Accurate stresses in the thin-layer method

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SUMMARY

A method is described by means of which accurate strains and stresses can be obtained for problems of wave motion in laminated media modelled with the thin layer method (TLM), a semi-discrete procedure that combines the power of finite elements with that of analytical solutions. It is shown that when the displacements in the TLM are combined with the consistent stresses at the layer interfaces, strains and stresses anywhere in the medium can be obtained with the same level of accuracy as the displacements. The proposed method thus circumvents the intrinsic problem that arises when strains are obtained via differentiation. As a bonus, it also renders the stresses continuous across layer interfaces, which is not the case when stresses are obtained via differentiation of the primary interpolation field. Copyright © 2004 John Wiley & Sons, Ltd.

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INTRODUCTION

The thin-layer method (TLM) is a powerful tool for the dynamic (and static) analysis of laminated media. It consists of a discretization in the direction of layering—commonly plane layers, but cylindrical or spherical thin layers can also be modelled—and analytical solutions for the remaining co-ordinate directions [1–6]. In analogy to the finite element method (FEM), the thin layers together with the layer interfaces and the number of sub-layers in the TLM can be interpreted as the elements, nodes, and mesh refinement in the FEM. The method is now widely used, for example in soil dynamics and soil–structure interaction, and for the non-destructive evaluation of laminated plates and shells. A brief review of the historical development of this method can be found, for example, in References [5, 6].

Like in other discrete procedures, such as the FEM, strains in the TLM are normally obtained by differentiation of the discrete displacement field within the thin layer (i.e. element), which
means that strains and, therefore, stresses are intrinsically less accurate than displacements. Thus, strains within the layers exhibit a slower convergence rate, a situation that is undesirable when accurate stresses are needed, for it imposes the use of highly refined layered models (‘meshes’). Unlike in the FEM, however, the nodal ‘forces’ in the TLM are actually consistent stresses across the interfaces, which are continuous surfaces and not isolated nodal points. Also, because these consistent interface stresses are computed directly from a dynamic equilibrium equation, namely the dynamic stiffness matrix relating interface stresses with interface displacements, the consistent interface stresses—and at first only these—converge as fast as the displacements to their corresponding values in the continuum. The question is then, How can these three consistent and relatively accurate interface stresses, namely two tangential and one normal component, be used together with the interface displacements to obtain any of the nine stress components anywhere within the thin layers with comparable accuracy? This issue is taken up in this paper, and it is shown that accurate stresses with the same convergence rate as the displacements can indeed be obtained in the TLM, an advantage that has no direct counterpart in the FEM.

To demonstrate this assertion, we begin by analysing the exact stresses within a layer of arbitrary thickness subjected to anti-plane (SH wave) motion, generate thereafter the discrete counterparts for thin layers modelled with linear, quadratic and cubic elements, evaluate the accuracy of the latter against the exact benchmark, and then generalize these results for plane strain (SV-P wave) problems.

SH WAVES IN A HOMOGENEOUS LAYER

Exact solution

Consider a homogeneous horizontal layer of arbitrary thickness $h$, shear modulus $G$, and mass density $\rho$, which is bounded by two horizontal planes, an upper and lower plane, which we identify by indices 1 and 2, respectively. We define the origin of co-ordinates at the centre of the layer, and take $z$ as the vertical co-ordinate, positive up. This layer is subjected to harmonic anti-plane shear waves of frequency $\omega$ and horizontal wavenumber $k$. As shown by Kausel and Roësset [7], the exact relationship between stresses and interface displacements at the two external interfaces (via a dynamic stiffness or impedance matrix), is of the form

$$\begin{pmatrix} \tau_1 \\ -\tau_2 \end{pmatrix} = \frac{G}{h \sinh \theta} \begin{pmatrix} \cosh \theta & -1 \\ -1 & \cosh \theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

in which

$$\theta = h \sqrt{k^2 - k_0^2}, \quad k_0 = \omega / C_S$$

with $C_S$ being the shear wave velocity. The negative sign for $\tau_2$ above is because at the lower interface, which has a downward normal, the internal stresses are opposite to the external tractions. If $k < k_0$, then $\theta$ becomes a positive imaginary number of the form

$$\theta = ih \sqrt{k_0^2 - k^2} = i\eta$$
\[
\begin{bmatrix}
\tau_1 \\ -\tau_2
\end{bmatrix} = \frac{G \eta}{h \sin \eta} \begin{bmatrix}
\cos \eta & -1 \\ -1 & \cos \eta
\end{bmatrix} \begin{bmatrix}
u_1 \\ \nu_2
\end{bmatrix}
\]

When \( \theta \) is real, we have vertically evanescent waves, while when \( \eta \) is real, the waves propagate. When the material has hysteretic damping \( \xi \), the shear wave velocity is a complex number of the form \( C_S/(1-i\xi) \), so \( k_0 \) acquires a negative imaginary part, \(-k_0^2\) has a positive imaginary part, and the square root returns a complex number in the first quadrant. Hence, \( \theta \) has positive real and imaginary parts, while \( \eta \) has a positive real part and a negative imaginary part. Either way, the waves attenuate as they propagate or evanesce.

On the other hand, the stresses at the interfaces can also obtained by differentiation of the displacement field, which corresponds to the operation

\[
\tau_{yz}(z_j) \equiv \tau_j = G \frac{\partial u}{\partial z} \bigg|_{z = z_j} = Gu_j'
\]

in which \( j = 1, 2, \ z_j = \pm h/2 \) and the prime denotes differentiation with respect to \( z \). Hence, the derivatives with respect to \( z \) at the two interfaces are

\[
u_j' = \frac{\tau_j}{G}
\]

Define the auxiliary displacement variables

\[
u_m = \frac{\nu_1 + \nu_2}{2}, \quad \Delta \nu = \frac{\nu_1 - \nu_2}{2}
\]

in terms of which

\[
u_1 = \nu_m + \Delta \nu, \quad \nu_2 = \nu_m - \Delta \nu
\]

Also

\[
u_m' = \frac{\nu_1' + \nu_2'}{2}, \quad \Delta \nu' = \frac{\nu_1' - \nu_2'}{2}
\]

in which a prime denotes differentiation with respect to \( z \). Also, define the dimensionless vertical co-ordinate \( \zeta \) as

\[
\zeta = \frac{2z}{h} = \frac{z}{a}
\]

where, for convenience, we have introduced the auxiliary parameter \( a = h/2 \). Now, from considerations of analytic continuation, it can be shown that in the absence of internal sources, the exact displacements and shearing stresses within the layer are given by the expressions

\[
u(\zeta) = \nu_m \cosh \frac{1}{2} \zeta \theta + \frac{\Delta \nu}{\cosh \frac{1}{2} \theta} \sinh \frac{1}{2} \zeta \theta \]

\[
u(\zeta) = \nu_m \cosh \frac{1}{2} \zeta \theta + \frac{\Delta \nu}{\cosh \frac{1}{2} \theta} \sinh \frac{1}{2} \zeta \theta
\]
\[ \tau_{yz}(\zeta) = \frac{2G}{h} \left\{ \frac{u_m}{\sinh \frac{1}{2} \zeta \theta} + \frac{\Delta u}{\cosh \frac{1}{2} \zeta \theta} \right\} \frac{1}{2} \theta \]

\[ = 2G \left\{ \frac{u_m'}{\cosh \frac{1}{2} \zeta \theta} + \frac{\Delta u'}{\sinh \frac{1}{2} \zeta \theta} \right\} \]  

(9b)

When \( \zeta \) is set to ±1, these expressions return the correct values for displacements and stresses at the two interfaces. Also, observe that the displacement and stress fields within the layer are composed of a symmetric part and an anti-symmetric part, which are associated with the average displacement \( u_m \), the differential displacement \( \Delta u \), and their derivatives with respect to \( z \). These expressions are exact, and have thus no limitation in either frequency or wavenumber. Thus, they can be used as benchmarks to judge the accuracy of the TLM stresses obtained by means of the procedure to be described in the ensuing.

**Discrete solution with the TLM**

In the TLM discrete formulation, displacements within a specific thin layer are expressed in terms of interpolation functions of the form

\[ u(x, z) = N^T U \]

in which \( N(z) \) contains the interpolation functions, and \( U(x, \omega) \) is the vector of displacements at the external interfaces together with an appropriate number of interior planes or nodes needed for quadratic and higher-order expansions. As before, we take the origin for the vertical co-ordinate \( z \) at the centre of the thin layer, that is, at the mid-plane. For consistency with the continuous formulation, we denote the upper and lower external interfaces by indices 1 and 2, respectively, and the internal nodes (if any) by the indices \( x, \beta, \) etc. Thus, \( U \) is of the form

\[ U(x, k, \omega) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_2 \end{bmatrix} \]

In the case of a linear element, no internal nodes exist, while for a quadratic element, there is one internal node, for a cubic element two, and so on for higher-order elements. The displacements in the discrete solution have errors of order \( h^{n+1} \) when compared to the exact solution, with \( h \) being the thickness of the layer, and \( n \) the expansion order. Thus, linear elements \( (n = 1) \) have quadratic convergence, quadratic element have cubic convergence, and so forth.

For a given frequency \( \omega \) and horizontal wavenumber \( k \), the consistent external tractions at the upper and lower interfaces of a thin layer are of the form

\[ S = K(k, \omega) U \]
in which $K$ is the dynamic stiffness matrix of the layer, and

$$S(x, k, \omega) = \begin{bmatrix} \tau_1 \\ 0 \\ 0 \\ \vdots \\ -\tau_2 \end{bmatrix}$$

with $\tau_1, \tau_2$ being the consistent internal stresses at the upper and lower external interfaces. The zeros in $S$ reflect the absence of external sources (i.e. tractions) applied at the inner nodes. Because the two consistent stresses at the outer boundaries are obtained directly from the displacements by virtue of equilibrium considerations via the dynamic stiffness matrix, and not by carrying out any differentiation on the displacements, they exhibit the same convergence rate as the displacements, i.e. $h^{n+1}$. By contrast, strains and stresses within the thin layer obtained by differentiation of the displacement field converge only as fast as $h^n$, so they are intrinsically less accurate. This suggest that after obtaining the solution of the discrete system, an auxiliary (secondary) interpolation scheme that considers both the computed interface displacements and the interface stresses could provide a more accurate displacement and stress field within the thin layer than the original expansion used for the weighted residual formulation, and this is indeed the case, as will be shown. Because the consistent stresses $\tau_j$ have convergence rates $h^{n+1}$, so do also the two derivatives of the displacements at the interfaces. Hence, we can apply Hermitian interpolation to recover a more accurate representation of the displacement field within that layer. In the ensuing, we first present the relevant equations for linear, quadratic and cubic elements, and then proceed to evaluate the accuracy of the proposed interpolation scheme.

(a) **Linear element.** The known field quantities in this case are $u_1, u_2$ and $\tau_1, \tau_2$, (i.e. $u_1', u_2'$) so a cubic interpolation scheme is in order:

$$u = A + B\zeta + C\zeta^2 + D\zeta^3$$

with $\zeta = 2z/h \equiv z/a$ being the dimensionless vertical co-ordinate such that $\zeta = \pm 1$ represent the upper and lower interfaces, respectively, and $a = h/2$. Hence, the vertical derivative is

$$au' = B + 2C\zeta + 3D\zeta^2$$

Evaluating these two expressions at $\zeta = \pm 1$, we obtain

$$\begin{bmatrix} u_1 \\ au_1' \\ u_2 \\ au_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$
whose solution is

\[
\begin{pmatrix}
A \\
B \\
C \\
D
\end{pmatrix} = \frac{1}{4}
\begin{pmatrix}
2 & -1 & 2 & 1 \\
3 & -1 & -3 & -1 \\
0 & 1 & 0 & -1 \\
-1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u'_1 \\
u_2 \\
u'_2
\end{pmatrix}
\]

Hence

\[
u = \frac{1}{4} \left( (2u_1 - au'_1 + 2u_2 + au'_2) + (3u_1 - au'_1 - 3u_2 - au'_2) \zeta \right) + a(u'_1 - u'_2)\zeta^2 + (-u_1 + au'_1 + u_2 + au'_2)\zeta^3 \]

With these definitions, the interpolated displacement together with its vertical derivative can be expressed as

\[
u = \frac{1}{2} \left( (2u_m - au'_m) + (3\Delta u - au'_m) \zeta + a\Delta u' \zeta^2 + (-\Delta u + au'_m)\zeta^3 \right)
\]

\[
au' = \frac{1}{2} \left( (3\Delta u - au'_m) + 2a\Delta u' \zeta + 3(-\Delta u + au'_m)\zeta^2 \right)
\]

Observe that these expressions satisfy the displacements and vertical derivatives at the two interfaces \( \zeta = \pm 1 \).

Now, for a linear element, the consistent stresses obtained from the dynamic stiffness matrix for a given wavenumber \( k \) and frequency \( \omega \) are

\[
\begin{pmatrix}
\tau_1 \\
-\tau_2
\end{pmatrix} = \frac{G}{h} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \left( k^2 - \omega^2/C_s^2 \right) h^2 + \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\
u_2
\end{pmatrix}
\]

\[
(10)
\]

in which \( C_s = \sqrt{G/\rho} \) = the shear wave velocity of the layer. These expressions can be written as

\[
\tau_1 = \frac{2G}{h} \left( u_m (\frac{1}{2} \theta)^2 + \Delta u [1 + \frac{1}{3} (\frac{1}{2} \theta)^2] \right) = Gu'_1
\]

\[
\tau_2 = \frac{2G}{h} \left( -u_m (\frac{1}{2} \theta)^2 + \Delta u [1 + \frac{1}{3} (\frac{1}{2} \theta)^2] \right) = Gu'_2
\]

where, as before, \( \theta = h \sqrt{k^2 - k_0^2} \). Adding and subtracting the above two expressions, we obtain

\[
au'_m = \Delta u [1 + \frac{1}{3} (\frac{1}{2} \theta)^2]
\]

\[
a\Delta u' = u_m (\frac{1}{2} \theta)^2
\]
Finally, introduction of the above expressions into the hermitian interpolation equations yields

\[
\begin{align*}
u &= \left[1 - \frac{1}{2} \left( \frac{1}{6} \theta \right)^2 (1 - \zeta^2) \right] u_m + \left[1 - \frac{1}{6} \left( \frac{1}{8} \theta \right)^2 (1 - \zeta^2) \right] \zeta \Delta u \\
au' &= \left( \frac{1}{2} \theta \right)^2 \zeta u_m + \left[1 - \left( \frac{1}{2} \theta \right)^2 (\frac{1}{6} - \frac{1}{2} \zeta^2) \right] \Delta u
\end{align*}
\] (11a) (11b)

Before comparing these expressions for displacements and strains within the layer against the exact ones presented earlier in Equations (9a) and (9b), we first go on developing the corresponding equations for the quadratic and cubic elements.

(b) Quadratic element. The known field quantities in this case are \(u_1, u_2, u_3\), and \(\tau_1, \tau_2\), (i.e. \(u'_1, u'_2\)) with \(u_2\) being the displacement at the centre of the layer. Hence, a fourth-order interpolation scheme is in order

\[
u = A + B \zeta + C \zeta^2 + D \zeta^3 + E \zeta^4
\]

whose vertical derivative is

\[
au' = B + 2C \zeta + 3D \zeta^2 + 4E \zeta^3
\]

so that

\[
\begin{align*}
\begin{bmatrix} u_1 \\ au'_1 \\ u_2 \\ u'_2 \\ au'_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix}
\end{align*}
\]

whose solution is

\[
\begin{align*}
\begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} &= \frac{1}{4} \begin{bmatrix} 0 & 0 & 4 & 0 & 0 \\ 3 & -1 & 0 & -3 & -1 \\ 4 & -1 & -8 & 4 & 1 \\ -1 & 1 & 0 & 1 & 1 \\ -2 & 1 & 4 & -2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ au'_1 \\ u_2 \\ u'_2 \\ au'_2 \end{bmatrix}
\end{align*}
\]

Substituting these constants into the displacement expansion, and recombining the known parameters in terms of the mean and differential displacements, we obtain

\[
\begin{align*}
u &= u_x + \frac{1}{2} (3\Delta u - au'_m) \zeta + (2u_m - \frac{1}{2} a \Delta u' - 2u_x) \zeta^2 + \frac{1}{2} (-\Delta u + au'_m) \zeta^3 \\
&\quad + (-u_m + \frac{1}{2} a \Delta u' + u_x) \zeta^4 \\
au' &= \frac{1}{2} (3\Delta u - au'_m) + (4u_m - a \Delta u' - 4u_x) \zeta + \frac{3}{2} (-\Delta u + au'_m) \zeta^2 \\
&\quad + 2(-2u_m + a \Delta u' + 2u_x) \zeta^3
\end{align*}
\]
On the other hand, the dynamic equilibrium equation for a quadratic element is

\[
\begin{bmatrix}
\tau_1 \\
0 \\
-\tau_2
\end{bmatrix} = \frac{G}{3h} \begin{bmatrix}
1 & 10 \\
2 & 16 & 2 \\
-1 & 2 & 4 \\
1 & -8 & 7
\end{bmatrix} \begin{bmatrix}
\theta^2 \\
\phi^2 \\
\beta^2
\end{bmatrix} + \begin{bmatrix}
7 & -8 & 1 \\
-8 & 16 & -8 \\
1 & -8 & 7
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_x \\
u_2
\end{bmatrix}
\]

(12)

Using the middle row in Equation (12) to condense out the internal degree of freedom and extracting the vertical derivatives from the stresses \(\tau_1, \tau_2\), we obtain

\[
u_x = \frac{1 - \frac{1}{10}(\frac{1}{2} \theta)^2}{1 + \frac{3}{8}(\frac{1}{2} \theta)^2} \phi_m
\]

\[au'_1 = \frac{1 + \frac{1}{15}(\frac{1}{2} \theta)^2}{1 + \frac{3}{8}(\frac{1}{2} \theta)^2} (\frac{1}{2} \theta)^2 \phi_m + (1 + \frac{1}{3}(\frac{1}{2} \theta)^2) \Delta \phi
\]

\[au'_2 = \frac{1 + \frac{1}{15}(\frac{1}{2} \theta)^2}{1 + \frac{3}{8}(\frac{1}{2} \theta)^2} (\frac{1}{2} \theta)^2 \phi_m + (1 + \frac{1}{3}(\frac{1}{2} \theta)^2) \Delta \phi
\]

Hence, adding and subtracting the last two expressions, we obtain

\[au'_m = (1 + \frac{1}{3}(\frac{1}{2} \theta)^2) \Delta \phi
\]

\[a \Delta u' = \frac{1 + \frac{1}{15}(\frac{1}{2} \theta)^2}{1 + \frac{3}{8}(\frac{1}{2} \theta)^2} (\frac{1}{2} \theta)^2 \phi_m
\]

Substitution of these expressions into the interpolation equation finally yields

\[u = \frac{\phi_m}{1 + \frac{3}{8}(\frac{1}{2} \theta)^2} \left[1 - \frac{1}{10}(\frac{1}{2} \theta)^2 + \frac{1}{2}(\frac{1}{2} \theta)^2 \psi^2 \right] \left[1 - \frac{1}{15}(\frac{1}{2} \theta)^2(1 - \psi^2)\right]
\]

+ \Delta \phi \psi \left[1 - \frac{1}{6}(\frac{1}{2} \theta)^2(1 - \psi^2)\right]

(13a)

\[au' = \frac{\phi_m(\frac{1}{2} \theta)^2}{1 + \frac{3}{8}(\frac{1}{2} \theta)^2} \left[1 - \frac{1}{15}(\frac{1}{2} \theta)^2(1 - 2\psi^2)\right] + \Delta \phi \left[1 - \frac{1}{2}(\frac{1}{2} \theta)^2(\frac{1}{6} - \frac{1}{2} \psi^2)\right]
\]

(13b)

(c) Cubic element. The known field quantities in this case are \(u_1, u_2, u_x, u_\beta\) and \(\tau_1, \tau_2\), (i.e. \(u'_1, u'_2\)) with \(u_2, u_\beta\) being the displacements at the internal elevations \(\zeta = \pm \frac{1}{3}\). In this case, a fifth order interpolation scheme is in order

\[u = A + B \zeta + C \zeta^2 + D \zeta^3 + E \zeta^4 + F \zeta^5
\]

whose vertical derivative is

\[au' = B + 2C \zeta + 3D \zeta^2 + 4E \zeta^3 + 5F \zeta^4
\]

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so that

\[
\begin{pmatrix}
  u_1 \\
  au_1' \\
  u_2 \\
  u_\beta \\
  au_2'
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & 1 & 1 & 1 \\
  0 & 1 & 2 & 3 & 4 & 5 \\
  1 & 3^{-1} & 3^{-2} & 3^{-3} & 3^{-4} & 3^{-5} \\
  1 & -3^{-1} & 3^{-2} & -3^{-3} & 3^{-4} & -3^{-5} \\
  0 & 1 & -2 & 3 & -4 & 5
\end{pmatrix}
\begin{pmatrix}
  A \\
  B \\
  C \\
  D \\
  E \\
  F
\end{pmatrix}
\]

whose solution is

\[
\begin{pmatrix}
  A \\
  B \\
  C \\
  D \\
  E \\
  F
\end{pmatrix} = \frac{1}{128}
\begin{pmatrix}
  -17 & 4 & 81 & 81 & -17 & -4 \\
  -21 & 4 & 243 & -243 & 21 & 4 \\
  162 & -40 & -162 & -162 & 162 & 40 \\
  202 & -40 & -486 & 486 & -202 & -40 \\
  -81 & 36 & 81 & 81 & -81 & -36 \\
  -117 & 36 & 243 & -243 & 117 & 36
\end{pmatrix}
\begin{pmatrix}
  u_1 \\
  au_1' \\
  u_2 \\
  u_\beta \\
  u_2' \\
  au_2'
\end{pmatrix}
\]

Substituting these constants into the displacement expansion and recombining the known parameters in terms of the mean and differential displacements, we obtain

\[
u = \frac{1}{128}
\left[
-34u_m + 4a\Delta u' + 81(u_x + u_\beta) \\
-42\Delta u + 4au_m' + 243(u_x - u_\beta)
\right]z^1
\]

\[
\times
\left[
324u_m - 40a\Delta u' - 162(u_x + u_\beta)\right]z^2
\] \[+ [204\Delta u - 40au_m' - 486(u_x - u_\beta)]z^3
\]

\[+ [-162u_m + 36a\Delta u' + 81(u_x + u_\beta)]z^4
\] \[+ [234\Delta u + 36au_m' + 243(u_x - u_\beta)]z^5
\]

Now, the dynamic equilibrium equation for the cubic element is

\[
\begin{pmatrix}
  \tau_1 \\
  0 \\
  -\tau_2
\end{pmatrix} = \frac{G}{40h}
\begin{pmatrix}
  128 & 99 & -36 & 19 \\
  99 & 648 & -81 & -36 \\
  -36 & -81 & 648 & 99 \\
  19 & -36 & 99 & 128
\end{pmatrix}
\begin{pmatrix}
  148 & -189 & 54 & -13 \\
  -189 & 432 & -297 & 54 \\
  54 & -297 & 432 & -189 \\
  -13 & 54 & -189 & 148
\end{pmatrix}
\begin{pmatrix}
  u_1 \\
  u_x \\
  u_\beta \\
  u_2
\end{pmatrix}
\]

in which we have used the shorthand

\[
z = \frac{1}{42}(k^2 - k_0^2)h^2 \equiv \frac{2}{27} \left(\frac{1}{2} \theta\right)^2
\]  

(15)
Condensing out the internal degrees \( u_x, u_\beta \) of freedom using MATLAB’s symbolic manipulation capabilities, which demands extremely tedious work if done by hand, we obtain

\[
u_x + u_\beta = \frac{2(15 - 7x)}{3(5 + 21x)} u_m \quad (16a)
\]

\[
u_x - u_\beta = \frac{2(9 - 5x)}{27(x+1)} \Delta u \quad (16b)
\]

On the other hand, eliminating these internal degrees of freedom from the dynamic equilibrium equation, we obtain

\[
au'_1 = \frac{(588x^3 + 1890x^2 + 960x + 50)u_1 + (147x^3 - 105x^2 + 90x - 50)u_2}{20(21x + 5)(x + 1)}
\]

\[
-au'_2 = \frac{(147x^3 - 105x^2 + 90x - 50)u_1 + (588x^3 + 1890x^2 + 960x + 50)u_2}{20(21x + 5)(x + 1)}
\]

from which the semi-sum and semi-difference yield

\[
u'_m = \frac{(21x^2 + 90x + 20)}{10(x+1)} \Delta u \quad (17a)
\]

\[
a\Delta u' = \frac{21(7x + 10)x}{2(21x + 5)} u_m \quad (17b)
\]

Finally, substituting Equations (16) and (17) into the Hermitian interpolation equation for \( u \), we obtain the displacement within the layer in terms of \( u_m \) and \( \Delta u \) together with its derivative as

\[
u(\zeta) = \frac{1}{320(21x + 5)(x + 1)} \{ (1600 - 80x - 945x^2 + 735x^3)u_m
\]

\[
+ (1600 + 5520x - 4935x^2 + 441x^3)\Delta u \zeta
\]

\[
+ 50(168 + 21x - 147x^2)u_m \zeta^2 + 70(40 + 153x - 63x^2)x \Delta u \zeta^3
\]

\[
+ 6615(1 + x)x^2u_m \zeta^4 + 189(21x + 5)x^2 \Delta u \zeta^5 \} \quad (18a)
\]

\[
au'(\zeta) = \frac{1}{320(21x + 5)(x + 1)} \{ (1600 + 5520x - 4935x^2 + 441x^3)\Delta u
\]

\[
+100(168 + 21x - 147x^2)x u_m \zeta + 210(40 + 153x - 63x^2)x \Delta u \zeta^2
\]

\[
+ 26460(1 + x)x^2u_m \zeta^3 + 945(21x + 5)x^2 \Delta u \zeta^4 \} \quad (18b)
\]
where the parameter $\alpha$ is given by Equation (15). The correctness of these expressions can be verified by setting $\zeta = \pm 1$; one then recovers the displacements at the two external interfaces as $u = u_m \pm \Delta u$.

**ACCURACY OF INTERPOLATED FIELD**

We proceed next to evaluate the accuracy of the interpolated displacements and stresses (Equations (11), (13) and (18)) against the exact benchmark for spatially and temporally harmonic displacements (i.e. in the frequency–wavenumber domain). To this effect, we first recall from Equation (9) that the exact solution has a symmetric and an anti-symmetric component. Since the problem is linear, it suffices to examine the accuracy of the solution for these two components separately. For this purpose, we take either $u_m = 0.5(u_1 + u_2)$ or $\Delta u = 0.5(u_1 - u_2)$ as reference values for the discrete and continuum solution. Also, in most cases the cubic expansion virtually coincides with the exact solution, so is not included in all figures to avoid clutter.

(a) *Displacement at centre of element.* Figures 1, 2 show the displacement at the centre of the layer predicted by the proposed interpolation scheme (Equations (11a), (13a), (18a) with $\zeta = 0$), as a function of the vertical wavenumber $\theta$. Only the effect of $u_m$ needs to be considered here, because the anti-symmetric component in $\Delta u$ does not contribute to the displacement at the centre. In Figure 1, we take $k \geq k_0 = \omega/C_s$ which corresponds to low frequencies or high horizontal wavenumbers, while in Figure 2, $k \leq k_0$, which arises when the frequency is high and/or the horizontal wavenumber is low. The solid line is the exact solution, while the dashed lines are for the interpolated TLM. Only the linear and quadratic elements are shown, because the cubic element provides results that at the scale of the plots cannot be distinguished from the exact solution. Observe that the secondary interpolation scheme—when judged against a direct application of the element’s interpolation used for the weighted residual

![Figure 1: Displacement at centre of element ($u_0/u_m$), Hermitian interpolation for $k > k_0$.](image-url)
formulation—improves even the linear element, since the primary linear expansion would have predicted \( u_0 = u_m \). At first, this result may seem strange and arousing suspicions that we may be getting something for nothing, but the physics involved can be clarified by a mental experiment. Suppose that the layer is excited so that the upper and lower interfaces move in synchrony, i.e. \( u_1 = u_2 = u_m \). Because of inertia within the layer, the interior will lag behind the interfaces, and generally will exhibit smaller (or larger) displacements than at the exterior interfaces, and this is also true at the centre. While the primary linear expansion would have predicted a rigid body motion for the entire layer, this is not so for the secondary interpolation proposed. The reason is that as the wavenumber (or frequency) changes, so do the consistent interface stresses, and therefore, also the displacements (and stresses) within the element.

Incidentally, if the plots were continued further for even higher vertical wavenumbers to the right, increasingly larger discrepancies would be observed between the exact and discrete solutions. The reason is that as the wavelengths become rather short in comparison to the layer thickness, the discrete model breaks down—as expected—and no amount of secondary interpolation can cure that. This comment also apply to the comparisons to follow.

(b) Variation of displacements across an element. Figures 3, 4 show the variation of displacements across an element due to \( u_m \) for a vertical wavenumber value of 2. Figure 3 represents vertically evanescent waves (\( \theta = 2 \) is real) while Figure 4 is for propagating waves (\( \eta = 2 = -i\theta \) is real). As could be expected, the displacements within the element are smaller than those at the interfaces in the first case, and larger in the second. In both cases, the secondary interpolation does a superb job in predicting the correct variation of displacements, especially for the high-order elements. Observe once more that the secondary interpolation improves even the linear element, which *ab initio* would have predicted a constant value 1.

Figures 5, 6 show the displacement across the element due to \( \Delta u \) (i.e. the anti-symmetric component) for a vertical wavenumber value of 4. In this case, both the linear and quadratic elements give the same accuracy, and this accuracy is a substantial improvement over the direct

Figure 3. Variation of displacement due to $u_m$ for $\theta = 2$.

Figure 4. Variation of displacement due to $u_m$ for $\eta = 2$.

linear interpolation across the element, which would have given a straight line between $-1$ and 1. The cubic element, on the other hand, is very close to the exact solution.

(c) Exact vs consistent interface stresses. Figures 7–10 show a comparison between the exact and consistent interface shearing stresses due to $u_m$ and $\Delta u$. The exact stresses follow from Equation (9b), while the consistent stresses follow from Equations (10), (12) or (14) for the linear, quadratic and cubic elements, respectively. In this case, the stresses due to $u_m$ for the quadratic (and higher elements) are virtually exact in the range of wavenumbers examined.
Figure 5. Variation of displacement due to $\Delta u$ for $\theta = 4$.

Figure 6. Variation of displacement due to $\Delta u$ for $\eta = 4$.

(Figures 7, 8). On the other hand, the stresses due to $\Delta u$ (Figures 9, 10) follow reasonably close the results of the continuum, while the cubic element coincides with the exact.

(d) Stresses at internal points. Finally, Figures 11–14 show the stresses within the element obtained by the proposed secondary interpolation scheme. As can be seen, this algorithm works extremely well, especially for the high order elements, with the quadratic element already providing a substantial improvement over the linear element. In fact, tests not included herein demonstrate that a single quadratic element outperforms two linear elements having the same combined total thickness.
FREQUENCY–WAVENUMBER SOLUTION VS FREQUENCY–SPACE SOLUTION

So far, we have demonstrated the accuracy of the secondary interpolation for a harmonic space–time wave-field, that is, for a formulation in the frequency–wavenumber domain. However, in the normal implementation of the TLM, the displacements are provided either in frequency–space or in the time–space domain, and this is accomplished by recourse of modal superposition. The requisite normal modes can be obtained very efficiently by solving a block-tridiagonal eigenvalue problem [6]. Thus, it behooves to extend the demonstration for the typical
non-harmonic displacement fields encountered in the TLM. For this purpose, consider the generic \( j \)th modal displacement component \( u_j(k_j, \zeta, \omega) \) at an arbitrary point in the medium, which has an associated eigenvalue (wavenumber) \( k_j \). With reference to the modal expansion, the displacement in the space domain for a TLM based on linear elements has the form

\[
 u(x, \zeta, \omega) = \sum_j u_j(k_j, \zeta, \omega) = \sum_j (A_j + B_j \zeta + C_j \zeta^2 + D_j \zeta^3)
\]

\[
 = \sum_j A_j + \zeta \sum_j B_j + \zeta^2 \sum_j C_j + \zeta^3 \sum_j D_j
\]

\[
 = \left\{ \begin{array}{c}
 1 \\
 \zeta \\
 \zeta^2 \\
 \zeta^3 \\
 \end{array} \right\} \sum_j \begin{bmatrix}
 A_j \\
 B_j \\
 C_j \\
 D_j \\
 \end{bmatrix}
\]

\[
 = \frac{1}{4} \left\{ \begin{array}{c}
 1 \\
 \zeta \\
 \zeta^2 \\
 \zeta^3 \\
 \end{array} \right\} \sum_j \begin{bmatrix}
 2 & -1 & 2 & 1 \\
 3 & -1 & -3 & -1 \\
 0 & 1 & 0 & -1 \\
 -1 & 1 & 1 & 1 \\
 \end{bmatrix} \begin{bmatrix}
 u_{1j} \\
 au'_{1j} \\
 u_{2j} \\
 au'_{2j} \\
 \end{bmatrix}
\]
It is now apparent that an interpolation on the modally superposed field is exactly the same as the addition of the interpolated modal components. Hence, the previous considerations about the accuracy of the displacement and stress field in the frequency–wavenumber domain apply equally well to the displacement field in the frequency–space domain. We thus arrive at the important conclusion that is not necessary to carry out the auxiliary Hermitian interpolation at the level of the modes, but it suffices to do so instead for the complete displacement and stress field in the spatial domain.
SV-P WAVES

We now continue with the somewhat more complicated in-plane (SV-P) displacement field. In this case, the consistent stresses in horizontal planes in the frequency–wavenumber domain.

Figure 12. Variation of stress due to $u_m$ for $\eta = 2$.

Figure 13. Variation of stress due to $\Delta u$ for $\theta = 2$. 

SV-P WAVES

We now continue with the somewhat more complicated in-plane (SV-P) displacement field. In this case, the consistent stresses in horizontal planes in the frequency–wavenumber domain.
(i.e. with an implied factor \( \exp(i\omega t - kx) \)) are

\[
\tau_{xz} = G \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = G(u' - ikw)
\]

\[
\sigma_z = (\lambda + 2G) \frac{\partial w}{\partial z} + \lambda \frac{\partial u}{\partial x} = (\lambda + 2G)w' - i k \lambda u
\]

Solving for the derivatives of the displacements at the interfaces, we obtain

\[
u' = \frac{\tau_{xz}}{G} + ikw
\]

\[
\sigma_z = (\lambda + 2G) \frac{\partial w}{\partial z} + \lambda \frac{\partial u}{\partial x} = (\lambda + 2G)w' - i k \lambda u
\]

Application of a modal superposition to transform the components into space–frequency yields

\[
u'(x, \omega) = \frac{1}{G} \sum_j \tau_{xz}^j + i \sum_j k_j \omega_j
\]

\[
\omega' = \frac{1}{\lambda + 2G} \sum_j \sigma_z^j + i \frac{\lambda}{\lambda + 2G} \sum_j k_j u_j
\]

At this point, it is apparent that we can once more apply the secondary Hermitian interpolation technique of the previous section separately to \( u \) and \( w \) and with these obtain displacements and
stresses at any arbitrary point within the layer, but in this case we need an auxiliary quantity, namely the summation of the modal displacements weighed by the characteristic wavenumber (or eigenvalues) \( k_j \), which is the second term in the summations above. Other than that, the expressions for \( u \) in terms of \( u_m, \Delta u \), and for \( w \) in terms of \( w_m, \Delta w \), are exactly the same as for the anti-plane case. Once we have obtained the interpolated displacement field, we can proceed to compute any arbitrary stress component.

**CONCLUSION**

In this paper, we have shown that a secondary interpolation scheme based on the combined use of the interface displacements and consistent interface stresses in the TLM provides greatly improved field quantities within the layers. Indeed, it not only improves substantially on the accuracy of displacements and stresses provided by a naïve direct application of the primary interpolation polynomials combined with differentiation, but it also eliminates the stress discontinuities entailed in the latter. For example, a linear element in anti-plane motion would predict a constant stress field within the layer, while the proposed secondary interpolation provides a cubic variation of stress instead, and in addition guarantees continuous variation of stresses across interfaces for neighbouring layers (except, of course, at the location of sources). Thus, we believe that the proposed algorithm should be the method of choice whenever field quantities, and especially stresses, are needed in the TLM.

**REFERENCES**