STOCHASTIC RESPONSE OF RIGID FOUNDATIONS

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SUMMARY

While the study of kinematic interaction (i.e. the dynamic response of massless foundations to seismic loads) calls, in general, for advanced analytical and numerical techniques, an excellent approximation was proposed recently by Iguchi.1.2 This approximation was used by the authors to analyse embedded foundations subjected to spatially random SH-wave fields, i.e. motions that exhibit some degree of incoherence. The wave fields considered ranged from perfectly coherent motions (resulting from seismic waves arriving from a single direction) to chaotic motions, resulting from waves arriving simultaneously from all directions. Additional parameters considered were the shape of the foundation (cylindrical or rectangular) and the degree of embedment. It was found that kinematic interaction usually reduces the severity of the motions transmitted to the structure, and that incoherent motions do not exhibit the frequency selectivity (i.e. narrow valleys in the foundation response spectra) that coherent motions do.

INTRODUCTION

Earthquake ground motions have a high variability in time and space; thus, they are usually treated as stochastic (or random) processes. In the case of earthquakes of long duration, these processes are, in general, assumed to be stationary in time and homogeneous in space, at least over small distances. The free-field motion at the site is then described either by its power spectral density function or by its response spectrum. For extended structures, in addition, the variability of the ground motion in space may be an important attribute to consider. Thus, rigorous dynamic analyses for such structures should consider the spatial variability of the ground motion, at least in an approximate way.

In general, earthquakes result from abrupt dislocations deep within the ground, from where the seismic energy emanates. The celerity of seismic waves usually increases with depth, which implies that these waves will be refracted to a nearly vertical travel path as they approach the ground surface. However, since other propagation patterns and alternate travel paths may develop because of material heterogeneities of the rock, the waves may arrive at the ground surface from various angles at the same time. This may cause structures in the travel path of the waves to respond in ways that would not be observed if the waves came along the vertical. For example, the torsion of symmetrical structures observed during earthquakes is a consequence of obliquely incident seismic waves;3,4 hence, such waves should be taken into account in the analysis of structures for earthquakes.

On a dimension scale comparable to the size of the structure, the seismic motion can be thought of as the superposition of several waves travelling in the soil along different directions. If the deformations of the ground are sufficiently small so that linearity holds, then the principle of superposition applies and the total

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response of the structure can be obtained as the sum of the effects produced by each single wave. In the general case, the earthquake motion will be caused by many types of body and surface waves. For simplicity, however, the seismic fields to be considered in this paper will be restricted to antiplane SH-wave models.

A rigorous approach to the problem of ideally massless foundations excited by earthquakes (i.e. kinematic interaction) is rather involved, so it is not surprising that analytical solutions are available for only very simple configurations. When dealing with other geometries, numerical solutions with finite elements or boundary elements are necessary. Although these methods can be used to solve the kinematic interaction problem without difficulty, they entail a relative computational expense, which is not always warranted; hence, approximate solutions are often more desirable. One alternative, which was recently proposed by Iguchi, provides good approximations to the kinematic interaction problem, even for embedded foundations. While this method requires knowing a priori the dynamic stiffness of the foundation, it is also possible to use it with engineering approximations to these functions (see, for example Pais and Kausel so that the need for a sophisticated program can be obviated. A brief description and assessment of this method can be found in Pais and Kausel.

In this paper, Iguchi’s method is used to compute the response of foundations subjected to variegated SH-wave patterns. The stochastic response of the foundation is obtained by superimposing the effects of the various wave-trains—characterized each by a spectral density function—and considering the cross-correlations between pairs of different wave-trains. The results presented here are extracted from a more extensive report on the subject by Pais and Kausel. The technique used differs from that employed in similar studies by Luco and Wong and Luco and Mita in that the latter authors solve the kinematic interaction problem exactly (in a numerical sense) and postulate a spatial coherence function for the ground motion; the writers, on the other hand, use approximate solutions for kinematic interaction, but determine the spatial coherence of the motion from the superposition of the assumed incident wave trains. Of course, it would be easy to accomplish also a numerically exact solution with stochastic wave-trains, but the qualitative results would probably not change much while the computational expense would increase substantially.

STOCHASTIC RESPONSE

Consider a rigid, massless foundation which may or may not be embedded in the ground. Its motion is described by a vector \( \mathbf{u}_t \) having at most six components, i.e. three displacements and three rotations:

\[
\mathbf{u}_t^T = [u_x, u_y, u_z, \phi_x, \phi_y, \phi_z]
\]

with the superscript \( T \) denoting the transposed vector. For a solution in the frequency domain, the motion vector \( \mathbf{u}_t \) elicited by a harmonic wave-train propagating at an angle \( \theta \) is of the form (a factor \( e^{i\omega t} \) is implied)

\[
\mathbf{u}_t = T_{\theta} \mathbf{u}_g
\]

where \( u_g \) = the displacement observed at the ground surface in the free field; and \( T_{\theta} = \{t_{\theta l}\} = \) the transfer functions vector relating the free-field motion produced by a harmonic wave-train propagating along direction \( \theta \), with the six motion components \( (t_{\theta l}, l = 1, \ldots , 6) \) of the foundation. These transfer functions may be estimated by Iguchi’s method (see Pais and Kausel).

If, on the other hand, the foundation supports a structure with a discrete number of degrees of freedom, the motion of the structure will be characterized by the well-known equation

\[
\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} = \mathbf{CEu}_t + \mathbf{Ku}_t
\]

in which \( \mathbf{M}, \mathbf{C}, \mathbf{K} \) are the mass, damping and stiffness matrices of the structure; \( \mathbf{E} \) is a geometric transformation matrix representing the rigid body displacements of the structure due to unit displacements and rotations of the foundation; \( \mathbf{u} \) is the vector of absolute displacements; and \( \mathbf{u}_t \) is the motion of the foundation, including kinematic interaction. Inertial interaction, on the other hand, has been deliberately excluded, so as to be able to clearly discern the effects of motion incoherence when compared to those of
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A simple wave passage (indeed, inertial interaction is identically zero for all examples involving massless foundations).

In a formulation in the frequency domain of this problem, the time derivatives are simply \( \dot{u}_r = i\omega u_r \), \( \dot{u}_s = -\omega^2 u_s \), \( \ddot{u}_s = i\omega u \) and \( \ddot{u} = -\omega^2 u \) [omitting the common factor \( \exp(i\omega t) \)]. Equations (2) and (3) then yield for a unit harmonic free field motion \( (u_\phi = 1) \):

\[
(K + i\omega C - \omega^2 M)u = (i\omega C + K)ET_\phi
\]  
(4)

If the damping matrix is assumed to be 'proportional' then equation (4) can be diagonalized by the modal transformation \( u = \Phi x \), with \( \Phi = \{ \phi_{ij} \} \) being the modal matrix for the undamped structure on a fixed base (normalized so that \( \Phi^T \Phi = I \)). After some algebra, this substitution leads to

\[
x = H\Gamma T_\phi = \{ x_j \}
\]
(5a)

\[
u = \Phi H\Gamma T_\phi = \{ u_i \}
\]
(5b)

in which

\[
\Gamma = \{ \gamma_{ji} \} = \Phi^T ME = \Phi^{-1} E
\]
(6)

is the matrix of participation factors, and

\[
H = \text{diag}\left\{ \frac{\omega_j^2 + 2i\beta_j\omega_j}{\omega_j^2 + 2i\beta_j\omega_j - \omega^2} \right\}
\]
(7)

is the diagonal matrix of modal amplification functions. (For hysteretic damping, the term \( 2i\beta_j\omega_j \) should be replaced by \( 2i\beta_j\omega_j^2 \text{ sign}(\omega) \).)

The response of the \( i \)th component of the structure is then

\[
u_i = \sum_{j=1}^{n} \phi_{ij} x_j = \sum_{j=1}^{n} \phi_{ij} h_j \sum_{j=1}^{6} \gamma_{ji} \theta_l
\]
(8)

with \( n \) being the number of structural degrees of freedom. Equation (8) is the transfer function for the absolute motion of the structure due to a unit free field motion; caused by a wave-train with inclination \( \theta \) with respect to the vertical. If a stochastic wave-train with power spectral density function \( S_\theta = S_\theta(\omega) \) is considered instead of a purely harmonic motion, then the spectral density of the response \( S_{\theta\phi} \) will be given by (an asterisk denotes the complex conjugate)

\[
S_{\theta\phi} = |u_i|^2 S_\theta = u_i u_i^* S_\theta
\]
(9)

From equation (8) it can be seen that the squared amplitudes of \( u_i \) will involve cross-products of all modes of the system. However, if the structure has well separated natural frequencies, then it is possible to neglect these cross-products, and to estimate the square of the transfer function simply as

\[
|u_i|^2 = \sum_{j=1}^{n} \phi_{ij}^2 x_j^2 = \sum_{j=1}^{n} \phi_{ij}^2 x_j x_j^*
\]
(10)
or considering equation (8)

\[
x_j x_j^* = |h_j|^2 \sum_{j=1}^{6} \sum_{k=1}^{6} \gamma_{ji} \gamma_{jk} \theta_l \theta_k
\]
(11)

Assume now that wave-trains are arriving at the foundation simultaneously from several directions. In principle, the evaluation of the stochastic response of the foundation under these conditions would need to consider the cross-correlation that may exist between these multiple wave-trains. However, since the latter will have experienced different travel paths as well as travel times from the source (the earthquake focus) to the receiver (the foundation), it follows that their cross-spectral density functions cannot be significant. For this reason, it will be assumed in the following that the cross-correlation between any two wave-trains can be
neglected; more rigorous analyses in which such cross-correlations were included did not produce significantly different response statistics.

Assume, then, a continuum of uncorrelated wave-trains arriving at angles \(-\pi/2 \leq \theta \leq \pi/2\) with respect to the vertical; the spectral response of the structure is thus

\[
S_{u} = \int_{-\pi/2}^{\pi/2} |S_{u}|^2 S_{\theta} d\theta = \int_{-\pi/2}^{\pi/2} |u_{i}|^2 S_{\theta} d\theta
\]  

(12)

Assume further that the waves arriving with the various inclinations have the same variation with frequency \(S_{\omega}\), but different relative amplitude \(g_{\omega}\), i.e. that the spectral density of the wave-trains is of the form

\[
S_{\theta} = g_{\theta}^2 S_{\omega}, \quad \text{with} \quad \int_{-\pi/2}^{\pi/2} g_{\theta}^2 d\theta = 1
\]  

(13)

(This spectral density function is a special case of a more general cross-spectral density function considered by the authors,\(^5\)–\(^10\) namely \(S_{\eta, \theta; \omega} = g_{\theta} g_{\eta} f(\theta' - \theta')S_{\omega}\)). Combining equations (10), (11) and (12), the spectral density of the response is

\[
S_{u} = S_{\omega} \sum_{j=1}^{n} \phi_{i j}^2 |h_j|^2 \sum_{l=1}^{6} \sum_{k=1}^{6} \gamma_{j l} \gamma_{j k} \int_{-\pi/2}^{\pi/2} g_{\theta}^2 t_{i j} t_{i k} d\theta
\]  

(14)

or defining the cross-spectral coefficients

\[
\tau_{i k} = \int_{-\pi/2}^{\pi/2} g_{\theta}^2 \text{Re}(t_{i j} t_{i k}^*) d\theta
\]  

(15)

with Re being the real part operator, and \(i, k = 1, 2, \ldots, 6\). From the symmetry of the double summations in equation (14), it follows that

\[
S_{u} = S_{\omega} \sum_{j=1}^{n} \phi_{i j}^2 |h_j|^2 \sum_{l=1}^{6} \gamma_{j l} \left( \gamma_{j l} \tau_{i l} + 2 \sum_{k=l+1}^{6} \gamma_{j k} \tau_{i k} \right)
\]  

(16)

Notice that the \(\tau_{i k}\) can be computed for all \(i, k\) independently of the structure, since they are only a function of the relative amplitude of the waves and the geometry of the foundation. Indeed, in the absence of a superstructure, the product \(S_{\omega} \{\tau_{i k}\}\) represents the real part of the cross-spectral density matrix for the motion of the foundation. (The imaginary part of the CSDF is not needed, since it cancels out in the summations in equation (14); hence, the summation in equation (16) extends only over the upper triangle.)

**EXAMPLES OF APPLICATION**

Consider first a massless cylindrical foundation with some arbitrary embedment ratio \(E/R\) (\(E = \) depth of embedment, and \(R = \) radius of foundation). The soil has Poisson’s ratio \(\nu = 0.25\) and zero internal damping (\(\beta = 0\)). Referring to equation (15), the values of the transfer functions \(t_{i j}\) (and \(t_{i k}\)) can be evaluated using Iguchi’s approximation (for details, see Pais and Kausel,\(^5\) which includes also analyses for rectangular embedded foundations subject to waves with arbitrary azimuth). If it is further assumed that the seismic waves arrive uniformly (i.e. \(g_{\theta} = \text{constant}\)) in the angular interval \(\theta_1 \leq \theta \leq \theta_2\) (where \(\theta_1, \theta_2\) are arbitrary limits), and that the travel paths of the waves are contained in the same vertical plane, then the cross-spectral coefficients \(\tau_{i k}\) can be computed. (While waves having different azimuths could be modelled with ease, this is not done here, to reduce the number of parameters studied.)

The response of the massless structure under these assumptions is shown in Figures 1 7 for a surface foundation and for an embedded foundation with \(E/R = 1\) and considering several wave patterns. For instance in Figure 1, the waves arrive with equal intensity in the range \(0 \leq \theta \leq 45^\circ\).

The motions shown are referred to the centre of the bottom of the foundation. Also, the rocking and torsional response functions have been multiplied by \(R\) so as to express them in dimensionless form.
Figure 1. Response functions for surface foundations subjected to steeply incident SH-waves

Figure 2. Response functions for surface foundations subjected to shallowly incident SH-waves

Figure 3. Response functions for surface foundations subjected to broadly incident SH-waves
Figure 4. Response functions for embedded foundations subjected to steeply incident SH-waves

Figure 5. Response functions for embedded foundations subjected to shallowly incident SH-waves

Figure 6. Response functions for embedded foundations subjected to broadly incident SH-waves
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Figures 1 to 6 display the variation of the cross-spectral coefficients $\tau_{22}$ (swaying), $\tau_{44}$ (rocking) and $\tau_{60}$ (torsion) with dimensionless frequency $a_0$ ($a_0 = \omega R/C_s$, where $\omega$ = angular frequency; $R$ = radius of foundation; and $C_s$ = shear wave velocity of soil). Figure 7 illustrates, on the other hand, the coupling terms $2\tau_{24}$ (swaying–rocking), $2\tau_{26}$ (swaying–torsion) and $2\tau_{46}$ (rocking–torsion) for one specific wave pattern.

As can be seen, shallowly arriving waves induce more torsion and less rocking than steep waves, while surface foundations do not experience rocking for any angle of incidence. This could have been anticipated from an analysis of the motion pattern in the free field, since the response of the foundation depends importantly on the extension of the foundation in the direction of travel of the waves.

It is important to note that the cross-spectral coefficients cannot, in general, be neglected. Thus, they should not be discarded from equation (16) to obtain the spectral density function $S_{u_i}$.

INFLUENCE OF KINEMATIC INTERACTION ON A SIMPLE STRUCTURE

Filter functions

A simple structural system is studied to assess the influence of kinematic interaction and angle of arrival of the seismic waves on the dynamic response of the system.

The structure is idealized as a single degree of freedom system, i.e., a point mass connected to a cylindrical embedded foundation by a massless column of height $b$. The mass of the foundation is also neglected so that the only dynamic degree of freedom of the system is the horizontal displacement of the structural mass. As a result, any torsion of the foundation will have no effect on the motion of the mass; hence the terms associated with torsion in equation (16) vanish. Also, owing to the axisymmetry of the structure, rocking and horizontal translation occur only in the direction of the free-field motion.

Clearly, the horizontal displacement of the top mass produced by a unit translation or rotation of the foundation is 1 and $b$ respectively; it follows that $\gamma_{12} = 1$ and $\gamma_{14} = b$.

Also, $\phi_{11} = 1$, since the structure on a fixed base has only one mode and its amplification function $h$ is that of a one degree of freedom system. Hence, equation (16) reduces to

$$S_{u_i} = S_0 |h|^2 \left[ \tau_{22} + \left( \frac{b}{R} \right)^2 \tau_{44} + 2 \frac{b}{R} \tau_{24} \right]$$

Equation (17) gives the spectral density function for the mass at the top of the structure.

The term in square brackets in equation (17) represents the influence of kinematic interaction on the response of the system. It is plotted versus frequency $\omega$ in Figures 8–10, for several ratios $b/R$ (0, 1 and 4), wave patterns and an embedment ratio $E/R = 2$. The case $b/R = 0$ presents the influence on the structure of
Figure 8. Response functions at top of 1-dof structure for steeply incident SH-waves, deep embedment

Figure 9. Response functions at top of 1-dof structure for shallowly incident SH-waves, deep embedment

Figure 10. Response functions at top of 1-dof structure for broadly incident SH-waves, deep embedment
the horizontal translation of the foundation alone, since the terms corresponding to rocking vanish. In the examples analysed it was assumed that the foundation had a radius of 20 m, and the celerity of the SH-waves, \( C_s \), was 200 m/s. However, the same results would have been obtained for any other combination of physical parameters satisfying the ratio \( R/C_s = 0.1 \). Three wave patterns were considered: a broad incidence range with waves arriving uniformly within the first quadrant; a steep incidence range, with waves arriving uniformly between 0° and 22.5° (with respect to the vertical); and a shallow incidence range, with waves arriving uniformly between 67.5° and 90°.

From Figures 8–10, it may be observed that, as expected, the influence of rocking increases with height of the structure, especially for vertically propagating waves.

**Response spectra**

In practice, the spectral density function of the response is not a very meaningful quantity to the engineer who, in general may prefer to know the maximum expected value of the response. Thus, he may want to assess the effects of wave passage in terms of response spectra.

The evaluation of the maximum expected value of some stochastic variable, given its spectral density function, is a complex problem, and exact solutions are not available. However, under certain reasonable assumptions, acceptable results can be found. First, it may be assumed that the spectral density function represents a random process whose variation in space for a given time has a Gaussian distribution. This assumption is justified in our case because the seismic motion results from the sum of several uncorrelated components; hence, the probability density functions of the motion will approach, according to the central limit theorem, a Gaussian distribution. Also, the dynamic equation used is a linear equation, and the response of a linear system to a zero-mean Gaussian input is also a zero-mean Gaussian process.

Under these assumptions, it is possible to use relatively simple models to obtain the distribution of the maximum of a random process for a specified direction. Indeed, Vanmarcke\(^{11}\) and Der Kiureghian\(^{12,13}\) proposed some approximate formulae for this purpose.

The procedure referred to previously was applied to the computation of the displacement response spectrum, \( S_d \), for the one d.o.f. system considered earlier. It was assumed that the oscillator had 5 per cent of critical damping. A process duration equal to 100 s was used; the sensitivity of the results to duration, however, is not very pronounced.

The seismic input is given in equation (16) in terms of the spectral density function \( S_{\omega} \), associated with the soil displacements. In practice, however, the spectral density of the ground acceleration \( S_{a_g} \) is more widely used. Hence, in this example, \( S_{a_g} \) will be employed instead of \( S_{\omega} = S_{\omega} \). This also requires substituting the appropriate transfer function for relative displacement \( h_\z \) in place of \( h \) in equation (16).

When kinematic interaction is not taken into account, then equation (17) reduces to

\[
S_{d_\z} = |h_\z| S_{a_g}
\]

(18)

To describe the variation of \( S_{a_g} \) with frequency, the following arbitrary equation was chosen:

\[
S_{a_g} = \begin{cases} 
  f^4 & f \leq 8 \text{ Hz} \\
  0.5 + f^4 (1 - f^2/64) S_\omega & f > 8 \text{ Hz}
\end{cases}
\]

(19)

where \( f \) represents the frequency of the motion (in Hz). This function (normalized by \( S_\omega \)) is shown in Figure 11. Note that, in the low frequency range, \( S_{a_g} \) varies at least as \( f^4 \) or \( (\omega^4) \) so that \( S_{a_g} = (1/\omega^4) S_{a_g} \) does not become unlimited as \( f \) approaches zero.

The response spectra for the relative displacement of the oscillator, \( S_d \), are plotted in Figures 12–15. The full line was obtained neglecting kinematic interaction, while the dashed line includes such interaction. Also shown in these figures are \( S_\z \) and \( S_d \), which represent the pseudo-spectra of velocity and acceleration respectively.

From Figures 12–14, it can be seen that, in this example, kinematic interaction generally reduces the maximum response of the system except when the natural frequency of the oscillator is less than 0.5 Hz, in
which case the difference between the two response spectra is small. However, for tall structures with deeply embedded foundations and vertically propagating waves, the response spectra may increase in certain frequency ranges as a result of kinematic interaction (see Figures 12, 15), as could be anticipated from the results in Figures 8–10.

CONCLUSIONS

Using approximate procedures, the influence of kinematic interaction was evaluated for structures with rigid foundations subjected simultaneously to several obliquely incident SH-waves. For numerical simplicity, it
Figure 13. Response spectra for motion at top of structure, shallowly incident waves, normal embedment

Figure 14. Response spectra for motion at top of structure, broadly incident waves, normal embedment

Figure 15. Response spectra for motion at top of structure, steeply incident waves, deep embedment
was assumed that waves arriving from different directions were completely uncorrelated, although the formulation permits the incorporation of an arbitrary degree of correlation between the seismic waves.

Although only SH-waves in the context of very simple examples were used in this paper to illustrate the possible effects of kinematic interaction, this formulation can be extended easily to other types of waves. It is shown how simple approximations can be of help in understanding the complex kinematic interaction problem and, at least for a preliminary design, how they can be used to estimate the effective seismic input.

As was noted also by Luco and Wong, the effects of spatial incoherence and simple wave passage of a coherent wave are qualitatively similar, i.e. induced rotational motions in the structure and filtering of translational components. There is, however, an additional important effect associated with incoherent motions that wave passage alone does not impersonate, namely the frequency selectivity of the motion at points below the ground surface. When one looks at the Fourier or response spectra of the motions produced by coherent waves at certain locations, one finds that selected frequencies are suppressed because of the existence of standing waves (produced by reflections at the ground surface). If the motion is produced by incoherent waves, however, such nodal points do not exist, and as a result, the spectra below the ground surface do not exhibit narrow valleys at selected frequencies. This can have important consequences for the seismic response and design of structures of limited spatial extent which are buried in the ground.

REFERENCES