DECONVOLUTION OF STOCHASTIC SH-WAVE MOTIONS IN SOIL DEPOSITS

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July 1984
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Sponsored by the National Science Foundation
Grant CEE-8211021

Research Report R84-09 Order No. 769
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Abstract

The spatial and temporal variabilities of seismic ground motions that may be expected in soil deposits are investigated by means of simple physical models for stochastic SH waves travelling in a homogeneous medium. It is assumed that the motion observed at the surface of the soil deposit is the result of many plane, stationary waves coming at differing angles, each having a characteristic spectral density function, and exhibiting a prescribed degree of cross-correlation. The statistical properties of the motions at two different points on the surface and/or within the soil mass are then computed and analyzed.
Acknowledgement

This is a research report issued under the project entitled "Improved Ground Motion Models for Aseismic Design". The project is supported by the National Science Foundation under Grant No. CEE-8211021, of which Dr. K. Thirumalai is the cognizant NSF Program Official.

The opinions, findings and conclusions or recommendations expressed in this report are those of the authors, and do not necessarily reflect the views of the National Science Foundation.
1. **Introduction**

The forces induced in a structure by an earthquake are not only caused by the inertia of the structure in reaction to the motion, but also by the variability in space of the ground motion itself. The effect of this variability is particularly important in spatially extended facilities such as bridges or pipelines, for which the relative displacement between two points in the ground can result in important additional stresses. While empirical data provided by arrays of strong motion instruments may be used to study this problem from a direct statistical point of view (Harichandran, 1984), it is also desirable to develop mechanistic models to study the ground motion problem from an analytical standpoint. Such models could be used not only to interpret seismic array data in terms of wave content, but as an aid in the development of site-specific seismic design parameters.

The changes in earthquake ground motions that are observed from one location to another at a site are the result of numerous factors, such as distance to the source and size of the seismic event, site geology and stratigraphy (refraction, reflection, amplification), wave content (multiple travel paths) as well as attenuation (hysteretic and geometric damping), dispersion (frequency-dependent phase velocity of waves) and scattering (variability of soil properties at a small scale; random media), etc. The relative importance of these factors depends on the spatial scale at which the observations are made. For example, at a scale of a few kilometers (or more) attenuation may not be neglected, whereas it is probably less important than wave content at a scale of a few hundred meters. While one could develop, at least in principle, stochastic models with high degrees of complexity, it is not practical to do so in a preliminary investigation such as this one, for the large number of parameters would obscure
the proper interpretation of the results obtained with those models.

Is it not the goal of this report to describe credible stochastic models of earthquake ground motions in realistic soil deposits, accounting for some or all of the factors referred to above. Instead, the aim of this study is to formulate very simple wave-content models for which the output statistics at any two observation points (and specifically, the coefficient of cross-correlation) can readily be computed; analysis of the response statistics vis-a-vis the known incident wave field would permit the deciding whether the former still contains enough information to estimate the latter. A positive answer to this question would imply that by studying the statistics of ground motion at grade level of actual sites one could perhaps obtain information on the wave content, and as a result, determine also statistical properties for the motions underneath the soil surface. A negative answer, on the other hand, would indicate that the averaging process of computing statistics has erased substantial information on wave content: one could then reasonably expect that response statistics alone would not suffice to evaluate more complex three dimensional wave fields involving body and surface waves. As we shall see in the following, the answer seems to lie somewhere between these two extremes.

This report describes the results obtained with a stochastic wave propagation model that is based on the superposition of plane SH waves traveling within a homogeneous soil at varying angles. While the model is admittedly simple, it sheds significant light on the problem of the spatial coherence of earthquake ground motions, and on the question of the usefulness of statistical models in deriving empirically the wave contents of actual motions. In addition, it may be useful in extrapolating the known statistical data to regions for which no information is readily available, such as points within the soil mass (i.e., underneath the observation horizon).
The equations derived herein could be generalized rather straightforwardly to three-dimensional situations, considering combinations of body waves (P, SV, SH) and surface waves; horizontal layering could be incorporated into the model as well. Such developments will not, however, be considered here.

2. Theoretical Background

2.1 SH-Wave superposition

If one assumes that the soil near the surface is spatially homogeneous, and that the earthquake is the result of a composite of plane SH waves propagating through the soil at varying angles with respect to the vertical, then the motion of a point at the surface, in the absence of material damping, is given by the integral

\[ u(x, t) = \int_{-S}^{S} v(s, t - sx) ds \]  \hspace{1cm} (1)

where \( s \) = apparent slowness of the waves at the surface

\[ S = \frac{1}{c_{s}} = \text{maximum slowness (with } c_{s} \text{ being the shear wave velocity of the soil).} \]

The kernel \( v(s, t) \) in the integral represents the time history of an individual wave component travelling with an apparent celerity \( c = \frac{1}{s} \).

Representing by \( \theta \) the angle between the vertical and the trajectory of the incident wave (Fig.1), one can write the slowness of the waves at the surface as a function of \( \theta \) as:

\[ s = S \sin \theta = \sin \frac{\theta}{c_{s}} \]  \hspace{1cm} (2)
By Fourier transforming $u(x, t)$ both in space and time, one obtains (see Appendix 1)

$$u(x, t) \rightarrow U(\xi, \omega) = 2\pi \frac{V(\xi/\omega, \omega)}{\omega}$$  \hspace{1cm} (3)

in which $\omega$ represents the angular frequency and $\xi$ is the wavenumber.

The term $V(\xi/\omega, \omega)$ is the Fourier Transform with respect to time of the wave traveling at the surface with a slowness $\xi/\omega$, or the wave arriving at an angle $\theta$ such that $\sin \theta = c_s \xi/\omega$.

The motion at any depth can be computed by adding the incident and reflected waves:

$$u(x, t, z) = \frac{1}{2} \int_{-S}^{S} \left[ V(s, t-sx-\omega z) + V(s, t-sx + \omega z) \right] ds$$  \hspace{1cm} (4)
where $\mu = \tilde{S} - s_0^2$ is the slowness in the vertical direction.

The factor $\frac{1}{2}$ is due to the fact that we already took into account the reflected waves when describing the motion at the surface; in fact, for $z = 0$, equation (4) reduces to eq. (1).

Again, Fourier-Transforming $u(x, t, z)$ in $x$ and $t$, we obtain (see Appendix 2)

$$U(\xi, \omega, z) = 2\pi V(\xi/\omega, \omega) \cos \left( \sqrt{\omega^2 s_0^2 - \xi^2} z / \omega \right)$$

(5)

We notice that $|\xi/\omega|$ cannot be greater than $S$, otherwise we would have the square root of a negative number; a value of $\xi/\omega$ greater than $S$ would imply the existence of waves traveling with an apparent velocity less than $c_s$, which is contrary to our hypothesis.

2.2 Statistical properties of the wave field at the free surface

Assuming stationarity and ergodicity in time of the incident waves, then the covariance function between two points on the surface at a distance $L$ can be obtained from the limiting value of the temporal average (spatial homogeneity cannot be assumed a priori, hence the dependence on $x$):

$$C(x, L, \tau) = \mathbb{E}[u(x, t) u(x+L, t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(x, \xi) \cdot u(x+L, \xi+\tau) \, d\xi$$

(6)

substituting equation (1) into (6), one obtains (Appendix 3)

$$C(x, L, \tau) = \int \int \mathcal{R}[s, s', \tau + sx - s'(x+L)] \, ds' \, ds$$

(7)

or

$$C(x, L, \tau) = \int \int \int S(s, s', \omega) e^{i\omega(\tau + sx - s'(x+L))} \, d\omega \, ds' \, ds$$

(8)
where \( R[s, s', \tau + sx - s'x + L] \) represents the crosscorrelation function for waves traveling with slownesses \( s \) and \( s' \), respectively, for a lag time \( \tau = \tau + sx - s'x + L \) and \( S(s, s', \omega) \) represents the corresponding cross-spectral density function. The coefficient of correlation between two points is obtained by dividing equation (8) by \( C(x, 0, 0) \):

\[
\rho(x, L, \tau) = -S \int_{-\infty}^{\infty} S(s, s', \omega) e^{i\omega(\tau + sx - s'x + s'L)} ds ds'
\]

(9)

On the other hand, the spectral density function of the motion at an arbitrary point \( x \) is given by:

\[
S(x, \omega) = \int_{-S}^{S} \int_{-S}^{S} S(s, s', \omega) e^{i\omega(s - s')} ds ds'
\]

(10)

The spatially inhomogeneous variance of the motion is then simply

\[
\sigma^2(x) = \int S(x, \omega) d\omega.
\]

Let us now change variables, from \( s \) and \( s' \) to \( \theta \) and \( \gamma \),

where the latter two represent the angles between the directions of propagation of the incident waves and the vertical; (Note that \( S(\theta, s', \omega) \) and \( S(\theta, \gamma, \omega) \) are not the same functions; however, we define these functions in such a way that \( S(s, s', \omega) ds ds' = S(\theta, \gamma, \omega) d\theta d\gamma \))

\[
\rho(x, L, \tau) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} S(\theta, \gamma, \omega) e^{i\omega[\tau + x (\sin \theta - \sin \gamma)/c_s - L \sin \gamma/c_s]} d\omega d\theta d\gamma
\]

(11)

and

\[
S(x, \omega) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} S(\theta, \gamma, \omega) e^{i\omega(\sin \theta - \sin \gamma)/c_s} d\theta d\gamma
\]

(12)
In general, these expressions are difficult to integrate and one has to make some reasonable simplifications. Let's assume first that the waves coming at the various angles have spectral density functions exhibiting the same variation with frequency, although not necessarily the same amplitude. Furthermore, we can also assume that the cross-spectrum between two waves depends only on the autospectrum of each wave and on their relative angle of travel. These assumptions imply:

\[ S(\theta, \gamma, \omega) = S(\omega) \cdot g(\theta) \cdot g(\gamma) \cdot f(|\theta-\gamma|) \]  

(13)

Substituting (13) into (12), we obtain

\[ S(x, \omega) = S(\omega) \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(\theta) \cdot g(\gamma) \cdot f(|\theta-\gamma|) e^{i\omega x (\sin \theta - \sin \gamma)/c} \sin \theta d\theta d\gamma \]  

(14)

From the evenness of the integrand, it follows that the imaginary part cancels out; hence, equation (14) can be written as

\[ S(x, \omega) = S(\omega) \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(\theta) \cdot g(\gamma) \cdot f(|\theta-\gamma|) \cos \omega x (\sin \theta - \sin \gamma)/c \sin \theta d\theta d\gamma \]  

(15)

Also, we can write the coefficient of cross-correlation as

\[ \rho(x, L, \tau) = \frac{\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(\theta) \cdot g(\gamma) \cdot f(|\theta-\gamma|) \int_{-\infty}^{\infty} S(\omega) e^{i\omega L} d\omega \sin \theta d\theta d\gamma}{\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(\theta) \cdot g(\gamma) \cdot f(|\theta-\gamma|) \int_{-\infty}^{\infty} S(\omega) e^{i\omega L} d\omega \sin \theta d\theta d\gamma} \]  

(16)
in which
\[ \tilde{\lambda} = \lambda + x(\sin \theta - \sin \gamma) / c_s - L \sin \gamma / c_s \]
and
\[ \lambda = x(\sin \theta - \sin \gamma) / c_s \]

These expressions can be evaluated numerically for arbitrary functions \( g(\beta) \) and \( f(|\theta - \gamma|) \).

In order to evaluate \( \rho \) we must also specify the function \( S(\omega) \). Let's assume \( S(\omega) \) can be described by the Kanai-Tajimi spectrum:

\[
S(\omega) = S_0 \left( 1 + \frac{4 \beta^2 \xi^2}{(1 - \xi^2)^2 + 4 \beta^2 \xi^2} \right) \xi = \frac{\omega}{\omega_0}
\] (17)

The plot of \( S(\omega) \) with \( \beta = 0.25 \), \( \omega_0 = 6\pi \) and \( S_0 = 1 \) is shown in Fig. 2. These values of \( \beta \) and \( \omega_0 \) are typical of actual earthquakes and can be used in preliminary studies to arrive at some qualitative conclusions. The value of \( S_0 \) is not important since it represents only a scaling factor.

The plot of the autocorrelation coefficient \( \rho(\tau) \) corresponding to the Kanai-Tajimi spectrum \( S(\omega) \) is shown in Fig. 3. For a unit area spectral density those functions are simply Fourier transform pairs.

Using the Kanai-Tajimi spectrum, we obtain

\[
B(\tau) = \int_{-\infty}^{\infty} S(\omega) \ e^{i\omega \tau} \ d\omega = \pi \omega_0 \ P(\tau) / (2 \beta \sqrt{1 - \beta^2})
\] (18)

in which

\[
P(\tau) = e^{-\beta |\tau| \omega_0} [a \cos (|\tau| \omega_d) + b \sin (|\tau| \omega_d)]
\] (19a)
\[ S(\omega) = \frac{1+4\beta^2\xi^2}{(1-\xi^2)^2 + 4\beta^2\xi^2} \]

\[ \xi = \frac{\omega}{\omega_0} \]

\[ S_{\text{max}} = \frac{1+4\beta^2\xi^2}{(1-\xi^2)^2 + 4\beta^2\xi^2} \]

\[ \bar{\xi} = \frac{\sqrt{1+8\beta^2}-1}{4\beta^2} \]

---

**Fig. 2**

\[ \beta = 0.25 \quad \omega_0 = 6\pi \text{ rad/sec} \]

\[ \rho(\tau,0) = e^{-1.5\pi |\tau|} \frac{[1.2103073 \cos(5.809475\pi |\tau|) + 0.1875 \sin(5.809475\pi |\tau|)]}{1.2103073} \]

---

**Fig. 3**
with 

\[ a = (1 + 4 \beta^2) \sqrt{1-\beta^2} \]  

(19b)

\[ b = \beta(1 - 4 \beta^2) \]  

(19c)

and 

\[ \omega_d = \sqrt{1 - \beta^2} \omega_0 \]  

(19d)

Hence, for 

\[ \lambda = \tau + x(\sin \theta - \sin \gamma)/c_s - L \sin \gamma/c_s \]  

(20)

and 

\[ \lambda = x(\sin \theta - \sin \gamma)/c_s \]

we obtain

\[
\rho(x, L, \tau) = \frac{\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \rho(\theta) g(\gamma) f(|\theta - \gamma|) \ p(\lambda) \ d\theta d\gamma}{\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \rho(\theta) g(\gamma) f(|\theta - \gamma|) \ p(\lambda) \ d\theta d\gamma}.
\]  

(21)

Again, we can evaluate this expression by numerical integration for arbitrary function \( g(\theta), f(|\theta - \gamma|) \), and selected values of \( x \). Some simple models will be examined in the following:

a) Constant cross-correlation

In this alternative, we assume that the waves coming at the various angles are fully correlated, i.e., \( f(|\theta - \gamma|) = f(0) = 1 \). Then

\[
S(x, \omega) = S(\omega) \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(\theta) g(\gamma) \cos \left[ \omega x(\sin \theta - \sin \gamma)/c_s \right] d\theta d\gamma + S(\omega) \cdot A(\omega x)
\]  

(22)

where \( A(\omega x) \) is the result of the double integral.
If the waves arrive with equal amplitude within the dihedron \(0 < \theta < \frac{\pi}{2}\), then \(g(\theta) = 1\) for \(0 \leq \theta \leq \frac{\pi}{2}\), and \(g(\theta) = 0\) for \(-\frac{\pi}{2} \leq \theta < 0\). Evaluating the double integral numerically using Simpson's rule, we obtain the inferior curve shown in Fig. 4.

b) Exponentially decaying cross correlation

Here, we assume that the cross-correlation decays exponentially with the angle between any two waves, i.e., \(f(|\theta - \gamma|) = e^{-2|\theta - \gamma|}\) (\(\theta, \gamma\) in radians).

Then

\[
S(x, \omega) = S(\omega) \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(\theta) \cdot g(\gamma) \cdot e^{-2|\theta - \gamma|} \cos[\omega \chi \sin(\theta - \sin \gamma / c)] d\theta d\gamma = S(\omega) \cdot A(w \chi)
\]

(Equation 23)

Evaluating the above expression for the same function \(g(\theta)\) used earlier, we obtain the intermediate curve depicted in Fig. 4.

c) Uncorrelated waves

If the incident waves are fully uncorrelated, then \(f(\theta - \gamma) = \delta(\theta - \gamma)\) (the Dirac delta function), and

\[
S(x, \omega) = S(\omega) \int_{-\pi/2}^{\pi/2} g^2(\theta) \, d\theta = S(\omega) \cdot A
\]

(Equation 24)

Thus, the autospectrum is no longer a function of the abcissa, so that the motion is spatially homogeneous. This is indicated by the upper horizontal line in Fig. 4.
Fig. 4

Inspection of Fig. 4 reveals that the more correlated the incident waves are, the more the high frequencies attenuate as $\omega x$ increases. In this sense, the origin represents a privileged observation point. Since the temporal correlation of the wave components decays with time lag ($\tau$), and since each wave component travels with different apparent speeds, it follows that the favorable conditions at the origin should be lost as the waves progress in the horizontal direction. Hence, the correlation should be lost with increasing abcissa. In other words, the ratio $\rho(\omega x') = A[\omega(x_0 + x')]/A[\omega x_0]$, which relates the auto-spectra at $x_0 + x'$ and $x_0$ (i.e., shifting the origin by an amount $x_0$), should (and indeed does) approach the constant value 1 as $x_0$ is increased, (appendix 6). Thus, the random field is homogeneous for completely uncorrelated waves, and approaches homogeneity in the far field in the case of perfect correlation near the origin ("close to the source").

2.3 Cross-correlation with Arbitrary Points in the Soil Mass

The coefficient of correlation for the motions at a point on the free surface, and another at a depth $H$ and horizontal distance $L$ from the first, for the particular case of waves with no cross-correlation, is given by (Appendix 4):
\[
\begin{align*}
\rho(\tau, L, H) &= \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} S(\omega) e^{i\omega(\tau-L \sin \theta/c_s) \cos(\omega L \cos \theta/c_s)} d\omega d\theta \\
&= \int_{-\pi/2}^{\pi/2} S(\omega) d\omega \int_{-\pi/2}^{\pi/2} g^2(\theta) d\theta
\end{align*}
\] (25)

If the spectral density function of the waves is given by the Kanai-Tajimi spectrum, then the integral in \( \omega \) above can be evaluated analytically:

\[
\begin{align*}
\rho(\tau, L, H) &= \int_{-\pi/2}^{\pi/2} g^2(\theta) \rho(\theta, \tau, L, H) d\theta \\
&= \int_{-\pi/2}^{\pi/2} g^2(\theta) d\theta
\end{align*}
\] (26)

with

\[
\rho(\theta, \tau, L, H) = \frac{1}{2a} [P(\lambda) + P(\bar{\lambda})]
\] (27)

in which \( P(\lambda) \) is given by equation 19a), (replacing \( \lambda, \bar{\lambda} \) for \( \tau \)), and the constant \( a \) is defined by 19b). Also

\[
\lambda = \tau - L \sin 0/c_s - H \cos 0/c_s \quad , \quad \bar{\lambda} = \tau - L \sin 0/c_s + H \cos 0/c_s
\] (28)

For arbitrary amplitude functions \( g(\theta) \), equation 26) can be evaluated numerically by Simpson's rule. In the following, (Figures 5-20), we compute the coefficient of cross-correlation for several shapes of the function \( g(\theta) \), in a soil with a shear wave velocity of \( c_s = 300 \text{ m/s} \), and for a Kanai-Tajimi spectrum with \( \omega_0 = 6\pi \) and \( \beta = 0.25 \). The plots are drawn only for positive time lags \( \tau \), since the coefficient
of correlation exhibits polar symmetry (i.e., \( \rho(-\tau,-L,H) = \rho(\tau,L,H) \)) when the incident waves are uncorrelated. (The plots can be made valid for arbitrary dominant frequency and shear wave velocity by scaling the axes \( \tau, L \) by \( \omega_0 \) and \( \omega_0/c_s \), respectively.) Additional computations were also performed for waves having some degree of cross-correlation, using numerical quadrature. The results are presented and discussed in the next section.

3. **Evaluation of Cross-correlation Spectra**

3.1 **Effect of Correlation Between Incident Waves**

Figures 5 and 6 depict the cross-correlation spectra at \( x = 0 \) and \( x = 50 \) m, respectively, for an incident wavefield having a cross-correlation function \( f(\theta-\gamma) = \exp(-2|\theta-\gamma|) \). While the two plots exhibit an overall similarity, it can be observed that the symmetry with respect to \( L \), for \( \tau = 0 \), is lost in the case of Fig. 6. Fig. 7, on the other hand, presents the corresponding spectrum for the case of uncorrelated incident waves, which is independent of \( x \). Comparison of Figs. 5 and 7 indicates only minor differences; hence, the correlation between the incident waves seems to have only marginal effect on the cross-correlation spectrum. Conversely, inspection of a cross-correlation spectrum for an unknown incident wavefield should not be expected to reveal the degree of correlation, if any, between the components.

3.2 **Effect of Wave Content**

Figures 7 through 15 present the cross-correlation spectra at three different elevations \( (z = 0, z = 40 \) and \( z = 140 \) m) for wave fields having:

a) A broad incidence range: \( q(\theta) = 1 \) for \( 0 \leq \theta \leq \frac{\pi}{2} \) \( \Rightarrow \) diffuse seismic input (Figs. 7, 10, 13)

(continued on page 35)
$\omega_0 = 6\pi \text{ rad/sec}$

$\beta = 0.25$
$\omega_0 = 6\pi \text{ rad/sec}$

$\beta = 0.25$

Fig. 13
$\omega_0 = 6\pi \text{ rad/sec}$
$\beta = 0.25$

Fig. 16 a)
Fig. 17

$C_s = 300$

$H = 0$

$\Omega_0 = 6\pi \text{ rad/sec}$

$\beta = 0.25$
\( \omega_0 = 6\pi \text{ rad/sec} \)
\( \beta = 0.25 \)

Fig. 19
b) A shallow incidence range:

\[ g(\theta) = 1 \text{ for } 0 \leq \theta \leq \frac{\pi}{4} \] shallow, focused seismic input (Fig. 8, 11, 14).

c) A steep incidence range:

\[ g(\theta) = 1 \text{ for } \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \] steep, focused seismic input (Fig. 9, 12, 15).

With reference to Figs. 7, 8 and 9, we observe first that the intersection of the spectra with the \( \rho - \tau \) plane is the same in all 3 cases, since for \( L = 0 \), we have the autocorrelation spectrum \( \rho(\tau) \) corresponding to the Kanai-Tajimi power-spectral density function.

Next, we notice that the rate of decay of the spectrum with \( L \) is much faster for the diffuse (incoherent) input than for the more focused (coherent) shallow and steep inputs. In fact, for a seismic motion generated by an SH wave coming from a single direction (perfect coherence), the cross-correlation spectrum does not decay at all, but constitutes a cylindrical surface having the autocorrelation function \( \rho(\tau) \) as directrix and a generatrix with an inclination with respect to the \( L \) axis equal to \( \tan \alpha = L/\tau = \) phase velocity. This extreme case is approximated, to some degree, by the two more focused inputs, particularly the shallow incidence case. Furthermore, the inclination (in the first quadrant) of the projection of the hill crests onto the horizontal plane in all three cases is somewhat less than the phase velocity associated with the average incidence angle \( \bar{\theta} \). For example, in Fig. 7, the phase velocity associated with the average incidence angle \( \bar{\theta} = 45^\circ \) is \( c = c_s / \sin 45^\circ = 424 \text{ m/s} \); hence, a distance of \( L = 200 \text{ m} \) (edge of plot) is travelled in \( 200/424 = 0.47 \text{ s} \), which is somewhat less than the \( \tau \) coordinate of the first major peak observed along the edge \( L = 200 \text{ m} \). Observe also that while in both Figs. 8 and 9, the seismic energy arrives within a \( 45^\circ \) dihedron, Fig. 8 exhibits a much larger degree of coherence than Fig. 9; this follows from the fact that there is only a modest change in apparent phase velocity in the
range of $45^\circ < \theta < 90^\circ$, namely from 424 m/s to 300 m/s, whereas there is a very large change in the interval $0 < \theta < 45^\circ$, namely, from $\infty$ to 424 m/s.

Thirdly, we observe that the spectra are not symmetric with respect to the $\rho - \tau$ plane, and in fact, that the peaks (i.e., the largest cross-correlations) are substantially higher in the first quadrant than in the second; this is a consequence — indeed, an indication — of the fact that the waves are traveling in the positive x direction (from west to east) (the third and fourth quadrants are identical to the first and second quadrants, and can be obtained rotating the plot 180° about the $\rho$ axis). Furthermore, the direction of the hills and valleys in the second quadrant is more heavily influenced by the wave components with steeper angles of incidence. Indeed, for both Figs. 7 and 8, there are valleys in the second quadrant that run nearly parallel to the L axis. This follows from the fact that only the vertical components can contribute significantly to the correlation at large negative L and positive $\tau$, for waves propagating in the positive x direction.

3.3 Correlation with Points Underneath the Surface

Figs. 10, 11, and 12 present the cross-correlation spectra (relative to $x = 0, z = 0$) computed at a depth $z = 40$ m; also, Figs. 13, 14 and 15 show the spectra at a depth $z = 140$ m. The corresponding spectra at the free surface are, again, shown in Figs. 7, 8, and 9. While the spectra for points underneath the surface exhibit some of the same characteristics observed earlier in the surface spectra, they differ in that the principal peak, which is now less than one, is shifted by an amount $\tau$ approximately equal to the travel times of the waves between the elevation examined and the free surface.

3.4 Semi-Circle Noise with Kanai-Tajimi Spectrum

Fig. 16 shows the correlation spectrum for waves coming uniformly from
all directions (semi-circle noise with Kanai-Tajimi power spectrum), which is symmetric with respect to both the $\rho-T$ and the $\rho-L$ planes. It can be shown that this spectrum can be obtained also from the one in Fig. 7 by merely averaging the 1st and second quadrants, i.e., superimposing Fig. 7 onto its mirror image (w/r to the $\rho-T$ plane) and taking the average. This interpretation is useful for understanding the features exhibited by this spectrum: the waves travelling from west to east (in the positive x direction) contributes mostly to the spectrum in the first quadrant, while the waves traveling in the east-west direction (negative x direction) contribute mainly in the second quadrant. It is remarkable that, while the semi-circle noise input has no preferred directions, the cross-correlation spectrum nevertheless exhibits clearly the phase velocities associated with waves coming at approximately 50° with respect to the vertical. Thus, observation of similar features in an empirically obtained spectrum could lead to the erroneous conclusion that most of the seismic energy comes along those (or other) preferred directions.

3.5 Non-uniform Input (direction bias)

We consider next the case of SH waves coming in the interval $0 \leq \theta \leq 90^\circ$ with an amplitude bias

$$g(\theta) = e^{-\frac{(\theta-45^\circ)^2}{500}} \quad (\theta \text{ in degrees})$$

The spectrum for this case is shown in Fig. 17. Comparison with Fig. 7 (no bias) demonstrates that the spectra for the two cases are very similar, except in the second quadrant, where the valleys that are parallel to the L axis are missing in Fig. 17. The reason, of course, is that the biased input has little energy propagating vertically up. Also, the "mountain ranges" in the first quadrant associated with the 45° direction are stronger in Fig. 17, as expected.
3.6 Correlation Spectra Along Arbitrary Directions

Up to this point, we have examined cross-correlation spectra at three elevations (z = 0, 40, 140 m) in which the L parameter represents the horizontal distance from the origin. In Figs. 18, 19 and 20, on the other hand, we present spectra along other directions, namely $\theta = 45^\circ$, $\theta = 0$, and $\theta = -45^\circ$.

For example, in Fig. 18, the coordinate L represents the $45^\circ$ diagonal distance to the origin at the free surface. (Hence, negative values for L are not meaningful). Since the direction L opposes the average direction of propagation, no patterns associated with preferred directions are evident in this case.

Fig. 19, on the other hand, exhibits a preferred direction, namely, the vertical. This is caused by the fact that the reflected waves (but not the incident waves!) associated with the vertical components of the wave field contribute significantly to the correlation coefficient along this direction for positive $\tau$.

It is interesting to observe that Fig. 19 is identical to the first quadrant of Fig. 16 for semicircle noise. To understand why, consider the two cases shown in Fig. 21. In 21a), an SH wave coming from a single direction 0 relative to

![Diagram](image)

Fig. 21
the vertical reflects completely at the free surface. The motion along $z = L$ is

$$u'(L,t) = v(t - \mu L) + v(t + \mu L)$$  \hspace{1cm} (29)$$

where $\mu = \cos \theta / c_S = \text{vertical slowness}$. On the other hand, in Fig. 21b, the motion along $x = L$, generated by two SH waves coming from directions $\theta \pm (\pi/2 - \theta)$ relative to the vertical, is (the factor 2 arises as a result of the reflections)

$$u''(L,t) = 2[v(t - \mu L) + w(t + \mu L)]$$  \hspace{1cm} (30)$$

with a horizontal slowness $\mu$ which is identical to the one above, because of the complementary angle chosen.

Considering first equation (30), we note that both $v$ and $w$ have the same autocorrelation function $R(\tau)$, since we assumed that all the waves have a Kanai-Tajimi auto-spectrum. The cross-correlation coefficient is then

$$\rho = \frac{E[u''(L,t + \tau) u''(0, t)]}{E[u''(0, t) u''(0, t)]}$$  \hspace{1cm} (31)$$

Expanding the numerator, we obtain

\begin{align*}
E[v(t + \tau - \mu L) v(t)] &= R(\tau - \mu L) \\
E[w(t + \tau + \mu L) w(t)] &= R(\tau + \mu L) \\
E[v(t + \tau - \mu L) w(t)] &= \rho_{vw} R(\tau - \mu L) \\
E[v(t) w(t + \tau + \mu L)] &= \rho_{vw} R(\tau + \mu L)
\end{align*}

in which $\rho_{vw}$ gives the degree of cross-correlation between $v$ and $w$. The terms in the denominator are obtained setting $\tau, L$ equal to zero. Hence
\[
\rho = \frac{4 \left[ R(\tau - u_L) + R(\tau + u_L) \right] \left[ 1 + \rho_{vw} \right]}{4 \left[ R(0) + R(0) \right] \left[ 1 + \rho_{vw} \right]}
\]

\[
= \frac{[R(\tau - u_L) + R(\tau + u_L)]}{[2R(0)]}
\]

(33)

The above expression implies that the coefficient of cross-correlation \( \rho \) is independent of \( \rho_{vw} \). Hence, \( \rho \) does not depend on whether \( u, w \) are perfectly correlated (as in Fig. 21a and eq. 29) or perfectly uncorrelated (as in Fig. 21b and eq. 30). The cases of quarter- and semi-circle noise considered earlier in Figs. 16, 19 are generalizations of the above case, requiring additional integrations that do not affect the above conclusion. (Note that \( L \) is measured along different directions in Figs. 16, 19)

3.7 Influence of spectral density function shape

The cross-correlation spectra that were developed in the previous sections were based on a wavefield characterized by a Kanai-Tajimi spectrum, with \( \beta = 0.25 \) and \( \omega_0 = 6\pi \text{ rad/sec} \). Fig. 2 shows that this spectrum exhibits a sharp peak at \( \omega_0 \); this in turn implies an autocorrelation function (Fig. 3) with weakly attenuated oscillations. One may wonder then if the conclusions reached earlier are also valid for other spectral shapes, and particularly, for a white-noise input. The computations were thus repeated for a bandlimited white-noise with a cutoff frequency \( \Omega = 10\pi \left(5H_2\right) \), a realistic value for earthquakes on firm ground.

Fig. 22 shows the autocorrelation function for this case; as can be seen, it has more oscillations than that on Fig. 3, and it decays faster.

Fig. 23, on the other hand, presents the cross-correlation spectrum for band-limited white-noise waves with a broad incidence range (0-90°). Comparison with the corresponding spectrum for Kanai-Tajimi noise (Fig. 7) reveals some
differences, although the principal features are preserved. The more important
difference is that the zero time lag function $\rho(0, L)$ never changes sign; also,
Fig. 23 exhibits more secondary peaks and valleys. Nevertheless, the conclu-
sions stated earlier for the Kanai-Tajimi noise are valid also for the band-
limited white-noise, particularly the observation concerning the apparent angle
of incidence.

\[ \rho(\tau, 0) = \frac{\sin 10\pi |\tau|}{10\pi |\tau|} \]

Fig. 22
4. Deconvolution of Cross-correlation Spectra

The goal, in this chapter, is the recovery of the wave content (i.e., the function $g(\theta)$) and the determination of the underground spectra from an analysis of the cross-correlation spectrum at the site surface for the simple SH wave models considered earlier. A successful accomplishment of this goal would pave the way for eventual generalizations of the procedure for estimations of the wave content and underground spectra for more complex wave patterns (involving not only SH waves, but also P, SV and surface waves) from a statistical analysis of the motions recorded on the free surface at an actual site.

4.1 Wave content implied by surface spectrum:

Assuming that the cross-correlation spectrum corresponds to a displacement field generated by plane SH waves propagating at various angles in a halfplane, having all the same spectral shape, and being mutually uncorrelated, it is possible - at least in principle - to recover the wave amplitudes from that spectrum. In fact, applying a double Fourier Transform to the cross-correlation function, the following equation is obtained (see appendix 5):

$$g^2(\text{arc sin}(\frac{kC_s}{\omega})) = \frac{\omega}{C_s} \cdot \frac{\overline{\rho(\omega,k)}}{\overline{\rho(\omega,0)}} \sqrt{\frac{1}{\frac{|kC_s|}{\omega}}}$$

(34)

where $g$ represents the intensity of the wave coming at an angle $\theta$ such that $\sin \theta = \frac{kC_s}{\omega}$; $\overline{\rho(\omega,k)}$ is the double Fourier transform of $\rho(t,L)$ in time $(t-\omega)$ and space $(L-k)$; and $\overline{\rho(\omega,0)}$ is Fourier transform in time $(t)$ of the auto-correlation function $\rho(\omega) = \rho(\omega,L=0)$. The denominator $\overline{\rho}$ is equal to the unit area spectral density function of each wave. For given values of $\omega$, $k$, one could then determine $g(\theta)$ from the above equation.

Even though the above procedure is exact for the ideal wave model postulated, it is difficult to implement it in practice because of the lack of sufficient
information in the \( \tau - L \) space (plots are truncated at some arbitrary values); furthermore, in a real situation the information is available only at a restricted number of points, and errors (noise) are normally present in the data. The truncation error can be particularly important when the motion contains a significant fraction of waves arriving at a shallow angle, since they influence the cross-correlation spectrum up to large separations \( L \). Consider, for example, two waves having a small difference in their angles of incidence; then the difference in their slownesses is large only if the waves travel nearly vertically, whereas it is small if the waves travel nearly horizontally. Conversely, for a given difference in slownesses, the angular distance between travel paths is much larger for nearly horizontally propagation waves than for nearly vertically propagating waves. Hence, the wave components with shallow angles of incidence cannot be separated (by two observers at a distance \( L \) from one another) as well as those with steep angles of incidence, since the former "stay together" for greater distances on the free surface than the latter. Their resolution requires then that the cross-correlation spectrum be available for large separations \( L \), which may not be the case in a practical situation.

Equation 34 points also to other potential difficulties (Fig. 24). On the one hand, the term on the right should be constant for a constant ratio \( k/\omega \), since it corresponds to a single direction of travel, \( \sin \theta = kC_S/\omega \). On the other hand, \( \tilde{\varphi}(\omega, k) \) should be zero for \( \omega < |k|C_S \), since otherwise the corresponding angle of incidence would be imaginary. In practice, these restrictions will not be satisfied, in part because of truncation and numerical errors, but more importantly, because the simple SH-wave model postulated may not be appropriate to describe the situation at hand, since it neglects attenuation and scattering in the medium as well as the presence of waves with different real speeds (such as \( P \) and \( S \) waves).
With reference to Fig. 24, a section at a specific frequency $\omega = \bar{\omega}$ illustrates how the doubly transformed spectrum $\rho$ is band-limited in $k$; thus, $\overline{\rho(\omega,L)}$ must be band-unlimited in $L$. Since the truncation process during the numerical computation of the transform imposes a fictitious limit on the spectrum $\rho$ in $L$, it follows that $\rho$ will be truncated as well, and so $\rho$ will be band-unlimited; hence, the restriction $\overline{\rho(\omega,k)} = 0$ for $|k| > \omega/C_s$ will be violated.

$$\frac{\omega \overline{\rho(\omega,k)}}{C_s \overline{\rho(\omega,0)}} \sqrt{1 - \frac{kC_s}{\omega}}$$

$\omega = \omega/C_s$  

$\omega = \omega/C_s$

4.2 Underground cross-correlation spectra

Perhaps more interesting than the wave content itself is the direct determination of the cross-correlation spectrum at depth from the known surface spectrum. For the model at hand, this can be achieved, at least formally. Namely,
from equations 25), 34) and the results of appendix 5, we can write

\[ \rho(\tau, L, H) = \frac{\omega}{C_s} \int_{-\infty}^{\pi/2} \int_{-\infty}^{-\pi/2} \rho(\omega, k) \cos \theta e^{i(\omega \tau - kL)} \cos(\omega H \cos \theta / C_s) \frac{\omega}{C_s} d\omega d\theta \]

in which \( k = \frac{\omega}{C_s} \sin \theta \). Noting that \( \cos \theta d\theta = \frac{C_s}{\omega} \sin \theta d\theta \), we can write equation 35 as

\[ \rho(\tau, L, H) = \int_{-\infty}^{\infty} \rho(\omega, k) \cos \left( H \sqrt{\frac{\omega}{C_s}} k \right) e^{i(\omega \tau - kL)} \frac{\omega}{C_s} d\omega dk \] (36)

Since \( \rho(\omega, k) \) is (or should) be zero for \( k > \omega / C_s \), it follows that the limits in the second integral could be changed to \(-\infty, \infty\) without affecting the results of the operation. This equation is a two-dimensional Fourier Transform in \( \omega \) and \( k \); it is analogous to the deconvolution integral for a surface antiplane motion in a homogeneous medium, with the cosine term playing the role of a Transfer Function. This could have been anticipated, since the Expectation is a linear operation that is interchangeable with the convolution integral. Hence, the deconvolution of surface spectra in layered media, perhaps involving also SV-P waves, is expected to follow the rules of linear deamplification theory, along the lines of the Haskell-Thompson procedure.

Evaluation of the above equation with the FFT algorithm is not necessarily straightforward, since the cross-correlation function \( \rho \) and its associated transform \( \tilde{\rho} \) may not be defined properly for the reasons indicated in the previous section.

5. Conclusions

This report investigated some of the properties of cross-correlation spectra which are associated with simple wave fields, namely, plane SH waves propagating at various angles in a homogeneous halfplane. A number of factors that could be
important in a real situation, such as three-dimensional effects, the presence of multiple wave types, attenuation, scattering, etc. have been disregarded in this work to keep the formulation reasonably simple. Nevertheless, some important conclusions that are believed to be relevant for more complex cases can be drawn at this time:

a) Since the degree of correlation between the wave-trains propagating along distinct directions has only a minor effect on the cross-correlation spectra, and since in practice these wave-trains are at most only weakly correlated, then the random field can be assumed to be locally homogeneous, (except perhaps in the immediate vicinity of seismic sources).

b) Strong peaks in the first quadrant of the cross-correlation spectrum and weak ones in the second quadrant are evidence that the motion is the result of waves propagating from left to right. Shallow components contribute mostly to the correlation in the 1st quadrant, while steep (near vertical) components influence the second as well.

c) Rapid decay of the spectra points to diffuse (incoherent) input, while slow decay indicates a more focused input (waves originating mostly from a single direction).

d) Continuous "mountain ranges" and "valleys" in the spectra are not necessarily evidence of focused inputs, i.e., they are not good indicators of preferred directions, as the semicircle noise case (Fig. 16) clearly illustrates. Hence, observation of such features in empirically obtained spectra could lead to
the erroneous conclusion that most of the seismic energy comes from a single direction.

e) In principle, the cross-correlation surface spectrum still contains sufficient information to determine the incident wave amplitudes as well as the spectra at arbitrary depths. In practice, however, these computations are hampered by insufficient data, modeling errors and noise which lead to distortions in the correlation spectrum and its associated transform.

f) Deconvolution of surface spectra for general motion fields could be achieved with generalizations of the linear (de)-amplification theory formulated by Haskell and Thompson. The advantage of a direct evaluation of the underground spectra is that it avoids determining first the incident wave field.

g) Cross-correlation functions associated with different spectral density functions of the incident wave field exhibit similar characteristics. Hence, the assumption of a common shape for the waves coming at various angles is probably of little consequence (as far as the results of this study are concerned), and all of the above conclusions still apply.
References


Appendix 1

\[ u(x,t) = \int_{-\infty}^{\infty} v(s, t-sx) ds \]

\[ U(x, \omega) = \int_{-\infty}^{\infty} v(s, t-sx) e^{-i\omega t} ds dt = \int_{-\infty}^{\infty} v(s, t-sx) e^{-i\omega t} ds \int_{-\infty}^{\infty} v(s, t-sx) e^{-i\omega (t-sx)} e^{-i\omega s} ds \]

\[ = \int_{-\infty}^{\infty} V(s, \omega) e^{-i\omega sx} ds \]

\[ U(\xi, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(s, \omega) e^{-i\omega sx} e^{i\xi x} ds dx = \int_{-\infty}^{\infty} V(s, \omega) e^{-i\xi (\omega s-\xi)} ds \int_{-\infty}^{\infty} dx \]

\[ = \int_{-\infty}^{\infty} V(s, \omega) \int_{-\infty}^{\infty} e^{-i\xi (\omega s-\xi)} dx ds = 2\pi \int_{-\infty}^{\infty} V(s, \omega) \delta (\omega s-\xi) ds = 2\pi V(\xi, \omega)_\omega \]
Appendix 2

\[ u(x,t,z) = \frac{1}{2} \int_{-S}^{S} \left[ v(s,t-sx-twz) + v(s,t-sx+wz) \right] ds \]

\[ v(\xi,\omega,z) = \frac{1}{2} \int_{-S}^{S} \int_{-\infty}^{\infty} v(s,\omega) \left[ e^{-i(x(s-\xi)} e^{-i\omega wz} + e^{-i(x(s+\xi)} e^{i\omega wz} \right] d\omega ds = 2\pi \cdot v(\xi/\omega,\omega) \cos \left( \sqrt{\omega^2 s^2 - \xi^2} \right)/\omega \]
Appendix 3

\[ C(x, L, \tau) = E[u(x, t) \cdot u(x+L, t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(x, t) \cdot u(x+L, t+\tau) dt = \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v(s, t-sx) \cdot v(s', t+s'-(x+L)) ds \cdot ds' dt = \]

\[ = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} v(s, t) \cdot v(s', t+s+sx-s'(x+L)) dt \cdot ds \cdot ds' = \]

\[ = \int_{-S}^{S} \int_{-S}^{S} R(s, s', t+sx-s'(x+L)) ds \cdot ds' = \]

\[ = \int_{-S}^{S} \int_{-S}^{S} \int_{-\infty}^{\infty} S(s, s', \omega) e^{i\omega((t+sx-s'(x+L))} ds \cdot ds' \cdot d\omega = \]

The spectral density at a point is given by:

\[ S(x, \omega) = \int_{-S}^{S} \int_{-S}^{S} S(s, s', \omega) e^{i\omega((x-s-\omega s')} ds \cdot ds' \]

Appendix 4

\[ \eta = \sqrt{S^2 - s^2} \]

\[ u(t, x, z) = \frac{1}{2} S \int_{-\xi}^{\xi} v(s, t - sx - \eta z) \, ds + \frac{1}{2} S \int_{-\xi}^{\xi} v(s, t - sx + \eta z) \, ds \]

\[ C(x, z, \tau, L, H) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(t, x, z) \cdot u(t + \tau, x + L, z + H) \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{1}{4} \left[ v(s, t - sx - \eta z) v(s', t - s' x + \mu z + \lambda) + v(s, t - sx + \eta z) v(s', t - s' x + \mu z + \lambda) \right] ds \, ds' \]

where \[ \eta = \sqrt{S^2 - s^2}, \quad \mu = \sqrt{S^2 - \xi^2}, \quad \lambda = \tau \frac{s}{L - \mu H}, \quad \lambda' = \tau \frac{s}{L + \mu H} \]

Interchanging the integrals and substituting \( \mathcal{R} \) by its Fourier Transform

\[ C(x, z, \tau, L, H) = \frac{1}{4} \int_{-\infty}^{\infty} S(s, \omega) [e^{i\omega(s - t)} x e^{i\omega(\eta - \mu) z} e^{i\omega \lambda} + e^{i\omega(\eta + \mu) z} e^{i\omega \lambda'} + e^{i\omega(-s - \hat{\mu}) z} e^{i\omega \lambda} + e^{i\omega(-\hat{\mu} - s) z} e^{i\omega \lambda'}] \, ds \, ds' \]
Appendix 4 continued

Taking \( S(s,\omega) = S(\omega).g(s).g(s')\delta(s'-s) \) then

\[
\mathcal{C}(z,\tau, L, H) = \frac{1}{4} \int_{-S}^{S} g^2(s) S(\omega) \left( e^{i\omega} + e^{i\omega(2\sqrt{s^2-H^2}-z)} + e^{-i\omega(2\sqrt{s^2-H^2}+z)} + e^{i\omega} \right) d\omega ds
\]

We see that \( \mathcal{C} \) depends on \( z \). Let's take \( z = 0 \) (one point is at the surface)

\[
\mathcal{C}(\tau, L, H) = \frac{1}{2} \int_{-S}^{S} g^2(s) S(\omega) e^{i\omega(\tau-L)} \left( e^{i\omega\sqrt{s^2-H^2}} + e^{-i\omega\sqrt{s^2-H^2}} \right) d\omega ds
\]

\[
\theta(\tau, L, H) = \int_{-\infty}^{\infty} S(\omega) d\omega . \quad g^2(s) ds
\]

or changing \( s \) for \( \theta \) and \( S \) for \( 1/\sin \theta \)

\[
\frac{1}{2} \int_{-\pi/2}^{\pi/2} g^2(\theta) S(\omega) e^{i\omega(\tau-L \sin \theta)} \left( e^{i\omega \cos \theta/c_{s}} + e^{-i\omega \cos \theta/c_{s}} \right) d\omega d\theta
\]

\[
\theta(\tau, L, H) = \int_{-\pi/2}^{\pi/2} g^2(\theta) d\theta
\]

(Note that \( g(s) \) and \( g(\theta) \) are not the same function; however, we define these functions in such a way that \( g^2(s) ds = g^2(\theta) d\theta \).)
Appendix 5

At the surface $\phi(t,L)$ is given by (see eq. (25)):

$$
\rho(t,L) = \frac{\frac{\pi}{2}}{\int_{-\pi/2}^{\pi/2} \frac{g^2(\theta)}{d \theta} \int_{-\infty}^{\infty} S(\omega) e^{i \omega (t - \sin \theta) L/C_s} d \omega d \theta}
$$

Because $\rho(t,L)$ is a ratio between two quantities ($c(t,L)/c(o,o)$), $g^2(\theta)$ can only be found in shape and not its magnitude. For simplicity, assume $\int_{-\pi/2}^{\pi/2} g^2(\theta) d \theta = 1$. Also $\int_{-\infty}^{\infty} S(\omega) d \omega = S(\omega)$, the unit area spectral density ($s(\omega)$ is the Fourier transform of $\rho(t,o)$).

Hence, it can be written:

$$
\rho(t,L) = \frac{\frac{\pi}{2}}{\int_{-\pi/2}^{\pi/2} \frac{g^2(\theta)}{d \theta} \int_{-\infty}^{\infty} S(\omega) e^{i \omega (t - \sin \theta) L/C_s} d \omega d \theta}
$$

Fourier transforming $\rho(t,L)$ in $t$ (*)

$$
\rho(\omega,L) = \int_{-\infty}^{\infty} \frac{\pi}{2} \frac{g^2(\theta)}{d \theta} \int_{-\infty}^{\infty} s(\omega') e^{i \omega'(t - \sin \theta) L/C_s} e^{-i \omega t} d \omega' d \theta d t
$$

and interchanging integrals

$$
\rho(\omega,L) = \int_{-\infty}^{\infty} \frac{\pi}{2} \frac{g^2(\theta)}{d \theta} \int_{-\infty}^{\infty} s(\omega') e^{-i \omega' \sin \theta} \frac{L}{C_s} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{i (\omega' - \omega) t} d \tau d \theta d \omega'
$$

$$
\int_{-\infty}^{\infty} \frac{1}{2\pi} \delta(\omega' - \omega)
$$

(*): The factor $1/2\pi$ in the Fourier Transform is used here when transforming from $t$ to $\omega$ (and $L$ to $k$), for consistency with the Wiener-Khinchine relations.
Appendix 5 continued

$$
\tilde{\rho}(\omega, L) = \int_{-\pi/2}^{\pi/2} s(\omega) g^2(\theta) e^{-i\omega \sin \theta} \frac{L}{C_s} \, d\theta
$$

For $L = 0$, we have

$$
\tilde{\rho}(\omega, 0) = s(\omega)
$$

Furthermore, Fourier transforming in $L \rightarrow k$

$$
\tilde{\rho}(\omega, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\pi/2}^{\pi/2} s(\omega) g^2(\theta) e^{-i\omega \sin \theta} \frac{L}{C_s} e^{ikL} \, d\theta \, dL
$$

But

$$
\int_{-\infty}^{\infty} e^{iL(k-\omega \sin \theta/C_s)} dL = 2\pi \delta(k-\omega \sin \theta/C_s)
$$

$$
\tilde{\rho}(\omega, k) = \int_{-\pi/2}^{\pi/2} s(\omega) g^2(\theta) \delta(k-\omega \sin \theta/C_s) \, d\theta
$$

Changing variables, $\mu = \omega \sin \theta/C_s$, $d\mu = \frac{\omega \cos \theta}{C_s} \, d\theta$, we obtain

$$
\tilde{\rho}(\omega, k) = \int_{-\omega/C_s}^{\omega/C_s} s(\omega) g^2(\arcsin \frac{\mu}{\omega}) \frac{\delta(k-\mu)}{\sqrt{1-\mu^2}} \, d\mu
$$

Hence

$$
\tilde{\rho}(\omega, k) = s(\omega) \frac{C_s}{\omega} g^2(\arcsin \frac{kC_s}{\omega}) / \sqrt{1 - \left(\frac{kC_s}{\omega}\right)^2}
$$
\[ g^2(\text{arcsin} \left( \frac{kC}{\omega s} \right) = \frac{\omega \Omega(\omega, k)}{C_s s(\omega)} \sqrt{1 - \left( \frac{kC}{\omega} \right)^2} \]

\[ = \frac{\omega \Omega(\omega, k)}{C_s \rho(\omega, 0)} \sqrt{1 - \left( \frac{kC}{\omega} \right)^2} \]
Appendix 6

When the waves are fully correlated, $A(\omega x)$ can also be found explicitly, in the case where $g(\phi)$ is taken constant in the first quadrant (as shown in fig. 4).

$$A(\omega x) = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(\phi)g(\gamma) \cos[\omega x (\sin\phi - \sin\gamma)] C_s d\phi d\gamma =$$

$$= \int_{0}^{\pi/2} \int_{0}^{\pi/2} g^2 \left[ \cos \left( \frac{\omega x}{C_s} \sin\phi \right) \cos \left( \frac{\omega x}{C_s} \sin\gamma \right) \right. + \sin \left( \frac{\omega x}{C_s} \sin\phi \right) \sin \left( \frac{\omega x}{C_s} \sin\gamma \right) \] \ d\phi d\gamma$$

Splitting the integrals and dividing by $A(0) = \frac{\tau^2}{L} g^2$, one obtains

$$\frac{A(\omega x)}{A(0)} = \left[ \frac{2}{\pi} \int_{0}^{\pi/2} \cos \left( \frac{\omega x}{C_s} \sin\phi \right) \ d\phi \right]^2 + \left[ \frac{2}{\pi} \int_{0}^{\pi/2} \sin \left( \frac{\omega x}{C_s} \sin\phi \right) \ d\phi \right]^2 = J_0^2 \left( \frac{\omega x}{C_s} \right) + E_0^2 \left( \frac{\omega x}{C_s} \right)$$

Where $J_0$ represents the Bessel function of order zero and $E_0$ represents the Weber function of order zero. In this case $\rho(\omega x')$ will be given by:

$$\rho(\omega x') = \frac{A[\omega(x_0+x')] / A[\omega x_0]}{J_0^2 (\omega x_0/C_s) + E_0^2 (\omega x_0/C_s)} = \frac{J_0^2 (\omega x_0/C_s) + E_0^2 (\omega x_0/C_s)}{J_0^2 (\omega x_0/C_s) + E_0^2 (\omega x_0/C_s)}$$

Noting that $E_0 = -H_0$, the Struve function of order zero, and substituting them by their asymptotic expansions for large arguments, $J_0$ and $E_0$ become
Appendix 6 continued

\[ J_0(\alpha) \sim \sqrt{\frac{2}{\pi \alpha}} \cos \left( \alpha \cdot \frac{\pi}{4} \right) \quad E_0(\alpha) = -H_0(\alpha) \sim -\sqrt{\frac{2}{\pi \alpha}} \sin \left( \alpha \cdot \frac{\pi}{4} \right) - \frac{2}{\pi} \frac{1}{\sqrt{\alpha}} - \frac{1}{\alpha^3} + \frac{1^2 + 3^2}{\alpha^5} - \ldots \]

If only the term \( \frac{1}{\alpha} \) is kept in the \( E_0 \) approximation then

\[ J_0^2(\alpha) + E_0^2(\alpha) \sim \frac{2}{\pi \alpha} \left( \cos^2 \left( \alpha \cdot \frac{\pi}{4} \right) + \sin^2 \left( \alpha \cdot \frac{\pi}{4} \right) \right) + \frac{4}{\pi} \frac{1}{\alpha} \sqrt{\frac{2}{\pi \alpha}} \sin \left( \alpha \cdot \frac{\pi}{4} \right) + \frac{4}{\pi^2 \alpha^2} = \]

\[ \frac{2}{\pi \alpha} + \frac{4}{\pi} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\alpha}} \sin \left( \alpha \cdot \frac{\pi}{4} \right) + \frac{4}{\pi^2 \alpha^2} = \frac{2}{\pi \alpha} \]

Substituting in the expression for \( \rho(\omega x') \), one obtains

\[ \rho(\omega x') \sim \frac{2}{\pi \omega (x_0 + x')/c_s} = \frac{x_0}{x_0 + x'} + 1 \text{ if } x_0 \gg x' \]