theta
\]

and considering \(\cos^2(\theta + \pi) = \cos^2 \theta, \sin^2(\theta + \pi) = \sin^2 \theta\), we obtain

\[
u_x^* = \cos^2 \theta \ u^*_\rho + \sin^2 \theta \ u^*_\theta
\]

(163)

\[
u_x^* = \cos^2 \theta \ u^*_\rho + \sin^2 \theta \ u^*_\theta
\]

(164)

The above integrals are available in Tables 11 and 12.

b) For a horizontal y-load:

\[
u_{xy}^* = (u^*_\rho - u^*_\theta) \sin \theta \cos \theta = \frac{1}{2} (u^*_\rho - u^*_\theta) \sin 2 \theta
\]

(165)

Again, \(\sin 2(\theta + \pi) = \sin 2 \theta\), hence

\[
u_{xy}^* = \frac{1}{2} \sin 2 \theta \ \frac{1}{a} \int_0^a (u^*_\rho - u^*_\theta) d\rho
\]

(166)

\[
u_{xy}^* = \frac{1}{2} \sin 2 \theta \ (u^*_\rho - u^*_\theta)
\]
c) For a vertical z-load

\[ u_{xz}^* = \frac{u^*}{\rho} \cos \theta \quad (167) \]

and since \( \cos(\theta + \pi) = -\cos \theta \), the integrals over the two branches cancel. Hence

\[ u_{xz}^* = 0 \quad (168) \]

Repeating now the previous operations for the integrals \( \int u_{yz}^* t_y \, dz \, ds \), and \( \int u_{z}^* t_z \, dz \, ds \), we obtain the results presented in Table 6.

Consider now the situation in which the wall is discontinuous at the corner, as shown in Fig. 14. The virtual load is placed on the vertical line passing through the corner. We are interested in the integral over the shaded region A-0-B, with discontinuous (real) edge tractions \( q_x, q_y \), \( q_z \):

\[ q_x u_x^* \, ds = \int_{AO} q_x' u_x^* \, ds + \int_{OB} q_x'' u_x^* \, ds \quad (169) \]

\[ = q_x' \int_{AO} u_x \, ds + q_x'' \int_{OB} u_x \, ds \]

But from section a) of this chapter, we have the following results

\[ u_x^* = \frac{1}{2} (\cos^2 \theta_1 \frac{\bar{u}_x}{\rho} + \sin^2 \theta_1 \frac{\bar{u}_\theta}{\rho}) \]

\[ = \frac{1}{4} \left[(\frac{\bar{u}_x}{\rho} + \frac{\bar{u}_\theta}{\rho}) + \cos 2\theta_1 (\frac{\bar{u}_x}{\rho} - \frac{\bar{u}_\theta}{\rho})\right] \quad (170a) \]

\[ u_x^* = \frac{1}{2} (\cos^2 \theta_2 \frac{\bar{u}_x}{\rho} + \sin^2 \theta_2 \frac{\bar{u}_\theta}{\rho}) \]

\[ = \frac{1}{4} \left[(\frac{\bar{u}_x}{\rho} + \frac{\bar{u}_\theta}{\rho}) + \cos 2\theta_2 (\frac{\bar{u}_x}{\rho} - \frac{\bar{u}_\theta}{\rho})\right] \quad (170b) \]

With the tangential plane at an angle \( \theta = \frac{\theta_1 + \theta_2}{2} \) and the angle of change of direction, \( \alpha = \theta_1 - \theta_2 \), we can write
Figure 14
\[ \theta_1 = \theta + \frac{\alpha}{2}, \quad \theta_2 = \theta - \frac{\alpha}{2} \]

\[
\begin{align*}
\cos 2\theta_1 &= \cos 2\theta \cos \alpha - \sin 2\theta \sin \alpha \\
\cos 2\theta_2 &= \cos 2\theta \cos \alpha + \sin 2\theta \sin \alpha
\end{align*}
\]  

\[ (171) \]

Hence

\[
\int q_x u_x^* \, ds = \frac{a}{4} (q_x' + q_x'') \left[ (u_p^* + u_\theta^*) + \cos 2\theta \cos \alpha (u_p^* - u_\theta^*) \right] \\
- \frac{a}{4} (q_x' - q_x')(u_p^* - u_\theta^*) \sin 2\theta \sin \alpha
\]

\[ (172) \]

But \( a(q_x' + q_x') \cdot Q_x \) = total horizontal \( x \)-load on the node. We can write then

\[
\int q_x u_x^* \, ds = Q_x \frac{1}{2} \left[ \cos^2 \theta \tilde{u}_p^* + \sin^2 \theta \tilde{u}_\theta^* \right] - \\
\frac{a}{2} (u_p^* - u_\theta^*)[(q_x' + q_x'')\cos 2\theta \sin \frac{\alpha}{2} + (q_x' - q_x')\sin 2\theta \cos \frac{\alpha}{2}] \sin \frac{\alpha}{2}
\]

\[ (173) \]

For \( \alpha = 0 \), (continuous slope) \( \sin \frac{\alpha}{2} = 0 \) and \( q_x' = q_x'' \), so that the last term cancels. On the other hand, in the neighborhood of the (horizontal) point load \( u_p^* \neq u_\theta^* \); it follows that the last term can be neglected, even when \( \alpha \neq 0 \). Hence

\[
\int q_x u_x^* \, ds = Q_x \frac{1}{2} \left[ \cos^2 \theta \tilde{u}_p^* + \sin^2 \theta \tilde{u}_\theta^* \right]
\]

\[ (174) \]

which indicates that in the case of a discontinuous slope, the displacement integral must be evaluated in the tangential plane. Similar conclusions apply to the case of horizontal \( y \)-loads and vertical \( z \)-loads.
4.2.2.2 Integrals over horizontal cells

Again, we resort in this case to integration over circles rather than squares, since the results are considerably easier to obtain. The virtual work is (assuming the real tractions to be uniform over the cell,

\[
\iint t_x u_{xx}^* \, dS = t_x \iint \left( \frac{u_{\rho}^*}{\rho} \cos^2 \theta + \frac{u_{\theta}^*}{\rho} \sin^2 \theta \right) \rho \, d\theta \, d\rho
\]

\[
= Q_x \frac{1}{a^2} \int_0^a \left( \frac{u_{\rho}^*}{\rho} + \frac{u_{\theta}^*}{\rho} \right) \rho \, d\rho
\]

\[
= Q_x \frac{1}{a^2} \left( \bar{y}_\rho + \bar{y}_\theta \right)
\]

(175)

\[
\iint t_x u_{xy}^* \, dS = t_x \iint \left( u_{\rho}^* - \bar{u}_{\theta}^* \right) \sin \theta \cos \theta \rho \, d\theta \, d\rho
\]

\[
= 0 \quad \text{(since } \int_0^{2\pi} \sin 2\theta \, d\theta = 0) \]

(176)

\[
\iint t_x u_{xz}^* \, dS = t_x \iint \frac{u_{\rho}^*}{\rho} \cos \theta \, d\theta \, d\rho
\]

\[
= 0
\]

(177)

Similarly,

\[
\iint t_y u_{yy}^* \, dS = Q_y \frac{1}{a} \left( \bar{y}_\rho + \bar{y}_\theta \right)
\]

(178)

\[
\iint t_z u_{zz}^* \, dS = Q_z \frac{2}{a^2} \bar{y}_z
\]

(179)

and the other terms are zero. These results are summarized in Table 16.

For nodes that lie on the edges or corners defined by the intersection of horizontal and vertical planes, we can take weighted averages of the expressions presented in parts a) and b) of Table 15. The final
results should not be sensitive to this averaging, because to a first approximation we have that \( \bar{y} \approx \frac{1}{2} a \bar{x}_\rho \), \( \bar{y}_\theta \approx \frac{1}{2} a \bar{x}_\theta \), and \( \bar{x}_\rho \approx x_\theta \). Hence, parts a, b of table 16 give essentially the same results.
Positive sign:

Exposed surface:
upper edge

m

soil

m+1

lower edge

Negative sign:

Exposed surface

soil

m

soil

Excavated soil
(\( m^{th} \) layer)

m+1
5. Implementation of Method

The large number of equations presented up to this point may prove confusing to the uninitiated reader, who may not be able to distinguish the forest from so many trees. For this reason, an attempt will be made here to summarize the important results.

The boundary integral considered in this report (refer also to equation (23)) is

\[
\iint \text{(virtual tractions)(actual displacements)} + \{u_j\} = \]

\[
= \iint \text{(virtual displacements)(actual tractions)}
\]

The term in \(u_j\) corresponds to the displacement at the point of application of the load. If the load is evaded (i.e., the integration path distorted) in such a way that the point of application of the load is within the excavated soil, then \(u_j\) must be removed from the above equation (and from eq. (23)).

Example 1

As a first example, consider the case of a loaded small rectangular area on the surface of an elastic stratum (Fig. 16). The surface is discretized here with only two nodes to keep the presentation simple.

One alternative is to integrate above the nodes where the concentrated virtual forces will be applied (Fig. 15a). This implies that the concentrated loads are enclosed by the integration path, and so the term in \(u_j\) (the actual nodal displacements) must be kept. Since the virtual tractions (regular and singular parts!) are zero in this case, then the first integral drops out; on the other hand, the integral on the right-hand side gives the average virtual displacements at nodes I and II times the loads applied there. The boundary integral equation is then used six times in connection with the cases of two horizontal (x and y) and one vertical point loads applied successively on nodes I and II.
Figure 16
In the above table, the diagonal terms come from table 16, section b, with \( m=1 \) (first interface) and dummy index \( n=1 \) (loads are placed on interface I). The off-diagonal terms, on the other hand, correspond to the Green functions themselves (equations 25), computed at \( \rho-b \) (distance between node I and II). The negative signs (and some of the zeroes) are the result of the conversions from cylindrical coordinates to Cartesian coordinates, with \( \theta = 0 \) when the loads are placed on node I, and \( \theta = \pi \) when the loads are placed on node II. Notice that the above flexibility matrix is symmetric, since \( \overline{u}^I_\rho (\mu=0) = - \overline{u}^I_z (\mu=1) \) i.e., the radial displacement due to a vertical load is the negative of the vertical displacement due to a horizontal load on the same interface. Note also that \( \overline{u}^I_\rho (\mu=1) \neq \overline{u}^I_\rho (\mu=0) \) and \( \overline{u}^I_z (\mu=0) \neq \overline{u}^I_z (\mu=1) \). Combining the six equations into one matrix equation, we obtain

\[ \mathbf{U} = \mathbf{F} \mathbf{P} \]

where \( \mathbf{U} = \{U_j\} \) is the nodal displacement vector, \( \mathbf{P} \) is the load vector, and \( \mathbf{F} \) is the flexibility matrix given by the table above.
Alternatively, we can integrate underneath the virtual loads (i.e., the nodes) as shown in Fig. 15b. Since the points are not enclosed by the integration path, the term in \( u_j \) must be dropped. However, in this case the first integral will be nonzero: Table 14 gives the contribution of the regular part, while table 15 gives the singular part. Since the surface is stress-free for the virtual solution, the contribution of the regular part vanishes, and only the contribution of the singular part in table 15 need be considered. In this case, we would compute the virtual forces as per section a), with \( \lambda, G \) and \( h \) corresponding to layer 1, and \( \alpha = 2\pi \). Since there are no further overlying layers, it can be shown (eq. 95) that \( f_{xx}' = p = 1, f_{zz}' = p = 1 \). Hence, the boundary integral reduces again to \( u_j' \), which gives the displacement at the virtually loaded node in the direction of the load. Hence, we obtain the same result as before. (The right-hand side does not depend on whether the loads are eluded from above or below).

Example 2

Consider now the excavated situation shown in Fig. 17. For the sake of simplicity, we present in this case only a single plane cross-section; the lateral walls and the bottom are assumed flat.

Again, we have a choice as to the integration path to be taken. In Fig. 17a the nodes (i.e., the location of the virtual loads) are enclosed by the integration path, which includes the dotted line around infinity that does not contribute to the integral. Hence, the displacement term \( u_j \) must be kept in this case. Alternatively, in Fig. 17b the integration path excludes the point loads, and so the term in \( u_j \) would have to be dropped in this situation.

With reference to Fig. 17a, consider first the nodes on the vertical wall at the left (1 → 5). For these nodes, we have from Fig. 10 and eqs. (120)

\[
\begin{align*}
\theta_1 &= -\frac{\pi}{2} \\
\theta_2 &= \frac{\pi}{2}
\end{align*}
\]

\[\alpha = \pi\]

\[\bar{\theta} = 0\]
a) Integration around excavated soil

b) Integration on exposed walls

Figure 17
The $Z$ matrices (eqs. 123) are then (the negative sign follows from the fact that the positive normal of the exposed wall points to the interior of the prism defined in Figs. 10 and 7a, which corresponds to excavated soil, see footnote in table 13).

\[
Z_x = - \left\{ \begin{array}{ccc}
\frac{1}{2} \pi & \frac{1}{2} \pi & . \\
. & . & . \\
. & . & 2 \\
\end{array} \right\}
\]

\[
Z_y = - \left\{ \begin{array}{ccc}
. & . & . \\
\frac{1}{2} \pi & \frac{1}{2} \pi & . \\
. & . & . \\
\end{array} \right\}
\]

\[
Z_z = - \left\{ \begin{array}{ccc}
2 & . & . \\
. & . & . \\
. & . & \pi \\
\end{array} \right\}
\]

From table 13, we obtain the consistent virtual loads

\[
F^*_{\omega m} = Z \begin{pmatrix} f^m_x \\ f^m_y \\ f^m_z \end{pmatrix} = Z_{x,y,z} \begin{pmatrix} f^m_\rho \\ f^m_\theta \\ f^m_\zeta \end{pmatrix}
\]

and in view of the $Z$-matrices above, we obtain for the upper edge of the $m$-th layer ($\mu=1$)

\[
f^m_{xx} = - \frac{\pi}{2} \left\{ -a(i^m_{\rho \rho} + i^m_{\theta \theta}) + \frac{1}{2} a (\overline{v}^m_{\rho} + \overline{v}^{m+1}_{\rho}) + \frac{G}{h} \left[ (\overline{y}^m_{\rho} + \overline{y}^{m+1}_{\rho}) - (\overline{y}^{m+1}_{\rho} + \overline{y}^{m+1}_{\rho}) \right] \\
- \gamma \omega^2 \frac{h}{6} \left[ 2(\overline{y}^m_{\rho} + \overline{y}^{m+1}_{\rho}) + (\overline{y}^{m+1}_{\rho} + \overline{y}^{m+1}_{\rho}) \right] \right\}
\]

\[
f^m_{yx} = 0
\]
\[ f_{zx}^m = -2 \left\{ -a \frac{q_{\rho z}}{\rho} + \frac{1}{2} \lambda \left[ (a \frac{u_{\rho}^m - u_\theta^m}{\rho}) + (a \frac{u_{\rho}^{m+1} - u_\theta^{m+1}}{\rho}) \right] + \left( \frac{\lambda + 2G}{h} - \gamma \omega \frac{2}{3} h \right) y_z^m - \left( \frac{\lambda + 2G}{h} + \gamma \omega \frac{2}{6} h \right) y_z^{m+1} \right\} \]

(and similar expressions for the lower edge).

\[ f_{xy}^m = 0 \]

\[ f_{yy}^m = \text{same as } f_{xx}^m \]

\[ f_{zy}^m = 0 \).

Also, for the vertical load \((\mu=0!)\)

\[ f_{xz}^m = -2 \left\{ -a \frac{q_{\rho z}}{\rho} + \frac{1}{2} G (a \frac{u_{\rho}^m - u_\omega^m}{\rho}) + \frac{1}{2} G (a \frac{u_{\rho}^{m+1} - u_\omega^{m+1}}{\rho}) + \right. \]

\[ + \left. \frac{\gamma}{h} (y_{\rho}^m - y_{\rho}^{m+1}) - \gamma \omega \frac{2}{6} h \left( 2 y_{\rho}^m + y_{\rho}^{m+1} \right) \right\} \]

\[ f_{yz}^m = 0 \).

\[ f_{zz}^m = -\pi \left\{ -a \frac{q_{\rho z}}{\rho} + \frac{\alpha \lambda}{\rho} (u_{\rho}^m + u_{\rho}^{m+1}) + \left( \frac{\lambda + 2G}{h} - \gamma \omega \frac{2}{3} h \right) y_z^m \phantom{\frac{G}{h} (y_{\rho}^m - y_{\rho}^{m+1})} \right. \]

\[ \left. - \left( \frac{\lambda + 2G}{h} + \gamma \omega \frac{2}{6} h \right) y_z^{m+1} \right\} \]

(Again, note that the quantities such as \(y_z^m\) etc. are different for horizontal and vertical loads). The functions \(q_{\rho \rho}^m\), etc. are obtained from table 4, computed at \(\rho = a\).

The above expressions must be evaluated for loads placed on nodes 1 through 5. The contribution of each layer to a given node is obtained overlapping the contributions of the layers above and below the node (node 5 would only receive the contribution from above). For example, for node 2 we would compute \(f_{xx}^2\) etc. with table 13b (lower edge!), and
the properties of layer 1, and then $f_{xx}^2$, etc., with table 13a (upper edge → equations above) and the properties of layer 2. However, comparison with table 14 shows that in this particular case ($\alpha = \pi$), the consistent virtual forces $f_{xx}, f_{yy}, f_{zz}$ are proportional to the expressions for horizontal cells. Since the regular part of the horizontal tractions is continuous across the interfaces, it follows that the consistent virtual forces will cancel for nodes 1 through 4 (but not 5!), except for the coupling terms $f_{zx}$ and $f_{xz}$. We must add now the contribution of the singularities. For example, for node 2 we would obtain from table 15, section b) (using the properties of layer 1), and table 15, section a) (using the properties of layer 2), with $\alpha = \pi$, $p = \delta_{mn}$, and considering eqs. (45) and (102)

$$\Delta f_{xx}^m = \Delta f_{yy}^m = \Delta f_{zz}^m = -\frac{1}{2} \delta_{mn} = -\frac{1}{2}$$ if $m = n$.

That is, the contribution of the singularity to the loaded ($n^{th}$ node) is equal to $-\frac{1}{2}$ the applied virtual load (nodes 1 → 4). Again, the negative sign follows from Fig. 15. For node 5, we would consider only table 15b, with the properties of the fourth layer. Since the contribution of the neighboring $5^{th}$ layer is not overlapped, we will not obtain $-1/4$ of the load, but some other result.

For the horizontal cells (nodes 6 through 9), we would use tables 14b and 15b (with $\alpha = 2\pi$) and using the properties of the $4^{th}$ layer. The vertical wall to the right (nodes 10 to 14) follows a procedure similar to the one just described for the wall to the left, except that $\bar{\theta} = \pi$, which results in some sign changes.

So far, only the virtual forces contributed by the nodes on the same vertical line or the individual nodes on the horizontal planes have been discussed. To obtain the coupling terms between the two vertical walls, we would use tables 4, 5, 6, with $\theta = 0, \nu = \pi$, and multiplying the edge tractions (evaluated at $\rho = L =$ width of excavation) times the length of the edge at nodes 10 → 14 (2a in this case). The coupling terms between the left vertical wall and the floor are obtained from table 9b, multiplying the tractions times the area of the cells ($\pi a^2$ in our case), with the tractions evaluated at $\rho = \text{distance between left vertical wall and node}$.
under consideration (6→9). The virtual loads would be placed at nodes 1→5. From the last loading case (n = 5), we would be able to derive also the coupling terms between the floor nodes themselves.

The right-hand side is simply obtained by using the averages of the displacements when the loads are on the same vertical line, and the actual displacements when they are not on this line.

In the alternative presented in Fig.17b, the integration prisms are part of the soil, so that the signs in tables 13 through 15 are to be taken as positive. For the vertical wall to the left, $\theta_1 = \pi/2$ and $\theta_2 = 3\pi/2 \rightarrow \alpha = \pi$, $\bar{\theta} = \pi$. The loads are not enclosed by the integration path, so the term in $u_j$ must be dropped. Note that node 5 (and node 10) will receive in this case the contribution of a full horizontal cell. Also, the integration on the horizontal cells is performed with the aid of tables 14a, 15a, using the properties of the 5th layer underneath the floor.

A careful analysis of the procedures just described reveals that both alternatives (Fig.17a,b) yield the same results.
Appendix I

Deletion of singular term of $H_1^{(2)}(z)$:

Substitution of equations (30) into equations (29b), (29c), and (26c) yields the following contribution of the singular term of the first Hankel function to the Green's functions (horizontal load case):

$$U_{\rho} = \frac{1}{\rho} \frac{2i}{2\pi} \left( \phi_x \kappa_x^{-2} \phi_x^T - \phi_y \kappa_y^{-2} \phi_y^T \right)$$

$$= \frac{2i}{\rho \pi} \left( -C_x^{-1} + C_y^{-1} \right) \quad \text{(from eqs. E196).}$$

and since $C_x \equiv C_y$, this term vanishes. Similarly, for $U_z$ we have

$$U_z + \frac{2i}{\rho \pi} \phi_z \kappa^{-1} \phi_x^T = 0 \quad \text{(by eqs. E196).}$$

Thus, the singular term of $H_1$ does not contribute to the Green's functions. A similar demonstration holds for the vertical load case. Since the above observation also carries over to the derivatives and integrals of the Green's functions, it is convenient to redefine the Hankel functions as follows (see also table 10 for definition of terms):

$$H_0 = \frac{1}{4\pi} H_0^{(2)}(z)$$

$$= -\frac{1}{2\pi} \left\{ c_0 + \sum_{j=1}^{\infty} b_j^2 c_{j+1} \right\}$$

$$H_1 = \frac{1}{4\pi} \left[ H_1^{(2)}(z) - \frac{2i}{\pi} \frac{1}{z} \right] = \frac{1}{4\pi} H_1^{(2)}(z) - \frac{1}{2\pi z}$$

$$= -\frac{z}{4\pi} \left\{ -c_0 + \sum_{j=1}^{\infty} \frac{b_j^2}{i+1} \left( c_{j+1} - \frac{1}{2j+2} \right) \right\}$$

The derivatives of these functions are (from the recurrence relations for the Hankel functions):
\[
\frac{d}{d\rho} H_0 = -k (H_1 + \frac{l}{2\pi z}) = -k H_1 - \frac{l}{2\pi \rho}
\]

\[
\frac{d}{d\rho} H_1 = \frac{1}{4i} \frac{d}{d\rho} H^{(2)}_1(z) + \frac{1}{2\pi z^2} = \frac{1}{4i} k[H^{(2)}_0(z) - \frac{1}{4i} \rho H^{(2)}_1(z) + \frac{1}{2\pi z^2}
\]

= k(H_0 - \frac{1}{z} H_1).

The additional term in the derivative for \(H_0\) comes from the deletion of the singular element of \(H^{(2)}_1\).

The Hankel functions in equations (25), (35) can then be replaced by the modified Hankel functions (in combination with the 4i factor) without affecting the results.

References


\( A_m = \frac{h}{6} \)

\[
\begin{bmatrix}
2(\lambda+2G) & \lambda+2G & \cdot & \cdot & \cdot \\
\cdot & 2G & \cdot & G & \cdot \\
\cdot & 2G & \cdot & G & \cdot \\
\lambda+2G & \cdot & 2(\lambda+2G) & \cdot & \cdot \\
\cdot & G & \cdot & 2G & \cdot \\
\cdot & G & \cdot & 2G & \cdot \\
\end{bmatrix}
\]

\( B_m = \frac{1}{2} \)

\[
\begin{bmatrix}
\cdot & \cdot & \lambda-G & \cdot & -(\lambda+G) \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \lambda+G & \cdot & -(\lambda-G) \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
-(\lambda+G) & \cdot & -(\lambda-G) & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]

\( C_m = \frac{1}{n} \)

\[
\begin{bmatrix}
G & \cdot & -G & \cdot & \cdot \\
\cdot & G & \cdot & -G & \cdot \\
\cdot & 2G & \cdot & -(\lambda+2G) & \cdot \\
-G & \cdot & G & \cdot & \cdot \\
-G & \cdot & G & \cdot & \cdot \\
\cdot & -(\lambda+2G) & \cdot & \cdot & \lambda+2G \\
\end{bmatrix}
\]

\( M_m = \frac{\gamma h}{5} \)

\[
\begin{bmatrix}
2 & \cdot & 1 & \cdot & \cdot \\
\cdot & 2 & \cdot & 1 & \cdot \\
1 & \cdot & 2 & \cdot & \cdot \\
1 & \cdot & 2 & \cdot & \cdot \\
\cdot & 1 & \cdot & 2 & \cdot \\
\cdot & \cdot & 1 & \cdot & 2 \\
\end{bmatrix}
\]

\( \lambda = \text{Lame constant} \) for soil with damping,

\( G = \text{shear modulus} \) use complex values.

\( \gamma = \text{mass density} \)

\( h = \text{layer thickness} \)

\( m = \text{layer number} \)

\( K = \{K_m\} = A k^2 + B k + G - \omega^2 N \)

\( A = \{A_m\} \)

\( B = \{B_m\} \)

\( G = \{G_m\} \)

\( N = \{N_m\} \)
\[ D_m = \frac{1}{2} \begin{bmatrix} \ldots & \lambda & \ldots & -\lambda \\ \ldots & \ldots & \ldots & \ldots \\ G & \ldots & -G & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ G & \ldots & -G & \ldots \end{bmatrix} \]

\[ E_m = \frac{h}{6} \begin{bmatrix} 2\lambda & -2\mu\lambda & \lambda & -\mu\lambda & \ldots \\ 2\mu G & -2G & \mu G & -G & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda & -\mu\lambda & 2\lambda & -2\mu\lambda & \ldots \\ \mu G & -G & 2\mu G & -2G & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \]

\[ D = \{D_m\} \]

\[ E = \{E_m\} \]

\[ \lambda = \text{Lamé constant} \]  
\[ G = \text{shear modulus} \]  
\[ h = \text{layer thickness} \]  
\[ \mu = \text{Fourier index} \]  
\[ m = \text{layer number} \]

For soil with damping, use complex values.
Table 4

Circumferential edge tractions, contributions of m-th layer

\[
\begin{align*}
\bar{Q}^\theta &= \begin{cases} 
\bar{q}_m^\theta \\
\bar{q}_{m+1}^\theta 
\end{cases}, & \bar{Q}^\rho &= \begin{cases} 
\bar{q}_m^\rho \\
\bar{q}_{m+1}^\rho 
\end{cases}, & \bar{Q}^{\rho+1} &= \begin{cases} 
\bar{q}_m^{\rho+1} \\
\bar{q}_{m+1}^{\rho+1} 
\end{cases}
\end{align*}
\]

a) upper edge:

\[
\bar{q}_m^\rho = \left(\lambda + 2G\right) \frac{h}{6} \left(2 \frac{d}{d\rho} \bar{u}_\rho^m + \frac{d}{d\rho} \bar{u}_\rho^{m+1}\right) + \frac{\lambda}{2} \left(\bar{u}_z^m - \bar{u}_z^{m+1}\right) + \\
+ \frac{\lambda}{\rho} \frac{h}{6} \left[(2\bar{u}_\rho^m + \bar{u}_\rho^{m+1}) - \mu (2\bar{u}_\theta^m + \bar{u}_\theta^{m+1})\right]
\]

\[
\bar{q}_m^{\rho+1} = G \frac{h}{6} \left(2 \frac{d}{d\rho} \bar{u}_\rho^m + \frac{d}{d\rho} \bar{u}_\rho^{m+1}\right) + G \frac{h}{6} \left[\mu (2\bar{u}_\rho^m + \bar{u}_\rho^{m+1}) - (2\bar{u}_\theta^m + \bar{u}_\theta^{m+1})\right]
\]

\[
\bar{q}_m^{\rho+1} = G \frac{h}{6} \left(2 \frac{d}{d\rho} \bar{u}_\rho^m + \frac{d}{d\rho} \bar{u}_\rho^{m+1}\right) + \frac{G}{2} (\bar{u}_\rho^m - \bar{u}_\rho^{m+1})
\]

b) lower edge:

\[
\bar{q}_m^{\rho+1} = \left(\lambda + 2G\right) \frac{h}{6} \left(2 \frac{d}{d\rho} \bar{u}_\rho^m + \frac{d}{d\rho} \bar{u}_\rho^{m+1}\right) + \frac{\lambda}{2} \left(\bar{u}_z^m - \bar{u}_z^{m+1}\right) + \\
+ \frac{\lambda}{\rho} \frac{h}{6} \left[(\bar{u}_\rho^m + 2\bar{u}_\rho^{m+1}) - \mu (\bar{u}_\theta^m + 2\bar{u}_\theta^{m+1})\right]
\]

\[
\bar{q}_m^{\rho+1} = G \frac{h}{6} \left(2 \frac{d}{d\rho} \bar{u}_\rho^m + \frac{d}{d\rho} \bar{u}_\rho^{m+1}\right) + G \frac{h}{6} \left[\mu (\bar{u}_\rho^m + 2\bar{u}_\rho^{m+1}) - (\bar{u}_\theta^m + 2\bar{u}_\theta^{m+1})\right]
\]

\[
\bar{q}_m^{\rho+1} = G \frac{h}{6} \left(2 \frac{d}{d\rho} \bar{u}_\rho^m + \frac{d}{d\rho} \bar{u}_\rho^{m+1}\right) + \frac{G}{2} (\bar{u}_\rho^m - \bar{u}_\rho^{m+1})
\]

Note: For derivatives of displacements, see Table 8.
Meridional edge tractions, contributions of $m^{th}$ layer

\[ \overline{Q}_0 = \begin{cases} \overline{Q}_m \end{cases}, \quad \overline{Q}_m = \begin{cases} -q_{\theta\rho}^m \\ -q_{\theta\phi}^m \\ -q_{\theta z}^m \end{cases}, \quad \overline{Q}_{m+1} = \begin{cases} -q_{\theta\rho}^{m+1} \\ -q_{\theta\phi}^{m+1} \\ -q_{\theta z}^{m+1} \end{cases} \]

a) upper edge

\[ -q_{\theta\rho}^m = G \frac{h}{\rho} \left( 2 \frac{d}{d\rho} u_{\theta}^m + \frac{d}{d\rho} u_{\theta}^{m+1} \right) + \frac{G}{\rho} \frac{h}{6} \left[ \mu (2u_{\theta}^m + u_{\theta}^{m+1}) - (2u_{\theta}^m + u_{\theta}^{m+1}) \right] \]

\[ -q_{\theta\phi}^m = \lambda \frac{h}{\rho} \left( 2 \frac{d}{d\rho} u_{\phi}^m + \frac{d}{d\rho} u_{\phi}^{m+1} \right) + \frac{\lambda}{\rho} (u_{\phi}^m - u_{\phi}^{m+1}) + \]

\[ + \frac{\lambda + 2G}{\rho} \frac{h}{6} \left[ (2u_{\phi}^m + u_{\phi}^{m+1}) - \mu (2u_{\theta}^m + u_{\theta}^{m+1}) \right] \]

\[ -q_{\theta z}^m = \frac{G}{2} (u_{\theta}^m - u_{\theta}^{m+1}) + \mu \frac{G}{\rho} \frac{h}{6} (2u_{z}^m + u_{z}^{m+1}) \]

b) lower edge

\[ -q_{\theta\rho}^{m+1} = G \frac{h}{\rho} \left( \frac{d}{d\rho} u_{\theta}^m + 2 \frac{d}{d\rho} u_{\theta}^{m+1} \right) + \frac{G}{\rho} \frac{h}{6} \left[ \mu (u_{\theta}^m + 2u_{\theta}^{m+1}) - (u_{\theta}^m + 2u_{\theta}^{m+1}) \right] \]

\[ -q_{\theta\phi}^{m+1} = \lambda \frac{h}{\rho} \left( \frac{d}{d\rho} u_{\phi}^m + 2 \frac{d}{d\rho} u_{\phi}^{m+1} \right) + \frac{\lambda}{\rho} (u_{\phi}^m - u_{\phi}^{m+1}) + \]

\[ + \frac{\lambda + 2G}{\rho} \frac{h}{6} \left[ (u_{\phi}^m + 2u_{\phi}^{m+1}) - \mu (u_{\theta}^m + 2u_{\theta}^{m+1}) \right] \]

\[ -q_{\theta z}^{m+1} = \frac{G}{2} (u_{\theta}^m - u_{\theta}^{m+1}) + \mu \frac{G}{\rho} \frac{h}{6} (u_{z}^m + 2u_{z}^{m+1}) \]

Note: for derivatives of displacements, see Table 8.
Table 6

a) **Edge tractions in arbitrary plane (Fig. 6)** (refer to eqs. 24 for c, s)

\[
Q^\nu_m = \bar{T}^\mu_m \bar{Q}^\rho_m \cos \nu + \bar{T}^\theta_m \bar{Q}^\theta_m \sin \nu
\]

\[
q^m_{\nu\rho} = \bar{q}^m_{\rho\rho} c \cos \nu + \bar{q}^m_{\rho\theta} s \sin \nu
\]

\[
q^m_{\nu\theta} = \bar{q}^m_{\rho\theta} s \cos \nu + \bar{q}^m_{\theta\theta} c \sin \nu
\]

\[
q^m_{\nu z} = \bar{q}^m_{\rho z} c \cos \nu + \bar{q}^m_{\theta z} s \sin \nu
\]

b) **Cartesian components of edge tractions (forces per unit edge length)**

\[
q^m_{\nu x} = q^m_{\nu \rho} \cos \theta - q^m_{\nu \theta} \sin \theta
\]

\[
q^m_{\nu y} = q^m_{\nu \rho} \sin \theta + q^m_{\nu \theta} \cos \theta
\]

\[
q^m_{\nu z} = q^m_{\nu z}
\]
\begin{align*}
\overline{v}_\rho^m &= \frac{p}{4\bar{t}} \sum_{\ell=1}^{2N} \phi_x^{n_\ell} \alpha_{n_\ell} k_\ell^R \frac{d}{d\rho} H^{(2)}_{(k_\ell, \rho)} \\
\overline{v}_\theta^m &= \frac{p}{4\bar{t}} \sum_{\ell=1}^{N} \phi_y^{n_\ell} \phi_y^{n_\ell} k_\ell^L \frac{d}{d\rho} H^{(2)}_{(k_\ell, \rho)} \\
\overline{v}_z^m &= -\frac{p}{4\bar{t}} \sum_{\ell=1}^{2N} \phi_z^{n_\ell} \alpha_{n_\ell} (k_\ell^R)^2 H^{(2)}_{(k_\ell, \rho)} \\
\overline{w}_\rho^m &= \frac{p}{4\bar{t}} \sum_{\ell=1}^{2N} \phi_x^{n_\ell} \alpha_{n_\ell} k_\ell^R H^{(2)}_{(k_\ell, \rho)} \\
\overline{w}_\theta^m &= \frac{p}{4\bar{t}} \sum_{\ell=1}^{N} \phi_y^{n_\ell} \phi_y^{n_\ell} k_\ell^L H^{(2)}_{(k_\ell, \rho)} \\
\overline{w}_z^m &= \frac{p}{4\bar{t}} \sum_{\ell=1}^{2N} \phi_z^{n_\ell} \alpha_{n_\ell} \frac{d}{d\rho} H^{(2)}_{(k_\ell, \rho)} \\
\alpha_{n_\ell} &= \begin{cases} 
\phi_x^{n_\ell} & \text{for horizontal load (} \mu = 1 \text{)} \\
\phi_z^{n_\ell} & \text{for vertical load (} \mu = 0 \text{)}
\end{cases}
\end{align*}

Note: the index \( n \) on the left-hand side has been left implicit for notational convenience (i.e., \( \overline{v}_\rho^m \equiv \overline{v}^{mn}_\rho \), etc.).
Table 8

Derivatives of Green's functions

a) Horizontal point load ($\mu = 1$)

\[
\frac{d}{d\rho} \bar{u}_\rho^m = - \bar{w}_\rho^m - \frac{1}{\rho} \left( \bar{u}_\rho^m - \bar{u}_\theta^m \right)
\]

\[
\frac{d}{d\rho} \bar{u}_\theta^m = - \bar{w}_\theta^m + \frac{1}{\rho} \left( \bar{u}_\rho^m - \bar{u}_\theta^m \right)
\]

\[
\frac{d}{d\rho} \bar{u}_z^m = - \bar{w}_z^m
\]

b) Vertical point load ($\mu = 0$)

\[
\frac{d}{d\rho} \bar{u}_\rho^m = - \frac{d}{d\rho} \bar{u}_z^m \quad (\mu = 1) \quad \text{(horizontal point load!)}
\]

\[
\frac{d}{d\rho} \bar{u}_\theta^m = 0
\]

\[
\frac{d}{d\rho} \bar{u}_z^m = - \bar{w}_z^m
\]

Note that $\bar{w}_z^m \ (\mu = 1) \neq \bar{w}_z^m \ (\mu = 0)$
Consistent interface tractions

\[ \bar{t}_m = \begin{pmatrix} t^m_{z\rho} & t^m_{z\theta} & t^m_{zz} \end{pmatrix}^T \]

a) upper interface: (positive normal)

\[ \bar{t}^m_{z\rho} = \frac{(\lambda + 2G)h}{6} (2v^m_\rho + v^m_\rho) + \frac{Gh}{6} \left( 2w^m_\rho + w^m_\rho \right) + \frac{\lambda}{2} (w^m_z - w^m_z) - \frac{G}{2} (w^m_z + w^m_z) + \frac{G}{h} (u^m_\rho - u^m_\rho) - \gamma \omega^2 \frac{h}{6} (2u^m_\rho + u^m_\rho) \]

\[ \bar{t}^m_{z\theta} = (\mu) \frac{Gh}{6} (2v^m_\theta + v^m_\theta) + \frac{Gh}{6} \left( 2w^m_\theta + w^m_\theta \right) + \frac{\lambda}{2} (u^m_\rho - u^m_\rho) - \frac{G}{2} (u^m_z + u^m_z) + \frac{\lambda + 2G}{h} (u^m_\theta - u^m_\theta) - \gamma \omega^2 \frac{h}{6} (2u^m_\theta + u^m_\theta) \]

\[ \bar{t}^m_{zz} = \frac{Gh}{6} (2v^m_z + v^m_z) + \frac{\lambda}{2} (w^m_\rho - w^m_\rho) - \frac{G}{2} (w^m_z + w^m_z) + \frac{\lambda + 2G}{h} (u^m_z - u^m_z) - \gamma \omega^2 \frac{h}{6} (2u^m_z + u^m_z) \]

b) lower interface: (negative normal)

\[ -\bar{t}^m_{z\rho} = \frac{(\lambda + 2G)h}{6} (v^m_\rho + 2v^m_\rho) + \frac{Gh}{6} (w^m_\rho + 2w^m_\rho) + \frac{\lambda}{2} (w^m_z - w^m_z) + \frac{G}{2} (w^m_z + w^m_z) - \frac{G}{h} (u^m_\rho - u^m_\rho) - \gamma \omega^2 \frac{h}{6} (u^m_\rho + 2u^m_\rho) \]

\[ -\bar{t}^m_{z\theta} = (\mu) \frac{Gh}{6} (v^m_\theta + 2v^m_\theta) + \frac{\lambda}{2} (w^m_\rho + 2w^m_\rho) - \frac{G}{2} (u^m_\rho + u^m_\rho) - \frac{\lambda + 2G}{h} (u^m_\theta - u^m_\theta) - \gamma \omega^2 \frac{h}{6} (u^m_\theta + 2u^m_\theta) \]

\[ -\bar{t}^m_{zz} = \frac{Gh}{6} (v^m_z + 2v^m_z) + \frac{\lambda}{2} (w^m_\rho + 2w^m_\rho) + \frac{\lambda + 2G}{h} (u^m_z - u^m_z) - \gamma \omega^2 \frac{h}{6} (u^m_z + 2u^m_z) \]

Note: \( \bar{t}^m_{z\theta} = \bar{t}^m_{z\theta} = 0 \) for \( \mu = 0 \).
Table 10

Integrals of modified Hankel functions

\[ \xi = k^b \rho, \quad z = k^b a, \quad H_0 = \frac{1}{4i} H_0^{(2)}, \]

\[ H_1 = \frac{1}{4i} \left[ H_1^{(2)}(z) - \frac{2i}{\pi} \frac{1}{z} \right] \quad (*) \]

\[ b_j = \left( \frac{iz}{2} \right)^J \frac{1}{J!}, \quad c_j = c_{j-1} - \frac{1}{j}. \]

\[ \zeta_0 = 2n \frac{\pi}{2} + \gamma + i \frac{\pi}{2}, \quad \gamma = 0.5772 \ldots \quad \text{(Euler's constant)} \]

\[ g_{00} = \int_0^Z H_0 d\xi - \frac{z}{2\pi} \left\{ c_0 - 1 + \sum_{j=1}^{\infty} \frac{b_j^2}{2j+1} \left( c_j - \frac{1}{2j+1} \right) \right\} \]

\[ g_{01} = \int_0^Z \xi H_0 d\xi = -\frac{z^2}{4\pi} \left\{ c_0 - \frac{1}{2} + \sum_{j=1}^{\infty} \frac{b_j^2}{j+1} \left( c_j - \frac{1}{2j+2} \right) \right\} = z H_1 \]

\[ g_{1,-1} = \int_0^Z \xi H_1 d\xi = -\frac{z^3}{4\pi} \left\{ c_0 - \frac{3}{2} + \sum_{j=1}^{\infty} \frac{b_j^2}{(j+1)(2j+1)} \left( c_j - \frac{1}{2j+1} - \frac{1}{2j+2} \right) \right\} \]

\[ g_{10} = \int_0^Z H_1 d\xi = \frac{1}{2\pi} \sum_{j=1}^{\infty} b_j^2 c_j = -\frac{c_0}{2\pi} + H_0 \]

\[ g_{11} = \int_0^Z \xi H_1 d\xi = -\frac{z^3}{4\pi} \left\{ \frac{3}{2} \left( c_0 - \frac{5}{6} \right) + \sum_{j=1}^{\infty} \frac{b_j^2}{(j+1)(2j+1)} \left( c_j - \frac{1}{2j+2} - \frac{1}{2j+3} \right) \right\} \]

(*) The singular term of \( H_1^{(2)} \) (i.e. \( \frac{2i}{\pi} \frac{1}{z} \)) is excluded in the definition of \( H_1 \), since it does not contribute to the Green functions (because of orthogonality (see Appendix 1)).
Table II

Definitions

\[ \bar{U}_j = \{ \bar{u}_j^m \}, \quad j = \rho, \theta, z \]

\[ \bar{x}_j^m = \int_0^a \bar{u}_j \, d\rho = a \bar{u}_j^m \quad (\bar{u}_j^m = \text{average displacement}) \]

\[ \bar{y}_j^m = \int_0^a \bar{u}_j^m \rho \, d\rho \quad \text{(first moment)} \]

\[ G_{ij} = \text{diag} \{ g_{ij}^\ell \}, \quad \ell = \text{mode number (from table 10)} \]

\[ G_{ij}^R = \text{as above, computed with Rayleigh modes} \]

\[ G_{ij}^L = \text{as above, computed with Love modes.} \]

Note: While the integrals in table II are given in matrix form (for notational convenience), they should be evaluated as a summation. For example, the \( m^{th} \) element of the first integral \( \int \bar{U}_\rho \, d\rho \) can be written in full as \( (n \rightarrow \text{layer at which load is applied}): \)

\[ \bar{x}_\rho = \int_0^a \bar{u}_\rho \, d\rho = p \left\{ \sum_{\ell=1}^{2N} \phi_{\rho \ell}^R \phi_{\eta \ell}^R (g_{\rho \rho}^{R\ell} - a_{1,1}^{R\ell})/k_{\rho \ell}^R + \right. \]

\[ \left. + \sum_{\ell=1}^{N} \phi_{\rho \ell}^L \phi_{\eta \ell}^L g_{1,1}^{L\ell} / k_{\rho \ell}^L \right\} \]
Table 12

Integrals of Green's functions (see definitions in table 11)

a) Horizontal load (μ=1)

\[
\int_0^a U_\rho \, d\phi = p \left\{ \phi_x (G_{oo}^R - G_{11}^R) K_{R}^{-1} \phi_x^T + \phi_y G_{11}^{L} K_{L}^{-1} \phi_y \right\}
\]

\[
\int_0^a U_\theta \, d\phi = p \left\{ \phi_y (G_{oo}^L - G_{11}^L) K_{L}^{-1} \phi_y^T + \phi_x G_{11}^{R} K_{R}^{-1} \phi_x \right\}
\]

\[
\int_0^a U_z \, d\phi = -p \phi_z G_{11}^{R} K_{R}^{-1} \phi_x^T
\]

\[
\int_0^a \rho U_\rho \, d\phi = p \left\{ \phi_x (G_{oo}^R - G_{10}^R) K_{R}^{-2} \phi_x^T + \phi_y G_{10}^{R} K_{L}^{-2} \phi_y \right\}
\]

\[
\int_0^a \rho U_\theta \, d\phi = p \left\{ \phi_y (G_{oo}^L - G_{10}^L) K_{L}^{-2} \phi_y^T + \phi_x G_{10}^{R} K_{R}^{-2} \phi_x \right\}
\]

\[
\int_0^a \rho U_z \, d\phi = -p \phi_z G_{10}^{R} K_{R}^{-2} \phi_x^T
\]

b) Vertical load (μ=0) (U_\rho = 0)

\[
\int_0^a U_\rho \, d\phi = p \phi_x K_{R}^{-1} G_{10}^{R} \phi_z
\]

\[
\int_0^a U_z \, d\phi = p \phi_z G_{oo}^{R} K_{R}^{-1} \phi_z
\]

\[
\int_0^a \rho U_\rho \, d\phi = p \phi_x K_{R}^{-2} G_{11}^{R} \phi_z
\]

\[
\int_0^a \rho U_z \, d\phi = p \phi_x G_{01}^{R} K_{R}^{-2} \phi_z
\]
Table 13

Consistent virtual forces contributed by vertical walls
(point load in same vertical line, see figs. 8, 9, 10, 13 and 14)

Equation (159) can be written as (tilde \& cartesian coordinates)

\[
\mathbf{F}_m^* = Z \mathbf{F}_m^* , \quad \mathbf{F}_{m+1}^* = Z \mathbf{F}_{m+1}^* , \quad \mathbf{F}_m^* = \begin{bmatrix} f_{m}^* \\ f_{m1}^* \\ f_{m2}^* \end{bmatrix} \quad \text{etc.}
\]

All functions below must be evaluated at \( \rho = a \). Refer also to table 4, and to tables 10, 11 and 12 and eqs. 123.

a) upper edge:

\[
f_{m}^\rho = -a \frac{m}{\rho} - \frac{aG}{Z} (u_m^m + u_m^m) - \frac{G}{Z} (x_m^m + \chi_m^m + \chi_m^m) - \frac{G}{h} (y_m^m - \gamma_m^m) - \frac{2h}{6} (y_m^m + \gamma_m^m)
\]

\[
f_{m}^\theta = -a \frac{m}{\rho} + \frac{\mu G}{Z} (x_m^m + \chi_m^m + \chi_m^m) + \frac{G}{h} (y_m^m - \gamma_m^m) - \frac{2h}{6} (2y_m^m + \gamma_m^m)
\]

\[
f_{m}^z = -a \frac{m}{\rho} + \frac{\lambda}{Z} (u_m^m + u_m^m) - \frac{\mu \lambda}{Z} (x_m^m + \chi_m^m + \chi_m^m) + \frac{\lambda + 2G}{h} (y_m^m - \gamma_m^m) - \frac{2h}{6} (2y_m^m + \gamma_m^m)
\]

b) lower edge:

\[
f_{m+1}^\rho = -a \frac{m+1}{\rho} - \frac{aG}{Z} (u_m^m + u_m^m) + \frac{G}{Z} (x_m^m + \chi_m^m + \chi_m^m) - \frac{G}{h} (y_m^m - \gamma_m^m) - \frac{2h}{6} (y_m^m + 2y_m^m)
\]

\[
f_{m+1}^\theta = -a \frac{m+1}{\rho} + \frac{\mu G}{Z} (x_m^m + \chi_m^m + \chi_m^m - \frac{G}{h} (y_m^m - \gamma_m^m) - \frac{2h}{6} (2y_m^m + \gamma_m^m)
\]

\[
f_{m+1}^z = -a \frac{m+1}{\rho} - \frac{\lambda}{Z} (u_m^m + u_m^m) + \frac{\mu \lambda}{Z} (x_m^m + \chi_m^m + \chi_m^m) - \frac{\lambda + 2G}{h} (y_m^m - \gamma_m^m) - \frac{2h}{6} (2y_m^m + \gamma_m^m)
\]

Note: The above forces act on the prism shown in Fig. 10. If this prism corresponds to the excavated soil, then the sign of the above forces should be reversed to obtain the forces (equal and opposite) acting on the exposed walls. These expressions include the integration over the horizontal cells adjacent to the vertical walls, if the integration is carried around the excavated soil (see section 5, Figs. 11, 17a).
Table 14

Consistent virtual forces contributed by horizontal walls

The material properties are those of the $m^{th}$ layer. The functions are evaluated at $\rho = a$. Refer also to tables 4, 10, 11, 12.

a) Cell with positive normal ($m^{th}$ interface)

$$\frac{1}{\pi} f^m = -a(q^m_{\rho \rho} + q^m_{\rho \theta}) + \frac{aG}{h} (u^m_{\rho} + u^{m+1}_{\rho}) + \frac{C}{h} \left[ y^m_{\rho} + y^m_{\theta} - (y^{m+1}_{\rho} + y^{m+1}_{\theta}) \right]$$

$$- \gamma \omega^2 \frac{h}{6} \left[ 2(y^m_{\rho} + y^m_{\theta}) + y^{m+1}_{\rho} + y^{m+1}_{\theta} \right]$$

$$\frac{1}{\pi} f^m_{yy} = \text{same as above}$$

$$\frac{1}{2\pi} f^m_{zz} = -a(q^m_{\rho z} + \frac{a\lambda}{2} (u^m_{\rho} + u^{m+1}_{\rho}) + \frac{\lambda + 2G}{h} (y^m_{\rho} - y^{m+1}_{\rho}) + \gamma \omega^2 \frac{h}{6} (2y^m_{\rho} + y^{m+1}_{\rho})$$

b) Cell with negative normal ($m+1^{th}$ interface)

$$\frac{1}{\pi} f^{m+1} = -a(q^{m+1}_{\rho \rho} + q^{m+1}_{\rho \theta}) - \frac{aG}{h} (u^{m}_{\rho} + u^{m+1}_{\rho}) - \frac{G}{h} \left[ y^m_{\rho} + y^m_{\theta} - (y^{m+1}_{\rho} + y^{m+1}_{\theta}) \right]$$

$$- \gamma \omega^2 \frac{h}{6} \left[ y^m_{\rho} + y^m_{\theta} + 2(y^{m+1}_{\rho} + y^{m+1}_{\theta}) \right]$$

$$\frac{1}{\pi} f^{m+1}_{yy} = \text{same as above}$$

$$\frac{1}{2\pi} f^{m+1}_{zz} = -a(q^{m+1}_{\rho z} - \frac{a\lambda}{2} (u^m_{\rho} + u^{m+1}_{\rho}) - \frac{\lambda + 2G}{h} (y^m_{\rho} - y^{m+1}_{\rho}) - \gamma \omega^2 \frac{h}{6} (y^m_{\rho} + 2y^{m+1}_{\rho})$$

The subindices $x, y, z$ refer to the directions of the point loads. All coupling terms are zero (since corresponding entries in $Z$ are zero).

Note: The above expressions correspond to the contribution of the regular part only; the contribution of the singular term is presented in table 15. Also notice that $f_{zz}$ corresponds to $\mu = 0$, while $f_{xx}, f_{yy}$ are computed for $\mu = 1$. If the layer corresponds to the excavation, the signs should be reversed (Fig.15 ).
Table 15

Contribution of singularity (see Fig. 15)

a) Load evaded from below (upper edge, or cell with positive normal)

\[ \Delta f_{xx}^m = \pm \frac{1}{12} \text{ ph} \left[ (\lambda + 2\mu)(2a_{mn}^{-x} + a_{m+1,n}^{-x}) + G(2a_{mn}^{-y} + a_{m+1,n}^{-y}) \right] \frac{\alpha}{2\pi} \]

\[ \Delta f_{yy}^m = \text{ same as above} \]

\[ \Delta f_{zz}^m = \pm \frac{1}{6} \text{ ph} G (2a_{mn}^{-z} + a_{m+1,n}^{-z}) \frac{\alpha}{2\pi} \]

b) Load evaded from above (lower edge, or cell with negative normal)

\[ \Delta f_{xx}^{m+1} = \pm \frac{1}{12} \text{ ph} \left[ (\lambda + 2\mu)(a_{mn}^{-x} + 2a_{m+1,n}^{-x}) + G(a_{mn}^{-y} + a_{m+1,n}^{-y}) \right] \frac{\alpha}{2\pi} \]

\[ \Delta f_{yy}^{m+1} = \text{ same as above} \]

\[ \Delta f_{zz}^{m+1} = \pm \frac{1}{6} \text{ ph} G (a_{mn}^{-z} + 2a_{m+1,n}^{-z}) \frac{\alpha}{2\pi} \]

Note: The soil properties are those of \( m \)th layer under consideration. The quantities \( a_{mn}^{-x}, a_{mn}^{-y}, a_{mn}^{-z} \) are the elements of the inverses of \( A_x, A_y, A_z \) respectively. The sign of the above expressions (+ or -) is defined in Fig. 15.
Consistent virtual displacements

(All functions are evaluated at $\rho = a$)

a) Nodes on vertical walls:

\[
\begin{align*}
\tilde{u}_{xx}^m &= \cos^2 \theta \tilde{u}_{\rho}^m + \sin^2 \theta \tilde{u}_{\theta}^m \\
\tilde{u}_{xy}^m &= \frac{1}{2} (\tilde{u}_{\rho}^m - \tilde{u}_{\theta}^m) \sin 2\theta \\
\tilde{u}_{yz}^m &= \tilde{u}_{xy}^m \\
\tilde{u}_{yy}^m &= \sin^2 \theta \tilde{u}_{\rho}^m + \cos^2 \theta \tilde{u}_{\theta}^m \\
\tilde{u}_{zz}^m &= \tilde{u}_{z}^m \\
\tilde{u}_{xz}^m &= \tilde{u}_{yz}^m = \tilde{u}_{zx}^m = \tilde{u}_{zy}^m = 0
\end{align*}
\]

In all cases, $\tilde{u}_{ij} = \frac{1}{a} \tilde{u}_{ij}^m$, see table 11.

b) Nodes in horizontal planes: (see also table 11)

\[
\begin{align*}
\tilde{u}_{xx}^m &= \frac{1}{a} (\tilde{y}_{\rho}^m + \tilde{y}_{\theta}^m) \\
\tilde{u}_{yy}^m &= \tilde{u}_{xx}^m \\
\tilde{u}_{zz}^m &= \frac{2}{a} \tilde{y}_{z}^m
\end{align*}
\]

All coupling terms are zero.