BAUTEN IN ERDBEBENGBEBIETEN
STRUCTURES IN SEISMIC REGIONS

Edited by
Prof. Dr.-Ing. Rudolf Damrath
Technical University of Berlin

BERLIN 1984

REPORTS ON COOPERATIVE RESEARCH
INHALTSVERZEICHNIS DER VORTRÄGE ZUM SEMINAR

BAUTEN IN ERDBEBENGEBIETEN

R. Damrath  Dynamische Analyse von Ein-Massen-Systemen
R. Damrath  Dynamische Analyse von Mehr-Massen-Systemen
E. Vanmarcke  Seismic Risk and Stochastic Seismic Analysis of Structures
S. Savidis  Erdbebenwellen im Baugrund
S. Savidis  Dynamische Eigenschaften des Baugrundes

→ E. Kausel  Dynamics of Structures with Foundation Interaction
Ch. Meyer  Konstruieren für den Lastfall Erdbeben
G. Waas  Kernkraftwerke in der Bundesrepublik Deutschland

→ E. Kausel  Earthquakes and Nuclear Power Plants in the USA
Ch. Meyer  Hochhäuser
H. Falkner  Brücken
P. J. Pahl  Betonstaumauern
S. Savidis  Erddämme
G. Waas  Pfahlgründungen
Notes for lectures on
DYNAMICS OF STRUCTURES WITH FOUNDATION INTERACTION

by

Eduardo Kausel
Assoc. Prof. of Civil Engineering
Massachusetts Institute of Technology

presented at the TUB-MIT Seminar on BAUTEN IN ERDBEBENGEBIETEN at the Technische Universität Berlin June 9-11, 1982
Dynamics of Structures with Foundation Interaction

6.1 Introduction

While most engineers in the fields of Geotechnical and Earthquake Engineering are familiar with the concept of Soil-Structure Interaction, it is surprising that no widely accepted definition of the term exists at present. It is in fact difficult to provide an illuminating definition for Soil-Structure Interaction without giving a lengthy explanation instead. There is also disagreement as to the terminology used to describe the various facets of the phenomenon; hence the choice in these pages reflects the particular preference of the authors.

Briefly, Soil-Structure Interaction is a discipline of Applied Mechanics concerned with the development and investigation of theoretical and practical methods for the analysis of dynamically loaded structures, taking into consideration the flexibility and dynamic properties of the supporting soil. In its early stages of development, the theory of Soil-Structure Interaction focused attention on problems involving external loads, i.e., problems in which the dynamic excitation was applied directly onto the structure: examples are the study of (unbalanced) reciprocating machines on elastic foundations, railroad tracks, radar tracking stations and tall structures subjected to wind loads.

More recently, the design of massive or tall structures in seismic areas, and underground facilities resistant to blast loads required extension of the theory to internal loads—that is, to dynamic excitations born within the soil mass. In the first situation, interaction effects arise solely as a result of external and inertial forces being transmitted to the compliant soil; this can be referred to as Inertial Interaction. The mechanical energy thereby transmitted to the ground, and carried away from the structure in the form of stress waves, is called Radiation Damping. On the other hand, additional interaction effects can also arise for earthquake loads because the stiffer structural foundation cannot conform to the distortions of the soil generated by the incident earthquake waves. The structure (or "inclusion") acts like an opaque or reflective object in the path of the incident seismic rays, producing as a result a scattered wave field which modifies locally the motion in the vicinity of the foundation. This effect is referred to as "Wave Passage," "Scattering of Seismic Rays by a Rigid Inclusion," or "Kinematic Interaction"; it depends only on the geometry of the foundation, the soil configuration and the travel path of the
seismic excitation across the soil-structure interface. Finally, the motion that the ground would have experienced if neither the soil had been excavated nor the structure erected is normally called the Free Field solution.

The concepts of Kinematic and Inertial Interaction referred to above are best understood with the aid of the Superposition and Substructure Theorems that follow.

6.2 The Superposition Theorem

Consider a soil-structure system as shown in Fig. 6.1. For the purposes of this presentation, it will be assumed that the structure and the soil have been discretized into nodes connected by linear members or elements; however, no particular method of analysis is implied. The generalization to the continuum can be obtained as the limiting case of infinitely many nodes and elements. We assume then that the general equations of motion of the system are given by the matrix equation

\[
\ddot{\mathbf{M}} \ddot{\mathbf{U}} + \dot{\mathbf{C}} \dot{\mathbf{U}} + \mathbf{KU} = \mathbf{R}(t)
\]

6.1

in which M, C, K are the system's mass, damping and stiffness matrices (excluding the support degrees of freedom), U is the absolute displacement vector, and R is the (time-dependent) load vector resulting from the modification for boundary conditions of the equations corresponding to the support nodes at the edges of the model (at which the excitation is prescribed); the exact form of R is irrelevant for our purposes, except for the fact that it does not have components in the structural degrees of freedom. The solution of this equation of motion is equivalent to the solution of the two matrix equations

\[
M_1 \ddot{\mathbf{U}}_1 + \dot{\mathbf{C}} \dot{\mathbf{U}}_1 + \mathbf{KU}_1 = \mathbf{R}(t) \quad 6.2a
\]

\[
M \ddot{\mathbf{U}}_2 + \dot{\mathbf{C}} \dot{\mathbf{U}}_2 + \mathbf{KU}_2 = -M_2 \ddot{\mathbf{U}}_1 \quad 6.2b
\]

where U = U_1 + U_2 and M = M_1 + M_2. In particular, M_1 excludes the mass of the structure (massless structure), while M_2 excludes the mass of the soil (massless soil). The equivalence of equations 6.2 with 6.1 is demonstrated by simple addition.
FIGURE 6.1
SUPERPOSITION THEOREM
In the first step (equation 6.2.a) the massless structure is subjected to the same seismic environment as the original system (same load vector R) but no inertial forces arise in the structure. However, since the structure is generally embedded, or the motion pattern in the ground does not consist simply of plane waves propagating vertically, the structure will experience distortions resulting from the spatial variations of the motions of the soil surrounding the foundation. For ideally rigid foundations, these distortions do not take place, and so no deformations are transmitted to the structure; hence, the massless structure moves as a rigid body, and may thus be replaced by an ideally massless, rigid foundation. This motion, \( U_1 \), is given the name of Kinematic Interaction.

In the second step, equation 6.2b, the loading is applied on the structure only, and consists of fictitious inertial forces which are proportional to the accelerations \( \ddot{U}_1 \) computed in the first step. The additional motions \( U_2 \) obtained in this calculations are given the name of Inertial Interaction.

6.3 The Three-step Solution

Consider once more equation 6.2.b. For a solution in the frequency domain, this equation would be written

\[
(-\omega^2 M + i\omega C + K)U_2 = \omega^2 M_2 U_1
\]

or briefly,

\[
\tilde{K} U_2 = \omega^2 M_2 U_1
\]

with \( \tilde{K} = -\omega^2 M + i\omega C + K \) being the dynamic stiffness matrix.

Since the loading term has non-zero components only for the degrees of freedom associated with the structure (because the soil in \( M_2 \) has no mass), it is possible to condense dynamically all the degrees of freedom in the soil, thus modeling it with a complex (frequency-dependent) impedance matrix \( X = X(\omega) \), having as many rows and columns as the number of degrees of freedom along the soil-structure interface. Equation 6.4 could then be written for the structure as

\[
(\tilde{K}_S + Z)U_{2s} = \omega^2 M_S U_{1s}
\]

where \( \tilde{K}_S = K_S + i\omega C_S - \omega^2 M_S \) is the dynamic stiffness matrix of the structure, including the interface nodes, and
\[ Z = \begin{pmatrix} 0 \\ X \end{pmatrix} \]  

is the impedance of the interface augmented with zeroes to match the dimensions of \( K_s \). Also, \( U_{1s}, U_{2s} \) are the subvectors of \( U_1, U_2 \) corresponding to the structural degrees of freedom (including those at the interface) and \( M_s \) is the mass matrix of the structure (i.e., the non-zero portion of \( M_2 \)).

Since \( U_s = U_{1s} + U_{2s} \), it follows from eq. 6.5 by substitution

\[(\tilde{K}_s + Z)U_s = (\tilde{K}_s + Z + \omega^2 M_s)U_{1s} \]

\[= (K_s + i\omega C_s + Z)U_{1s} \]

So far, no approximations have been made, equations 6.4 and 6.7 being fully equivalent. We mentioned before, however, that the displacements \( U_1 \) in the structure, that is \( U_{1s} \), may be approximated by a rigid body-motion. Assuming then this approximation to be valid, it is possible to write \( U_{1s} \) as

\[ U_{1s} = T U_0 \]

where \( T \) is a rigid body transformation matrix of the form

\[
T = \begin{pmatrix}
1 & 0 & (z_1 - z_0) & -(y_1 - y_0) \\
1 & -(z_1 - z_0) & 0 & (x_1 - x_0) \\
1 & (y_1 - y_0) & -(x_1 - x_0) & 0 \\
1 & 1 & 0 & 1
\end{pmatrix}
\]

same as above for nodes \( i = 2, 3 \ldots \) of the structure (a) and interface (b). The last three rows may not exist if node has only translational degrees of freedom.

in which the elements \( x_i, y_i, z_i \) in \( T \) are the coordinates of the nodes and \( x_0, y_0, z_0 \) are the coordinates of the reference point at which the rigid-body motion is observed. Also,
\[
U_0 = \begin{pmatrix}
    u_x \\
    u_y \\
    u_z \\
    \theta_x \\
    \theta_y \\
    \theta_z
\end{pmatrix} = \text{rigid-body displacement vector.}\]

consisting of the translations and three rotations.

Substituting 6.8 into 6.7 yields

\[
(\ddot{K}_S + Z)U_s = (K_S + i\omega C_S + Z)TU_0
\]

\[
= (K_S T + i\omega C_S T + ZT)U_0 \quad 6.11
\]

Now, both the stiffness matrix \(K_S\) and the damping matrix \(C_S\) satisfy the rigid-body transformation condition, since no elastic or viscous forces are necessary to execute a rigid-body displacement in the structure. It follows that \(K_S T = C_S T = 0\), and

\[
(\ddot{K}_S + Z)U_s = ZTU_0 \quad 6.12
\]

This equation can be written in partitioned form as

\[
\begin{pmatrix}
    \dddot{\bar{K}}_{aa} & \dddot{\bar{K}}_{ab} \\
    \dddot{\bar{K}}_{ba} & \dddot{\bar{K}}_{bb} + \chi
\end{pmatrix}
\begin{pmatrix}
    U_a \\
    U_b
\end{pmatrix} =
\begin{pmatrix}
    0 \\
    X^T b U_0
\end{pmatrix} \quad 6.13
\]

with the subindex, \(a\), referring to the degrees of freedom in the structure, excluding the interface, and subindex \(b\) referring to the degrees of freedom along the interface. Defining the vector of forces \(F\)

\[
- T_b^T X U_b + T_b^T X_T b U_0 = F \quad 6.14
\]

and adding it to 6.13 yields the expression
FIGURE 6.2
THE 3-STEP SOLUTION
which can be interpreted (Fig. 6.2) as a rigid-body support motion \( U_b \) specified at the base of the "springs" \( X \) which hold the structure. Notice that this interpretation does not require the foundation to be rigid, but merely that Kinematic Interaction be an essentially rigid-body motion. The 6 components of the vector \( F \) can be interpreted, then, as the overall forces/moments needed at the base to excite the system. If the additional assumption is made that the foundation is rigid, it follows that

\[
U_b = T_b U_f
\]

where \( U_f \) is a vector whose 6 components describe the translations and rotations of the rigid foundation. Substitution of 6.16 into 6.15 yields then

\[
\begin{bmatrix}
\tilde{K}_{aa} & \tilde{K}_{ab} & T_b \\
T_b^T \tilde{K}_{ba} & T_b^T \tilde{K}_{bb} T_b + \tilde{K}_f & -\tilde{K}_f \\
-\tilde{K}_f & \tilde{K}_f & \end{bmatrix}
\begin{bmatrix}
U_a \\
U_f \\
U_0 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
T_b^T T_b F \\
\end{bmatrix}
\]

in which

\[
\tilde{K}_f = T_b^T X T_b
\]

is a 6 x 6 matrix containing the (coupled, frequency-dependent) soil "springs" and "dashpots." We have now all the formulae needed to determine the solution to the soil-structure interaction problem at hand. This solution follows in the following three steps:
a) Determination of the motion of a massless, rigid foundation, which is subjected to the same seismic input as the original structure and soil. In general, this calculation yields both translations and rotations, even if the foundation is not embedded. The only exception is the case of surface foundations subjected to vertically propagating waves, in which case the motion of the foundation is identical to that of the free field at grade level. This step corresponds to the computation of kinematic interaction.

b) Determination of the frequency-dependent subgrade impedance matrix $X$, or alternatively, the frequency-dependent subgrade stiffness matrix $\tilde{K}_F$ for the relevant degrees of freedom. This step yields the "soil springs".

c) Computation of the response of the real structure supported on the frequency-dependent soil springs or impedances, and subjected to the support motion determined in step (a).

The only approximation involved in this approach concerns the deformability of the structural foundation. If the full impedance matrix $X$ is used in steps b, c, then this simplification applies to the kinematic interaction step only. If the "lumped" springs represented by $\tilde{K}_F$ are used instead, then the rigid foundation approximation applies to the whole analysis. Care must then be exercised in using a structural model which is consistent with the rigid foundation assumptions used for the foundation.

It should be observed that the fictitious support excitation $U_o$ determined in step a) has no real existence; that is, it occurs nowhere in the real soil-structure system. In particular, it is not the motion of the ground (either grade, embedment level or rock) before the soil has been excavated or the structure built. This fact is of particular importance in interpreting comparisons made between results obtained by a direct solution of equation 6.1 and approximations obtained by using the surface or rock excitation directly at the base of the soil "springs". These comparisons are usually inconsistent because they neglect kinematic
interaction in the support excitation, and embedment effects on the foundation stiffnesses.

An alternative 3 step scheme proposed by some authors (which can be shown to be equivalent to the method described in these pages) consists in holding the massless, rigid foundation fixed in step a), and determining the overall forces and moments necessary to accomplish this. These forces are then applied in step c) as fictitious excitations above the soil springs. It should be noted that these forces exist even if the foundation is not embedded and the earthquake waves propagate vertically.

Procedures to determine the foundation stiffnesses, and soil springs for some frequently encountered geometries are presented in Chapter 7. On the other hand, the motions that the ground experiences before any structure has been built or soil excavated, (i.e., the free field problem) is taken up in Chapter 7. Approximations for the kinematic interaction problems are reviewed later in this Chapter.

6.4 The Substructure Theorem:

Substructuring is a technique by which the analysis of a composite structure is performed in various steps, separating the structure into simpler subunits (or substructures) which are more efficiently (or easily) handled than the complete system in one single step. The analysis of each unit is then followed by synthesis, to ensure compatibility and equilibrium across the interfaces separating the various substructures. The technique is based on the principle of superposition, so that it is restricted to linear systems. For dynamic problems, the procedure requires in addition that the equations of motion be expressed in the frequency domain.

The substructure technique can be applied either with or without substitution. In the first alternative, not only is the system separated into substructures, but each substructure is complemented with a fictitious addition which transforms the subunits into solids with desirable geometric or material properties. For example, in the case of a soil-structure interaction problem, it is convenient to separate the complete system into two substructures consisting of the structure on the one hand, and the subgrade on the other; since the soil with
FIGURE 6.3
SUBSTRUCTURE THEOREM

a) SOIL STRUCTURE INTERACTION PROBLEM

b) FREE FIELD PROBLEM
the excavation is not a simple problem to analyze, we fill in the void with soil and transform the subgrade into a halfspace with desirable properties (i.e., satisfying the "free-field" conditions). We have thus substituted soil for the structure. The presentation given below applies to this particular situation.

Consider Fig. 6.3, showing two discrete models: a) a soil-structure interaction problem and b) a free-field vibration problem for the same soil before the structure is put in place. In either case, the model has been separated into two substructures, consisting of the excavated soil and the structure in the first case, and the excavated soil and the soil removed in the second case. The equations of motion for the structure given in partitioned matrix form are

\[
\begin{pmatrix}
\tilde{K}_{aa} & \tilde{K}_{ab} \\
\tilde{K}_{ba} & \tilde{K}_{bb}
\end{pmatrix}
\begin{pmatrix}
U_a \\
U_b
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
-S_b
\end{pmatrix}
\]

where the superscript tilde (compare with eqs. 6.3, 6.4) in the submatrices \( \tilde{K} \) is again a reminder that they represent frequency-dependent dynamic stiffness matrices; the subindex \( a \) refers to the degrees of freedom in the structure, excluding the interfaces, and the subindex \( b \) refers to the degrees of freedom along the interface; \( U_a, U_b \) are the corresponding absolute displacements, while \( -S_b \) represents the internal forces along the interface which are necessary to preserve equilibrium after the structure is separated from the soil (the minus sign indicates that these forces are defined positive when acting on the soil, not when acting on the structure).

The soil-structure interaction problem is then characterized by interface stresses (forces) \( S_b \) and displacements \( U_b \). A similar decomposition of the free-field problem leads to two substructures characterized by interface forces \( S_b^* \) and displacements \( U_b^* \). Since the excitation at the far boundaries of the model (in the form of prescribed displacements) is exactly the same in the interaction problem (Fig. 6.3a) and in the free-field problem (Fig. 6.3b), it follows that the difference between the interface displacements \( \Delta U_b = U_b - U_b^* \) is solely the result of the difference in interface forces \( \Delta S_b = S_b - S_b^* \).

Considering now the subgrade subjected to forces \( \Delta S_b \) applied at the interface only (no seismic excitation), they will produce displacements \( \Delta U_b \) such that
FREE FIELD PROBLEM  
(a)  

SOIL-STRUCTURE INTERACTION PROBLEM  
(b)  

FIGURE 6.4
\[ \Delta S_b = X \Delta U_b \] (6.20)

where \( X \) is the dynamic impedance matrix of the excavation; i.e., the stiffnesses of the soil as seen from the interface. It is the same matrix that was used in the three-step approach. It must be emphasized that neither \( U_b \) nor \( U_b^* \) and \( S_b^* \) are related to each other by this impedance matrix \((S_b \neq XU_b, S_b^* \neq XU_b^*)\), because of dynamic forces being applied elsewhere in the model (the seismic excitation). From equation (6.20), it follows that

\[ -S_b = -XU_b + XU_b^* - S_b^* \] (6.21)

Substituting (6.21) into (6.19) and taking the term \( XU_b \) to the left-hand side yields

\[
\begin{pmatrix}
    \vec{K}_{aa} & \vec{K}_{ab} \\
    \vec{K}_{ba} & \vec{K}_{bb} + X
\end{pmatrix}
\begin{pmatrix}
    U_a \\
    U_b
\end{pmatrix} =
\begin{pmatrix}
    0 \\
    XU_b^* - S_b^*
\end{pmatrix}
\] (6.22)

This set of equations can be interpreted as the substitution of the soil by the "springs" and "dashpots" implied by \( X \), and application of fictitious forces \( XU_b^* - S_b^* \) along the interface (see fig. 6.4). Since the free-field solution can be computed readily for a variety of soil conditions and seismic wave patterns (Chapter ), the solution for the structure can be obtained if \( X \) is available. For structures with no embedment, determining \( X \) presents no difficulties when currently available techniques are used (Chapter ). For embedded foundations, however, difficulties are encountered as a result of the computational effort required for numerical solutions, and the great theoretical difficulties for closed-form solutions.

As it stands, equation (6.22) is valid for any arbitrary soil-structure system, rigid or flexible. Comparison with (6.13) shows that when the approximation for kinematic interaction implicit in that equation is admissible, then

\[ X T_b U_o = XU_b^* - S_b^* \] (6.23)

Multiplication by \( T_b^T \), and considering eq. (6.18), we obtain

\[ \vec{K}_f U_o = T_b^T (XU_b^* - S_b^*) \] (6.24)
Note: soil springs (stiffnesses have zero length. This is identified with the two points having the same coordinates $x_0, y_0, z_0$. 

Figure 6.6
where \( U_a \) = absolute displacements (rotations) of the structural nodes, excluding the interface;

\( U_f \) = absolute displacements of the rigid foundation, observed at the reference point having coordinates \( x_o, y_o, z_o \) (three translations and three rotations).

The absolute displacements of the interface nodes \( b \) are linear combinations of the foundation displacement components \( U_f \), since the foundation moves as a rigid body:

\[
U_b = T_b U_f \tag{6.27}
\]

with \( T_b \) being the submatrix of the matrix \( T \) defined by eq. (6.9) which is associated with the degrees of freedom corresponding to the interface.

We define further the vector of local displacements \( V \), and relative displacements \( Y \) as

\[
V = \begin{bmatrix} V_a \\ V_f \end{bmatrix} \quad \text{local displacement vector} \tag{6.28}
\]

\[
Y = \begin{bmatrix} Y_a \\ Y_f \end{bmatrix} \quad \text{relative displacement vector} \tag{6.29}
\]

with \( V_a \) = displacements of structural nodes relative to a coordinate system moving with the foundation (attached at the reference point having coordinates \( x_o, y_o, z_o \));

\( V_f \) = displacements of the foundation (observed at the reference point) relative to a coordinate system moving with the support of the springs;

\( Y_a, Y_f \) = displacements of the structure and foundation relative to a coordinate system moving with the support of the springs. It follows that \( Y_f \equiv V_f \).

The relationship between \( U, V \) and \( Y \) is given by

\[
U = \psi(V + V_0) = Y + \psi V_0 \tag{6.30a}
\]

\[
V = \psi^{-1}U - V_0 = \psi^{-1} Y \tag{6.30b}
\]

\[
Y = U - \psi V_0 = \psi V \tag{6.30c}
\]

in which the support motion vector \( V_0 \) is defined as
\[ V_0 = \begin{bmatrix} 0 \\ U_0 \end{bmatrix} \text{ = support motion vector} \] (6.31)

such that the only non-zero components \( U_0 \) contain the three displacements and three rotations of the moving support, observed at the point having coordinates \( x_0, y_0, z_0 \). Also, the square transformation matrix \( \psi \) is given by

\[ \psi = \begin{bmatrix} I_a & T_a \\ 0 & I_6 \end{bmatrix} \] (6.32)

in which \( I_a \) = identity matrix matching the number of degrees of freedom in the structure

\( I_6 \) = identity matrix of rank 6

\( T_a \) = submatrix of the matrix \( T \) defined by equation (6.9), which is associated with the structural degrees of freedom.

The inverse of the transformation matrix is given by

\[ \psi^{-1} = \begin{bmatrix} I_a & -T_a \\ 0 & I_6 \end{bmatrix} \] (6.33)

which the reader can easily verify computing the product \( \psi \psi^{-1} \).

b) Force-displacement relations

By D'Alambert's principle, the external loads acting on the structure and foundation must be in equilibrium with the elastic, damping and inertia forces. The load vector \( P \) can be partitioned into two subvectors \( P_a, P_f \), which are associated with the structural and the foundation degrees of freedom, respectively (paralleling the partition made for the displacements).

The elastic forces in the structure are given by the product of the stiffness matrix of the structure (assumed fixed in space) times the absolute displacements:

\[ \begin{bmatrix} P_a \\ P_f \end{bmatrix}_{\text{elastic in structure}} = \begin{bmatrix} K_{aa} & K_{ab} T_b \\ T_b K_{ba} & T_b K_{bb} T_b \end{bmatrix} \begin{bmatrix} U_a \\ U_f \end{bmatrix} \] (6.34)
Since the (singular) stiffness matrix of the structure satisfies the rigid-body condition $K_S^T = 0$, it follows that

\[
\begin{align*}
K_{aa} T_a + K_{ab} T_b &= 0 \\
K_{ba} T_a + K_{bb} T_b &= 0
\end{align*}
\]

so that (6.34) transforms into

\[
\begin{align*}
\begin{bmatrix}
K_{aa} & -K_{aa} T_a \\
-T_a^T K_{aa} & T_a^T \end{bmatrix} & \begin{bmatrix}
U_a \\
U_f
\end{bmatrix} = \begin{bmatrix}
P_a \\
P_f
\end{bmatrix}
\end{align*}
\]

which can be decomposed into the product

\[
\begin{align*}
\begin{bmatrix}
I_a & I_a \\
-T_a^T & 0
\end{bmatrix} & \begin{bmatrix}
K_{aa} & I_a \\
0 & I_6
\end{bmatrix} & \begin{bmatrix}
U_a \\
U_f
\end{bmatrix} = \begin{bmatrix}
P_a \\
P_f
\end{bmatrix}
\end{align*}
\]

and considering (6.30a), (6.31) and 6.33), we obtain

\[
\psi^{-T} \begin{bmatrix}
K_{aa} \\
0
\end{bmatrix} \psi^{-1} \begin{bmatrix}
U_a \\
U_f
\end{bmatrix} = \psi^{-T} \begin{bmatrix}
K_{aa} \\
0
\end{bmatrix} \begin{bmatrix}
V_a \\
V_f
\end{bmatrix} = \begin{bmatrix}
P_a \\
P_f
\end{bmatrix}
\]

with $\psi^{-T}$ being the transposed inverse of $\psi$.

On the other hand, the elastic forces in the soil springs are given by the product $K_f V_f$. Adding these forces to the elastic forces in the structure (6.38) we obtain then the elastic forces in the complete system:

\[
\psi^{-T} \begin{bmatrix}
K_{aa} & K_f \\
K_f^T & 0
\end{bmatrix} \begin{bmatrix}
V_a \\
V_f
\end{bmatrix} = \begin{bmatrix}
P_a \\
P_f
\end{bmatrix}_{\text{elastic}}
\]
or briefly
\[ \psi^{-T} K V = P_{\text{elastic}} \] (6.40)

with
\[ K = \begin{pmatrix} K_{aa} \\ \end{pmatrix} \]
(6.41)

Similarly, the damping forces are given by
\[ \psi^{-T} C \dot{V} = P_{\text{damping}} \] (6.42)
in which \( \dot{V} = \frac{d}{dt} V \), and
\[ C = \begin{pmatrix} C_{aa} \\ C_{f} \end{pmatrix} \] (6.43)

with \( C_{aa} \) and \( C_{f} \) being damping matrices which are the counterparts of the stiffness matrices \( K_{aa} \) and \( K_{f} \).

Finally, the inertia forces are proportional to the absolute accelerations, that is (neglecting the coupling terms \( M_{ab} \))
\[ P_{\text{inertia}} = \begin{pmatrix} M_{aa} \\ \end{pmatrix} \begin{pmatrix} \ddot{U}_{a} \\ \vdots \\ \ddot{U}_{f} \\ M_{f} \\ \end{pmatrix} = M \ddot{U} \] (6.44)
in which \( M_{aa} \) is mass matrix of the structure, and \( M_{f} \) is the \((6 \times 6)\) mass matrix of the foundation (containing masses, static mass moments and products of inertia relative to the reference point, which has coordinates \( x_{0}, y_{0}, z_{0} \)).

Adding then the elastic, damping and inertia forces together, we obtain the applied external loads:
\[ M \ddot{U} + \psi^{-T} C \dot{V} + \psi^{-T} K V = P \] (6.45)

If we substitute (6.30a) into (6.45), multiply by \( \psi^{T} \) and rearrange terms, we obtain the dynamic equilibrium equation in local coordinates
\[ \ddot{M} \ddot{V} + C \dot{V} + K V = \ddot{P} - \ddot{M} \dot{V}_{0} \] (6.46)

with modified mass matrix \( \ddot{M} \).
\[
\bar{M} = \psi^T M \psi = \begin{bmatrix}
M_{aa} & M_{aa} T_a \\
\text{sym.} & M_o
\end{bmatrix}
\]

(6.47)

in which

\[
M_o = M_f + T_a^T M_a T_a
\]

(6.48)

is a 6 x 6 matrix containing the masses, static mass moments and mass products (moments) of inertia of the complete structural system with respect to the reference point \(x_0, y_0, z_0\) (i.e., the mass matrix of the complete structure when assumed to be rigid.) Also, the modified load vector \(\bar{P}\) is given by

\[
\bar{P} = \psi^T P = \begin{bmatrix}
P_a \\
T_a^P + P_f
\end{bmatrix} = \begin{bmatrix}
P_a \\
P_o
\end{bmatrix}
\]

(6.49)

in which \(P_o\) is the resultant (3 forces and 3 moments) of all external loads applied on the structure and the foundation. This resultant is observed at the reference point.

Alternatively, if we substitute (6.30b) into (6.45), and consider the fact that \(\psi^{-T} K V_o = KV_o\), we obtain

\[
\ddot{\bar{U}} + \bar{\bar{C}} \ddot{\bar{U}} + \bar{K} \bar{U} = \bar{P} + \bar{C} \dot{V}_o + KV_o
\]

(6.50)

with modified stiffness and damping matrices \(\bar{K}, \bar{C}\) given by

\[
\bar{K} = \psi^{-T} K \psi^{-1} = \begin{bmatrix}
K_{aa} & -K_{aa} T_a \\
\text{sym.} & T_a^T K T_a + K_f
\end{bmatrix}
\]

(6.51)

\[
\bar{C} = \psi^{-T} C \psi^{-1} = \begin{bmatrix}
C_{aa} & -C_{aa} T_a \\
\text{sym.} & T_a^T C T_a + C_f
\end{bmatrix}
\]

(6.52)

Finally, we express \(U, V\) in equation (6.45) in terms of \(Y\) (eqs. (6.30a), (6.30b), obtaining
\[ M\ddot{Y} + C\dot{Y} + K\dot{Y} = P - M\ddot{\psi}_o \] (6.53)

which expresses the equations of motion in terms of the relative displacements.

**Summary of equations**

It is convenient at this point to make a distinction between the non-seismic case and the seismic case. In the former case there is no support motion \((V_o = 0)\), so that the relative displacements are identical to the absolute displacements \((Y \equiv U)\); in the latter case, there are no external forces applied on the structure, so that \(P = 0\). The equations of motion given below are expressed in terms of the absolute, relative and local displacement vectors \(U\), \(Y\) and \(V\) respectively.

a) Non-seismic case \((V_o = 0, Y \equiv U)\):

\[ M\ddot{U} + C\dot{U} + KU = P \] (6.54)
\[ M\ddot{V} + CV + KV = \bar{P} \] (6.55)

b) Seismic case

\[ M\ddot{U} + C\dot{U} + KU = C\dot{V}_o + KV_o = \begin{cases} 0 \\ C_f U_o + K_f U_o \end{cases} \] (6.56)

\[ M\ddot{Y} + C\dot{Y} + K\dot{Y} = -M\ddot{\psi}_o = -\begin{cases} M_a \ddot{a} \\ M_f \end{cases} \] (6.57)

\[ M\ddot{V} + C\dot{V} + KV = -M\ddot{\psi}_o = -\begin{cases} M_a \ddot{a} \\ M_f \end{cases} \] (6.58)

The matrices and vectors in these equations are given by (a \(\rightarrow\) structure, f \(\rightarrow\) foundation)
\[ M = \begin{bmatrix} M_{aa} & \cdot \\ \cdot & M_f \end{bmatrix} \]

\[ M_0 = T_a^T M_{aa} T_a + M_f \]

\[ K = \begin{bmatrix} K_{aa} \\ \cdot \end{bmatrix} \]

\[ K = \begin{bmatrix} K_{aa} & - K_{aa} T_a \\ \text{sym.} & K_0 \end{bmatrix} \]

\[ K_0 = T_a^T K_{aa} T_a + K_f \]

\[ C = \begin{bmatrix} C_{aa} \\ \cdot \end{bmatrix} \]

\[ C = \begin{bmatrix} C_{aa} & - C_{aa} T_a \\ \text{sym.} & C_0 \end{bmatrix} \]

\[ C_0 = T_a^T C_{aa} T_a + C_f \]

\[ \mathbf{U} = \begin{bmatrix} U_a \\ U_f \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} Y_a \\ Y_f \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} V_a \\ V_f \end{bmatrix}, \quad \mathbf{V}_0 = \begin{bmatrix} 0 \\ U_0 \end{bmatrix} \]

\[ \mathbf{U} = \psi(\mathbf{V} + \mathbf{V}_0) = \mathbf{Y} + \psi \mathbf{V}_0 \]

\[ \psi = \begin{bmatrix} I_a & T_a \\ \cdot & I_6 \end{bmatrix} \]

\[ P = \begin{bmatrix} p_a \\ p_f \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} p_a \\ p_0 \end{bmatrix}, \quad P_0 = T_a T_p + P_f \]

\[ \mathbf{T} = \begin{bmatrix} T_a \\ T_b \end{bmatrix} \quad \text{(see equation (6.9))} \]
Example:

A vertical cantilever beam of length $L$ and stiffness $EI$ has a mass $m$ attached at the upper end; the rotatory inertia of this mass is $J$. The beam is clamped to a foundation with mass $m_f$ and rotatory inertia $J_f$, which is supported on springs $k_x$, $k_\theta$ that correspond to the horizontal and rocking degrees of freedom of the foundation. The complete system is subjected to a horizontal support excitation $u_q$ (earthquake) at the base of the springs. Determine the equations of motion (for the relevant degrees of freedom).

Joint coordinates

- mass 1  $x = 0$  $y = 0$  $z = L$
- foundation  $x_o = 0$  $y_o = 0$  $z_o = 0$ (reference point)

Active degrees of freedom

$u_x \rightarrow 1$, $\theta_y \rightarrow 5$, hence only columns/rows one and five of the matrix $T_a$ are of interest.

Transformation matrix $T_a$:

$$T_a = \begin{bmatrix} 1 & \frac{L}{1} \\ 1 & 1 \end{bmatrix}$$
Mass matrix

\[
M = \begin{bmatrix}
m & J \\
J & m_f \\
mL & J_f \\
\end{bmatrix}
\]

\[
\bar{M} = \begin{bmatrix}
m & 0 & m & mL \\
0 & J & 0 & J \\
m & 0 & m+m_f & mL \\
mL & J & mL & J_f+mL^2 \\
\end{bmatrix}
\]

Stiffness matrix

\[
K = \begin{bmatrix}
6k & -3kL \\
-3kL & 2kL^2 \\
& k_x \\
& k_0 \\
\end{bmatrix}, \quad k = \frac{2EI}{L^3}
\]

\[
\bar{K} = \begin{bmatrix}
6k & -3kL & -6k & -3kL \\
-3kL & 2kL^2 & 3kL & kL^2 \\
-6k & 3kL & k_x + 6k & 3kL \\
-3kL & kL^2 & 3kL & k_0 + 2kL^2 \\
\end{bmatrix}
\]

Ground displacement vector

\[
\mathbf{U}_0 = \begin{bmatrix}
\mathbf{u}_g \\
0
\end{bmatrix}
\]

Load vector (R)

For eq. (6.56),

\[
R = \begin{bmatrix}
0 \\
0 \\
k_x \mathbf{u}_g \\
0
\end{bmatrix}, \quad \mathbf{u}_g = \begin{bmatrix}
0 \\
0 \\
k_x \mathbf{u}_g \\
0
\end{bmatrix}
\]
For eq. (6.57)
\[
R = -\begin{pmatrix}
m \\
0 \\
m_f \\
0
\end{pmatrix}\dddot{u}_g = -\begin{pmatrix}
m\dddot{u}_g \\
0 \\
m_f\dddot{u}_g \\
0
\end{pmatrix}
\]

For eq. (6.58)
\[
= -\begin{pmatrix}
m \\
0 \\
m + m_f \\
m_L
\end{pmatrix}\dddot{u}_g = -\begin{pmatrix}
m\dddot{u}_g \\
0 \\
(m + m_f)\dddot{u}_g \\
m_L\dddot{u}_g
\end{pmatrix}
\]

6.6 Modal Synthesis in the Frequency Domain

A frequently occurring situation in the analysis for soil-structure interaction effects is that the natural frequencies and modal shapes of the structure on fixed base are available, while the soil is characterized by frequency-dependent stiffness functions. It is then advantageous to use the information at hand and perform a modal synthesis in the frequency domain. Such a technique is attractive, because it allows the engineer to use many of the available programs for the dynamic analysis of structures, while requiring only a modest additional effort to account for interaction effects. Among the benefits of such a formulation are the possibility of including three-dimensional effects in the structure, without sacrifice in the representation of the soil.

We begin defining the transformation matrix
\[
\phi = \begin{pmatrix}
\phi_a \\
I_6
\end{pmatrix}
\]
(6.59)
in which \( \phi_a = \) modal matrix of structure as fixed base (normalized with respect to the mass matrix \( M_{aa} \)) and \( I_6 = \) identity matrix of rank 6. (The notation used in the following is the same as the one used in the previous section). We emphasize that \( \phi \) is not the modal matrix of the combined system, but merely a convenient transformation matrix. We express next the local displacements \( V \) as
\[ V = \phi A = \begin{cases} a \ A_a \\ V_f \end{cases}, \quad A = A(t) \] (6.60)

which denotes merely a linear transformation. Substituting this expression into equation (6.46), we obtain

\[ \ddot{M} \dddot{A} + C \ddot{A} + K A = \ddot{P} - \dot{M} \dot{V}_o \] (6.61)

which after premultiplication by \( \phi^T \) becomes

\[ (\phi^T M \phi) \dddot{A} + (\phi^T C \phi) \ddot{A} + (\phi^T K \phi) A = \phi^T \dddot{P} - \phi^T \dot{M} \dot{V}_o \] (6.62)

Considering now the orthogonality relations

\[ \phi^T a \ M a \ \phi_a = I_a \] (6.63)

\[ \phi^T a K a \ \phi_a = \Omega_a^2 = \text{diag} \left\{ \omega_n^2 \right\} \] (6.64)

the participation factors matrix for the structure on rigid base

\[ \Gamma_a = \phi^T a M a \ \Gamma_a \] (6.65)

and assuming the damping of the structure to be diagonalizable by the modal transformation

\[ \phi^T a C a \ \phi_a = 2 \beta \ a \ \Omega_a = \text{diag} \left\{ 2 \beta \ n \ \omega_n \right\} \] (6.66)

(in which \( \beta_n \) is the fraction of critical damping in the \( n \) th natural mode of the structure on fixed base having frequency \( \omega_n \)), we can write equation (6.62) in partitioned form as

\[ \begin{bmatrix} I_a & \Gamma_a \\ \Gamma_a^T & M_0 \end{bmatrix} \ddot{A} + \begin{bmatrix} 2 \beta \ a \ \Omega_a \\ C_f \end{bmatrix} \dot{A} + \begin{bmatrix} \Omega_a^2 \\ K_f \end{bmatrix} A = \begin{bmatrix} \phi^T a \ P_a \\ p_0 \end{bmatrix} - \begin{bmatrix} \Gamma_a \\ M_0 \end{bmatrix} \dot{U}_o \] (6.67)
For a solution in the frequency domain \((\omega)\), this equation transforms into

\[
\begin{bmatrix}
\Omega_a^2 - \omega^2 I_a + 2iB_a\Omega_a\omega & -\omega^2 \Gamma_a \\
-\omega^2 \Gamma_a^T & \overline{K}_f - \omega^2 M_o
\end{bmatrix}
\begin{bmatrix}
(A_a) \\
(V_f)
\end{bmatrix}
+ \omega^2
\begin{bmatrix}
(\Gamma_a) \\
(M_o)
\end{bmatrix} U_o
\]  
(6.68)

in which \(\overline{K}_f = K_f + i\omega C_f\) is the subgrade impedance matrix. At this point, it is possible to perform an ad-hoc generalization, admitting impedance matrices \(\overline{K}_f\) having arbitrary variation with frequency (i.e., \(K_f = K_f(\omega), C_f = C_f(\omega)\)).

On the other hand, the local (relative) displacement vector \(V_f\) is related to the absolute displacements \(U_f\) and ground displacements \(U_o\) (from equation (6.30a)) by the expression \(V_f = U_f - U_o\). Substituting this expression into (6.68) yields

\[
\begin{bmatrix}
\Omega_a^2 - \omega^2 I_a + 2iB_a\Omega_a\omega & -\omega^2 \Gamma_a \\
-\omega^2 \Gamma_a^T & \overline{K}_f - \omega^2 M_o
\end{bmatrix}
\begin{bmatrix}
(A_a) \\
(U_f)
\end{bmatrix}
= \begin{bmatrix}
\phi_T p_a \\
(P_o)
\end{bmatrix} + \begin{bmatrix}
0 \\
(\overline{K}_f U_o)
\end{bmatrix}
\]  
(6.69)

which has the solution

\[
U_f = (\overline{K}_f - \omega^2 M_o - \omega^4 \Gamma_a^T D_a \Gamma_a)^{-1} [(P_o + \omega^2 \Gamma_a^T D_a \phi_T p_a) + \overline{K}_f U_o] 
\]  
(6.70a)

and

\[
A_a = D_a (\omega^2 \Gamma_a U_f + \phi_T p_a) 
\]  
(6.70b)

in which

\[
D_a = (\Omega_a^2 - \omega^2 I_a + 2iB_a\Omega_a\omega)^{-1} = \text{diag}\left\{\frac{1}{\omega^2 - \omega^2 + 2iB_a\omega}\right\}
\]  
(6.71)

An alternate expression for equation (6.70a) can be obtained considering the identities

\[
\psi_a \phi_a^T M_{aa} = I_a \quad \text{(which follows from } \phi_a^T M_{aa} \phi_a = I) 
\]  
(6.72)

and
\[ M_0 = M_f + T_a^T M_{aa} T_a \]
\[ = M_f + T_a^T M_{aa} \left( \phi_a \phi_a^T M_{aa} \right) T_a \]
\[ = M_f + T_a^T \Gamma_a \Gamma_a \]

which we can substitute for \( M_0 \) in (6.70a) above:

\[ 2M_0^2 + \omega^4 \Gamma_a D_a \Gamma_a = 2M_f^2 + \omega^2 \Gamma_a^T (I_a + \omega^2 D_a) \Gamma_a \]

\[ = \omega^2 M_f^2 + \omega^2 \Gamma_a^T H_a \Gamma_a \]  \hspace{1cm} (6.74)

with a diagonal transfer matrix \( H_a \) given by

\[ H_a = \text{diag} \left\{ \frac{\omega^2 + 2i \beta_n \omega}{\omega^2 - \omega^2 + 2i \beta_n \omega} \right\} \]  \hspace{1cm} (6.75)

The elements of this transfer matrix are identical to the frequency response functions of single degree of freedom oscillators with natural frequencies \( \omega_n \) and damping \( \beta_n \) when subjected to unit support excitations. Substituting (6.74) into (6.70a), we obtain

\[ U_f = (K_f - \omega^2 M_f - \omega^2 \Gamma_a^T H_a \Gamma_a)^{-1} \left[ (P_o + \omega^2 \Gamma_a^T D_a \phi_a^T P_a) + K_f U_o \right] \]  \hspace{1cm} (6.76)

Equations (6.70a) and (6.76) give the response of the foundation (in the frequency domain) in terms of the foundation parameters \( K_f, M_f \) and the free vibration characteristics of the structure on fixed base. These equations can easily be coded into an auxiliary program that reads the modal shapes, frequencies and damping of the structure as input, and computes the response of the foundation \( U_f \) (three translations and three rotations) to the prescribed dynamic excitation. If this motion is converted by the same program back into the time domain, it can be used as a (generalized) support excitation (consisting of up to three translations and three rotations) to a fixed-base model of the structure, using for this purpose any standard program of the choice of the analyst. Such an option has the advantage that the solution for the structure can be obtained with conventional programs having many convenient output capa-
bilities, which need not be duplicated in the special purpose program. It should be pointed out, however, that the components of the base motion vector $U_f$ are not independent. Therefore, a combination of responses by the SRSS method (square root of the sum of the squares) in combination with response spectra for the components of the vector $U_f$ is not admissible. Furthermore, the structural model used in the second step must be the same as the one used to derive the modal shapes and frequencies $\phi_a, \Omega_a$, since otherwise inconsistency errors arise in the computation of the structural response. This follows from the fact that the foundation must remain essentially motionless (exactly motionless, if the structure is undamped) at the natural frequencies of the structure on fixed base; changing these frequencies in the second step may result in large amplitudes of motion for the structure because the amplification peaks may shift to frequencies at which the foundation does have substantial motion. Such inconsistencies are obviously avoided if equation (6.70b) is incorporated into the same program that computes the response of the foundation.

Modal synthesis in the frequency domain is particularly attractive when dealing with complicated, three-dimensional structures, and soil-structure interaction effects are believed to be important. Rather than developing advanced special purpose programs that can handle the sophistication in the structure, it is far preferable to use available programs to determine only the modes and frequencies of the structure on fixed base. The special purpose program is then solely used for the computation of the motion of the foundation. In this fashion, one can account for the true variation with frequency of the soil impedances, while ensuring that the structure has uniform damping throughout the frequency range of interest, and without sacrificing the detail in the structural model.

The equations show that it is possible to find the solution for the foundation to a seismic excitation by just knowing a) the rigid structure properties (total mass, mass moments of inertia and center of mass); b) the participation factors of the structure on fixed base (for normalized modes), for support motion components paralleling the degrees of freedom of the foundation (the modes themselves are not necessary); c) the foundation impedances; d) the support excitation.
Asymptotic behavior

The modal synthesis technique presented in this section is not only valuable in the formulation and solution of the equations of motion, but it also sheds light on the behavior of the foundation at low and high frequencies. To illustrate this, let us consider first equation (6.70a).

For very low frequencies, \((\omega \ll \omega_1 = \text{fundamental frequency of structure on fixed base})\), the term

\[
\omega^2 D_a = \text{diag} \left\{ \frac{\omega^2}{\omega_n^2 - \omega^2 + 2i\beta_n \omega_n \omega} \right\} \approx \text{diag} \left( \frac{\omega^2}{\omega_n^2} \right) \]

is very small, so that equation (6.70a) behaves approximately as

\[
U_f \approx (\mathcal{K}_f - \omega^2 M_0)^{-1} \left[ P_0 + \mathcal{K}_f U_0 \right] \tag{6.77}
\]

This expression says that the structure (and the foundation) is responding to the applied loads essentially as a rigid body on elastic foundation. In the limiting case when the frequency is identically zero \((\omega = 0)\), this expression collapses into the static solution \((U_f = \mathcal{K}_f^{-1}P_0 \), for the non-seismic case, \(U_f = U_0 \) for the seismic). In fact, equation (6.77) would be valid for every frequency if the structure were ideally rigid.

For high frequencies, on the other hand, it is convenient to look at equation (6.76) instead. Since

\[
\begin{align*}
H_a & \to 0 \quad \text{if } \omega \gg \omega_n = \text{highest structural frequency}, \\
\omega^2 D_a & \to -I_a
\end{align*}
\]

it follows that

\[
U_f \approx (\mathcal{K}_f - \omega^2 M_f)^{-1} \left[ (P_0 - \mathcal{R}_a^T \phi_a^T P_a) + \mathcal{K}_f U_0 \right] \\
= (\mathcal{K}_f - \omega^2 M_f)^{-1} \left[ P_f + \mathcal{K}_f U_0 \right] \tag{6.78}
\]

(since \(\mathcal{R}_a^T \phi_a^T \phi_a = T_a^T M_a \phi_a \phi_a^T = T_a^T\)).
This equation states that the foundation moves as if the structure did not exist; the motion is controlled entirely by the stiffness and mass of the foundation, as well as the loads applied directly onto it.

Comparison of equation (6.77) with (6.78) shows that, as the frequency increases, the foundation loses progressively effective mass; at low frequencies the entire structural mass contributes to the foundation response, while at high frequencies only the foundation mass offers resistance to motion. In fact, the term $\omega^2 \Gamma_a^T D_a \Gamma_a$ in equation (6.70a) can be interpreted as a "virtual mass" $\Delta M$, i.e.,

$$\Delta M = \omega^2 \Gamma_a^T D_a \Gamma_a = \Delta M(\omega) \quad (6.79)$$

With this notation, equation (6.70a) could be written (for seismic loads) as

$$U_f = (K_f - \omega^2 (M_o + \Delta M))^{-1} K_f U_o \quad (6.80)$$

The virtual mass, $\Delta M$ can in turn be written as

$$\Delta M = \Delta M_1 a_1 + \Delta M_2 a_2 + \ldots + \Delta M_N a_N = \sum_{n=1}^{N} \Delta M_n a_n \quad (6.81)$$

with $\Delta M_n$ being (frequency independent) modal virtual mass matrices given by

$$\Delta M_n = \Gamma_n^T \Gamma_n \quad (6.82)$$

with $\Gamma_n$ being the participation factor (row) vector corresponding to the $n$th structural mode (having at most 6 elements) (i.e., $\Gamma_n = \{r_{an}\}$). These matrices satisfy the relationship

$$\Delta M_1 + \Delta M_2 + \ldots + \Delta M_N = M_0 - M_f \quad (6.83)$$

Also, the amplification functions $a_n$ are given by

$$a_n = \frac{\omega^2}{\omega^2 - \omega_n^2 + 2i\beta_n \omega_n \omega} \quad (6.84)$$
Typically, for real structures, the virtual mass matrices of the fundamental modes have elements which are much larger than for the higher modes. If damping in the structure is at the same time not too low (say more than 2%), then the foundation behavior at intermediate frequencies is controlled almost entirely by the first few modes of the system. This is illustrated in the example that follows.

**Example**

A shear beam with shear modulus $G$, cross section $A$, and length $L$ is supported elastically on soil springs $k_x$, $k_\theta$ as shown. We are interested in the behavior of the "foundation," i.e., the beam section at $z = 0$, when excited by an earthquake with components $u_0$, $\theta_0$.

![Diagram of shear beam](image)

From standard structural analysis, it is known that the frequencies ($\omega_n$) and normalized modes ($\phi_n$) of the system on fixed base are given by

$$\omega_n = \frac{\alpha_n C_s}{L}, \quad C_s = \sqrt{\frac{G}{\rho}} = \text{shear wave velocity}$$

$$\alpha_n = \frac{n\pi}{2} (2n-1), \quad n = 1, 2, 3, \ldots$$

$$\phi_n = \sqrt{\frac{G}{\rho m}} \sin \alpha_n \frac{z}{L}, \quad m = \rho AL = \text{total mass of beam}.$$

The modal shapes satisfy the orthogonality condition, and are normalized in the sense that

$$\int_0^L \phi_n^2 \, dm = 1 \quad (dm = \rho Adz)$$
The participation factors are as follows:

\[
\gamma_{xn} = \int_0^L \phi_n^2 dm = \sqrt{\frac{2}{m}} \frac{m}{\alpha_n} = \frac{2}{\pi} \frac{\sqrt{2m}}{(2n-1)}
\]

= participation factor for the \( n \)th mode due to base \((x)\) motion

\[
\gamma_{\theta n} = \int_0^L \phi_n \theta dm = -(-1)^n \frac{4L}{\pi^2 (2n-1)^2} \sqrt{2m}
\]

= participation factor for the \( n \)th mode due to base rotation \((\theta)\) (positive clockwise).

Hence,

\[
\Delta M_n = \begin{pmatrix}
\gamma_{xn} & \gamma_{xn} & \gamma_{\theta n} \\
\text{sym} & \gamma_{\theta n}^2 \\
\end{pmatrix}
\]

\[
= \frac{8m}{\pi^2} \frac{1}{(2n-1)^2} \begin{pmatrix}
1 & \frac{-(-1)^n 2L}{\pi (2n-1)^2} \\
\text{sym} & \frac{4L^2}{\pi^2 (2n-1)^3} \\
\end{pmatrix}
\]

In particular,

\[
\Delta M_1 = \frac{8m}{\pi^2} \begin{pmatrix}
1 & \frac{2 \pi L}{\pi} \\
\frac{2 \pi L}{\pi} & \frac{4 \pi^2 L^2}{\pi^2} \\
\end{pmatrix}
\]

Also, the rigid-body mass matrix \( M_0 \) is

\[
M_0 = \begin{pmatrix}
\text{mass} & \text{static moment} \\
\text{(sym)} & \text{moment of inertia} \\
\end{pmatrix} = \begin{pmatrix}
m & m L/2 \\
\frac{mL}{2} & \frac{mL^2}{3} \\
\end{pmatrix}
\]

\[
= m \begin{pmatrix}
1 & L/2 \\
L/2 & L^2/3 \\
\end{pmatrix}
\]
The first mode effective mass matrix, $\Delta M_1$, is comparable to $M_0$, while the effective mass matrices for the higher modes are much smaller. Thus, the effective mass of the foundation essentially vanishes at frequencies higher than, say, twice the frequency of the first mode. Thereafter, the "foundation" essentially follows the ground motion; the response of the structure at these higher frequencies is similar to the motion that the structure would have experienced if it had been supported on a rigid base altogether (i.e., no soil springs). Note that for 5% structural damping, the maximum amplification is only a factor 10, which is not large enough to cancel the decay due to the terms in $(2n-1)$.

If no terms were neglected, the motion of the section $z=0$ would be given by the equation

$$ \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} k_x & k_0 \\ k_0 & k_0 \end{bmatrix} - \omega^2 (M_0 + \sum_{n=1}^{\infty} a_n \Delta M_n)^{-1} \begin{bmatrix} k_x \\ k_0 \end{bmatrix} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} $$

If soil dashpots were added to this problem we would merely replace $k_x$ by $k_x + i\omega C_x$, and $k_\theta$ by $k_\theta + i\omega C_\theta$.

6.7 Modal Superposition with Soil Springs and Dashpots

Modal analysis techniques for the computation of the response of a structural system to dynamic loads are well established in the engineering professions, with many commercial programs incorporating advanced capabilities being readily available. The application of this method to problems involving foundation interaction effects, however, requires soil stiffness and damping constants that are independent of frequency.

In other words, one must be able to model the foundation stiffnesses simply as springs and dashpots in parallel. While few, if any, foundation stiffnesses do satisfy this requirement, it is often possible to perform judicious approximations and model the stiffnesses by sets of springs and dashpots, with excellent results. The mere independence of the soil elements with frequency is not sufficient, however, to guarantee a modal solution, since the
damping in the soil may not permit a decoupling of the equations of motion. In other words, classical normal modes may not exist. This difficulty may be overcome in an approximate formulation using weighted modal damping techniques; an example is the rule proposed by Biggs and Roesset. We shall explore below the undamped solution first, and extend later on the formulation to the damped case.

a) Undamped case

To formulate the undamped modal solution, we define the transformation matrix

$$\phi = \begin{bmatrix} \phi_a \\ \phi_o \end{bmatrix}$$

(6.85)

in which $\phi_a$ = modal matrix of flexible structure on rigid base (normalized w/r to the mass matrix $M_{aa}$) and $\phi_o$ = modal matrix of the rigid structure on flexible base. This transformation matrix is similar to the one defined by equations (6.59), with $\phi_o$ substituting for $I_6$. The modal matrices $\phi_a, \phi_o$ satisfy the eigenvalue problems

$$\begin{bmatrix} K_{aa} & M_{aa} \phi_a \\ K_f & M_o \phi_o \end{bmatrix} \begin{bmatrix} \phi_a \\ \phi_o \end{bmatrix} = \begin{bmatrix} \Omega_a^2 \\ \Omega_o^2 \end{bmatrix}$$

(6.86)

in which $\Omega_a, \Omega_o$ are diagonal matrices containing the natural frequencies of the flexible structure on fixed support, and the natural frequencies of the rigid structure on flexible support (soil springs). We emphasize again that $\phi$ is not the modal matrix of the combined soil-structure system, but merely a convenient transformation matrix.

As in the preceding section, we define the linear transformation

$$V = \phi A = \begin{bmatrix} \phi_a A_a \\ \phi_o A_o \end{bmatrix}$$

(6.87)

which we substitute into equation (6.46), with $C = 0$. Premultiplication of the resulting expression by $\phi^T$ gives then

$$(\phi^T M_\phi) \ddot{A} \begin{bmatrix} \phi^T K_\phi \end{bmatrix} A = \phi^T P - \phi^T M \ddot{V}_o$$

(6.88)
Assuming the modes \( \phi_a, \phi_0 \) to be normalized in the sense

\[
\begin{align*}
\phi_a^T M_{aa} \phi_a &= I_a, & \phi_a^T K_{aa} \phi_a &= \Omega_a^2 \\
\phi_0^T M_0 \phi_0 &= I_0, & \phi_0^T K_0 \phi_0 &= \Omega_0^2
\end{align*}
\]  
(6.89)

and defining the participation factors matrix

\[
\Gamma = \begin{bmatrix} \Gamma_a \\ \Gamma_0 \end{bmatrix}
\]  
(6.90)

in which

\[
\Gamma_a = \phi_a^T M_{aa} T_a = \text{part. fact. of the struct. on rigid base}
\]

\[
\Gamma_0 = \phi_0^T M_0 = \text{part. fact. of rigid struct. on flex. base}
\]

we obtain from (6.83) the dynamic equation

\[
N\ddot{A} + \Omega^2 A = \phi^T \bar{P} - \Gamma \ddot{U}_o
\]  
(6.92)

in which

\[
N = \begin{bmatrix} I_a & \Gamma_a \phi_0 \\ \text{Sym.} & I_0 \end{bmatrix}
\]  
(6.93)

and

\[
\Omega = \begin{bmatrix} \Omega_a \\ \Omega_0 \end{bmatrix}
\]  
(6.94)

To solve equation (6.92), we must first find solutions to the homogeneous equation

\[
N\ddot{A} + \Omega^2 A = 0
\]  
(6.95)

Substituting the trial solution

\[
A = X_n \sin \lambda_n t
\]  
(6.96)

we obtain the eigenvalue problem

\[
\Omega^2 X_n = \lambda_n^2 N X_n
\]  
(6.97)
with positive definite matrices $\Omega^2$, $N$. This eigenvalue problem can be written alternatively as

$$\Omega^2 \chi = N \chi \Lambda^2 \quad (6.98)$$

with $\chi = \{X_n\}$ and $\Lambda^2 = \text{diag} \{\lambda_n^2\}$. These matrices contain the normal modes $(X_n)$ and natural frequencies $(\lambda_n)$ of the coupled soil-structure system. The eigenvalue problem defined by equation (6.98) is relatively simple to solve because of the structure of the matrices $\Omega$ and $N$; the first one is diagonal, while the second one is quasi-diagonal (the coupling terms in $N$ have at the most 6 columns). Thus some iterative procedures such as the Stodola-Vianello method require little computational effort. This relative simplicity in determining the coupled modes and frequencies is particularly attractive in parametric studies in which only the stiffnesses of the foundation are changed, because the frequencies and participation factors of the structure are not affected by the changes.

A quick estimate of the gravest frequency of the coupled soil-structure system can be obtained by a simple transformation of variables. Defining

$$\Omega X_n = Z_n \quad (6.99)$$

we obtain by substitution into (6.97)

$$Z_n = \Omega^{-1} N \Omega^{-1} Z_n \lambda_n^2 \quad (6.100)$$

or

$$L Z_n = \frac{1}{\lambda_n^2} Z_n \quad (6.101)$$

with

$$L = \Omega^{-1} N \Omega^{-1} = \begin{bmatrix} \Omega^{-2} & \Omega^{-1} \Gamma_a \Phi_o \Omega^{-1} \\ \Omega^{-2} & \Omega^{-2} \end{bmatrix}_{\text{Sym}} \quad (6.102)$$

The eigenvalue problem defined by (6.101) (with eigenvalue $1/\lambda_n^2$) satisfies the relationship

$$\sum \frac{1}{\lambda_n^2} = \text{Trace } L = \text{Trace } \Omega^{-2} \, \Gamma_a + \text{Trace } \Omega^{-2}_o \quad (6.103)$$
Assuming that the frequencies of both the structure on fixed base, and the coupled soil-structure system, are well separated, and that $\omega_{1a}$, $\lambda_1$ are the smallest (gravest) frequencies of these systems, then equation (6.103) can be approximated by

$$\frac{1}{\lambda_1^2} \approx \frac{1}{\omega_{1a}^2} + \sum \frac{1}{\omega_{no}^2}$$

(6.105)

or multiplying by $(2\pi)^2$ and substituting $\tau_1 = 2\pi / \lambda_1$, $\tau_{1a} = 2\pi / \lambda_{1a}$, $\tau_{no} = 2\pi / \omega_{no}$ (i.e., changing from frequency to period)

$$\tau_1^2 = \tau_{1a}^2 + \sum \tau_{no}^2$$

(6.106)

which is known as Dunkerley's Rule. It states that the fundamental period squared of the coupled soil-structure system is approximately equal to the fundamental period squared of the structure on rigid base plus the sum of the periods squared of the rigid structure on flexible support. A generalization of Dunkerley's rule is presented in Appendix.

While the frequencies (and periods) of the coupled system are independent of whether global or local coordinates are used to formulate the problem, it should be emphasized that the modal shapes do depend on the choice of reference frame made to measure displacements. The modal shapes $X_n$ determined above are in transformed local coordinates; to obtain the modal shapes in the actual local coordinates, we have to multiply the matrix $X = \{ X_n \}$ by the transformation matrix $\phi$ (equation 6.85), i.e., $\phi X$; to obtain the modal shapes in global coordinates, we have to multiply in addition the above result by the transformation matrix $\psi$ (eq. 6.32), i.e., $\psi \phi X$.

Once the natural frequencies and modal shapes of the coupled system are known, we can proceed with the analysis using standard techniques of structural dynamics.

Since the coupled soil-structure interaction problem involves the frequencies and modes of the rigid structure on soil springs, it is worthwhile to explore the behavior of rigid structures in more detail. Such an investigation is presented in section
b) Case with damping: the Biggs-Roesset Rule

The use of viscous damping matrices in dynamic analyses such as the matrix C in equation (6.46) (implying damping forces that are proportional and in phase with the velocities) is more a mathematical convenience than a faithful representation of the mechanism by which energy is dissipated in the structural prototype. In real structures and soil-structure systems, the damping forces vary both as to magnitude and type; in some parts they may be viscous, while in others they may be more nearly hysteretic. For a solution in the frequency domain, such as in the modal synthesis technique presented earlier, it is possible to consider the true nature of damping by formulating the viscosities as general functions of frequency. Thus, viscous damping would correspond to viscosities that are independent of frequency, while hysteretic damping would correspond to viscosities that are inversely proportional to frequency. For a modal solution, however, approximations are necessary to ensure the existence of classical normal modes. The weighted modal damping rule proposed by Biggs and Roesset is one of several alternatives which is based on considerations about energy dissipated and stored in the dynamic system. This technique is described below.

Consider a multidegree-of-freedom structure made up of many one-degree-of-freedom systems. The structure is forced (with external forces) to vibrate harmonically and in phase with some arbitrary frequency \( \omega \). As a result, each component will undergo cyclic deformation. If \( E_j \) is the maximum strain energy stored in the \( j^{th} \) component, the total energy stored in the system at the instant of maximum deformation is

\[
E_s = \sum_j E_j
\]

(since all components move in phase). On the other hand, each component will dissipate an amount of energy in one cycle of motion equal to \( (\text{sec.} \) ):

\[
E_{dj} = (\text{Energy ratio in } j^{th} \text{ comp.}) \times E_j
\]

\[
= 4\pi (\beta_j \frac{\omega_j}{\omega_j} + d_j) \times E_j
\]

with \( \beta_j \) being the fraction of viscous damping in the \( j^{th} \) component, defined relative to the frequency \( \omega_j \) which characterizes the component; also \( d_j \) is the
fraction of critical hysteretic damping in this component. The total energy
dissipated in the system is then

\[ E_d = \sum E_{d,j} = 4\pi \sum \left( \beta_j \frac{\omega_j}{\omega_j} + d_j \right) E_j \]  \hspace{1cm} (6.109)

and the overall energy ratio is

\[ E = \frac{E_d}{E_s} = 4\pi \frac{\left( \beta_j \frac{\omega_j}{\omega_j} + d_j \right) E_j}{\sum E_j} \] \hspace{1cm} (6.110)

which can be interpreted as the energy ratio of a 1-dof system with damping
\( \beta_e \), frequency \( \omega_e \) and hysteretic damping \( d_e \). It follows that

\[ \beta_e \cdot \frac{\omega}{\omega_e} + d_e = \frac{\sum \left( \beta_j \frac{\omega_j}{\omega_j} + d_j \right) E_j}{\sum E_j} \] \hspace{1cm} (6.111)

In particular, if we had chosen to excite the system in its \( n^{th} \) coupled mode
having frequency \( \lambda_n \), the equivalent damping would have been \( (\omega = \lambda_n = \omega_e) \)

\[ \beta_n + d_n = \frac{\sum \left( \beta_j \frac{\lambda_n}{\omega_j} + d_j \right) E_{nj}}{\sum E_{nj}} \] \hspace{1cm} (6.112)

with \( E_{nj} \) being the strain energies in the \( j^{th} \) component when the system vi-
brates in the \( n^{th} \) mode. Thus, if normal modes exist, then each mode should
have a viscous fraction of critical damping

\[ \beta_n = \frac{\sum \beta_j \frac{\lambda_n}{\omega_j} E_{nj}}{\sum E_{nj}} \] \hspace{1cm} (6.113)

and a hysteretic fraction of critical damping

\[ d_n = \frac{\sum d_j E_{nj}}{\sum E_{nj}} \] \hspace{1cm} (6.114)

Since for a one-degree-of-freedom system (as represented by a modal equa-
tion) interchanging a hysteretic damping ratio with a viscous damping ratio
\( (d = \beta) \) makes a negligible difference, we may define the equivalent viscous
fraction of damping for the \( n^{th} \) mode as
\[ \beta_{\text{neq}} = \frac{\sum (\beta_j \frac{\lambda_j}{\omega_j} + d_j) \cdot E_{nj}}{\sum E_{nj}} \]  

(6.115)

which constitutes the Biggs-Roesset equation. This formula states that the energy ratio in each mode at resonance (\( \omega = \lambda_n \)) is a weighted average of the energy ratios in each individual component (evaluated at the frequency of the mode) with weighting factors equal to the modal strain energies in each component.

It should be noted that equation (6.115) has been derived on the assumption that each mass moves synchronously with the others; that is, that all the motions are in phase. This does not imply, however, that the harmonic forces required to sustain the motion are in phase with the displacements, or even that the components of the external force vector are in phase with each other. Only if the system has normal modes will all the force components be in phase. (A system is said to have normal modes when the modes of the undamped structure are orthogonal with respect to the damping matrix; see sec. ). This means that if the system is left free to vibrate with initial conditions (displacements and velocities) matching a given undamped mode, then the system will execute an attenuated vibration in that mode, only if normal damped modes do exist; otherwise, other undamped modes will be excited as well, and the attenuated vibration will involve a combination of several or all the modes of the system.

It is of interest to show that the Biggs-Roesset criterion is equivalent to an enforcement of the normal mode condition; that is, to disregarding the off-diagonal terms in the modal transformation of the damping matrix. To demonstrate this, assume that the structure is forced to vibrate in the \( n^{\text{th}} \) normal mode. The displacement vector is then

\[ V = Q_n \sin \lambda_n t \]  

(6.116)

with \( Q_n = n^{\text{th}} \) normal mode of the system. If \( \bar{M}, K, C \) are the mass, stiffness and damping matrices of the structure, then the maximum strain energy stored in the system is

\[ E_s = \frac{\pi}{2} \lambda_n V^T \bar{K} \cdot V = \frac{\pi}{2} \lambda_n V^T K \ddot{V} \, dt \]

(6.117)

\[ = \frac{1}{2} Q_n^T K Q_n = \frac{1}{2} \lambda_n^2 Q_n^T \bar{M} Q_n \]
while the energy dissipated during one cycle of motion is

\[
E_d = \int_{\lambda_n}^{2\pi} \lambda_n \dot{V}^T C \dot{V} \, dt
\]

\[= \pi \lambda_n Q_n^T C Q_n \]

Therefore, the energy ratio can be computed as

\[
E = \frac{E_d}{E_s} = \frac{\pi \lambda_n Q_n^T C Q_n}{\frac{1}{2} \lambda_n^2 Q_n^T \ddot{M} Q_n} \]

(6.119)

which, according to the Biggs-Roesset criterion, is equal to \(4\pi \beta_n\). It follows that

\[
\beta_n = \frac{1}{2\lambda_n} \frac{Q_n^T C Q_n}{Q_n^T \ddot{M} Q_n} \]

(6.120)

and

\[
2\lambda_n \beta_n = \frac{Q_n^T C Q_n}{Q_n^T \ddot{M} Q_n} \]

(6.121)

The quotient on the right-hand side of this equation is precisely the diagonal term of the modal transformation of \(C\), divided by the corresponding term of the mass matrix. Thus, if the modes are orthogonal with respect to the damping matrix (i.e., if \(Q_n^T C Q_m = 0\) when \(n \neq m\)), then the Biggs-Roesset equation provides the true fraction of critical damping for the various modes, and does not constitute an approximation. Otherwise, the technique is equivalent to a modal transformation of the equations of motion, followed by a deletion of the off-diagonal terms. While the approximation works well even when these off-diagonal terms are not negligible, it deteriorates when there is not only a substantial difference in the components' dampings, but also in their stiffnesses (a couple of orders of magnitude or more). In these cases, a frequency solution is recommended.

Returning now to the soil-structure interaction problem as formulated in section 6.7, the normal modes of the system in local coordinates are

\[
Q_n = \phi X_n \]

(6.122)
The transformation matrix $\phi$ used above is defined by equation (6.85), in which the vectors $X_n$ follow from the eigenvalue problem expressed by equation (6.79). These modes can be written in partitioned form as

$$
Q_n = \begin{pmatrix}
\phi_n \\
\phi_0
\end{pmatrix}
X_n = \begin{pmatrix}
Q_n \\
Q_0
\end{pmatrix}
$$

(6.123)

As before, we force the system to vibrate in its $n^{th}$ coupled mode, having frequency $\lambda_n$. The energy thereby dissipated in the structure is then (compare with Eq. 6.151):

$$
E_{da} = \pi \left[ \left( \lambda_n C_{aa} + 2d_a K_{aa} \right) Q_{an} \right]
$$

$$
= \pi \left[ \phi_n T C_{aa} \phi_n + 2 d_a \phi_a T K_{aa} \phi_a \right] X_{an}
$$

(6.124)

$$
= 2\pi \left[ \phi_n T B_a \Omega_a^{-1} d_a \Omega_a \phi_n \right] X_{an}
$$

The last line is based on the assumption that the modal transformation, $\phi_n^T C_{aa} \phi_n = 2B_a \Omega_a$, produces a diagonal matrix $B_a = \text{diag} \{\beta_{aj}\}$. Factoring out the spectral matrix $\Omega_a$, we obtain

$$
E_{da} = 2\pi \left[ \phi_n^T B_a \Omega_a^{-1} + d_a \text{I}_a \right] \Omega_a \phi_n X_{an}
$$

(6.125)

$$
= 2\pi \left[ \phi_n^T B_a \Omega_a^{-1} + d_a \text{I}_a \right] Z_{an}
$$

with $Z_{an} = \Omega_a \phi_n = \{z_{anj}\}

(6.126)

Equation (6.125) can be written in terms of the components as

$$
E_{da} = 2\pi \sum_j \left( \beta_{aj} \frac{\lambda_n}{\omega_{aj}} + d_a \right) z_{anj}^2
$$

(6.127)

with $d_{aj} = d_a$.

On the other hand, the energy dissipated in the soil by radiation and hysteresis is
\[ E_{do} = \pi \mathbf{Q}_{on}^T \left( \lambda_n \mathbf{C}_f + 2 \mathbf{d}_o \mathbf{K}_f \right) \mathbf{Q}_{on} \]  
(6.128)

\[ = \pi \lambda_n \mathbf{Q}_{on}^T \mathbf{C}_f \mathbf{Q}_{on} + 2\pi \mathbf{d}_o \mathbf{Q}_{on}^T \mathbf{K}_f \mathbf{Q}_{on} \]

But
\[ \mathbf{Q}_{on}^T \mathbf{K}_f \mathbf{Q}_{on} = \mathbf{x}_{on}^T \mathbf{\phi}_o^T \mathbf{K}_f \mathbf{\phi}_o \mathbf{x}_{on} \]
\[ = \mathbf{x}_{on}^T \mathbf{\phi}_o^T \mathbf{x}_{on} \]
\[ = \mathbf{z}_{on}^T \mathbf{z}_{on} \]  
(6.129)

with
\[ \mathbf{z}_{on} = \mathbf{\Omega}_o \mathbf{x}_{on} = \{z_{onj}\} \]  
(6.130)

Hence
\[ E_{do} = \pi \lambda_n \mathbf{Q}_{on}^T \mathbf{C}_f \mathbf{Q}_{on} + 2\pi \mathbf{d}_o \mathbf{z}_{on}^T \mathbf{z}_{on} \]
\[ = \pi \lambda_n \mathbf{Q}_{on}^T \mathbf{C}_f \mathbf{Q}_{on} + 2\pi \mathbf{d}_o \sum_j^6 \sum_l^j z_{onj}^2 \]  
(6.131)

Finally, the strain energy in the soil-structure system is
\[ E_s = \frac{1}{2} \mathbf{Q}_{on}^T \mathbf{K}_n \mathbf{Q}_{on} = \frac{1}{2} \left( \mathbf{x}_{an}^T \mathbf{\phi}_a^T \mathbf{K}_a \mathbf{\phi}_a \mathbf{x}_{an} + \mathbf{x}_{on}^T \mathbf{\phi}_o^T \mathbf{K}_f \mathbf{\phi}_o \mathbf{x}_{on} \right) \]
\[ = \frac{1}{2} \left( \mathbf{z}_{an}^T \mathbf{z}_{an} + \mathbf{z}_{on}^T \mathbf{z}_{on} \right) = \frac{1}{2} \left( \sum_j^6 z_{anj}^2 + \sum_l^j z_{onj}^2 \right) \]  
(6.132)

The weighted modal damping \( \beta_n \) is then
\[ \beta_n = \frac{1}{4\pi} \frac{E_d}{E_s} = \frac{1}{4\pi} \frac{E_{da} + E_{do}}{E_s} \]  
(6.133)

and by substitution,
\[ \beta_n = \sum_j^6 \left( \frac{\lambda_n \omega_{aj} + d_{aj} \omega_{anj}}{\omega_{anj}^2} \right) z_{anj}^2 + \frac{1}{2} \lambda_n \mathbf{Q}_{on}^T \mathbf{C}_f \mathbf{Q}_{on} + \mathbf{d}_o \sum_l^j \sum_l^l z_{onj}^2 \]  
(6.134)

In the above equation we have not expanded the term in \( \mathbf{C}_f \), because the transformation \( \mathbf{\phi}_o^T \mathbf{C}_f \mathbf{\phi}_o \) does not in general diagonalize this matrix. Notice that the modal components
\[ Z_n = \Omega X_n = \begin{bmatrix} Z_{an} \\ Z_{on} \end{bmatrix} = \begin{bmatrix} \Omega_a & X_{an} \\ \Omega_o & X_{on} \end{bmatrix} \]

satisfy the eigenvalue problem defined by equation (6.101).

**Example**

A two-story lumped mass structure with the properties indicated has 5% structural damping in each of its modes on fixed base. The foundation is supported by a spring-dashpot system that models the lateral stiffness of the subgrade. Determine the coupled frequencies, modes and weighted modal damping ratios.

\[ \begin{align*}
  m & = \text{mass of each floor, } \frac{k}{2} \\
  k & = \text{lateral stiffness of subgrade, } \frac{d_a}{c/m} = \text{damping ratio} = \frac{0.05}{6.00} = \frac{1}{120} \\
  c & = \text{dashpot, } k = 100, \\
  \end{align*} \]

\[ \text{a) Modes of structure on fixed base: (Use closed-form solution)} \]

**Frequencies**

\[ \omega_a = 2 \sqrt{\frac{k}{m}} \sin \frac{\pi}{4N} \left(2j-1\right) \quad (N=2) \]

\[ \omega_{a1} = 2.10 \cdot \sin \frac{\pi}{8} = 7.6537 \text{ rad/sec} \]

\[ \omega_{a2} = 2.10 \cdot \sin \frac{3}{8} \pi = 18.477 \text{ rad/sec}. \]

**Modal shapes (normalized)**

\[ \Phi_a = \sqrt{\frac{2}{mn}} \begin{bmatrix} 1 & 1 \\ \cos \frac{\pi}{4} & \cos \frac{3}{4} \pi \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} \]
Participation Factors (for base translation)

\[
\Gamma_a = \phi_a^T M_{aa} \Gamma_a \\
= \frac{1}{\sqrt{m}} \left\{ \begin{array}{cc} 1 & \frac{\sqrt{2}}{2} \\ 1 & -\frac{\sqrt{2}}{2} \end{array} \right\} \left\{ \begin{array}{c} \frac{m}{2} \\ m \end{array} \right\} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} = \sqrt{m} \left\{ \frac{1 + \frac{\sqrt{2}}{2}}{2} \right\} \\
= \sqrt{m} \left\{ \begin{array}{c} 1.207 \\ -0.207 \end{array} \right\}
\]

b) Modes of rigid structure on soil springs (a 1-dof system)

\[
M_o \rightarrow \frac{5}{2} m \\
K_F \rightarrow k \\
\rightarrow \omega_{01} = \sqrt{\frac{k}{\frac{5}{2} m}} = 10 \frac{\sqrt{2}}{5} = 6.3246 \text{ rad/sec}
\]

The normalized mode is \( \phi_{01} = \sqrt{\frac{2}{5m}} = \frac{1}{\sqrt{m}} 0.6325 \)

c) Coupled frequencies and modes

We could derive these directly in closed form for this particular example. However, to illustrate the use of the material presented earlier, we shall formulate the problem via equations (6.97) and (6.101):

\[
\chi_2 \chi_1 \left\{ \begin{array}{c} \Gamma_a \phi_o \\ \text{sym} \end{array} \right\} X_n = \left\{ \begin{array}{c} \omega_a^2 \\ \Omega_o^2 \end{array} \right\} X_n
\]

but

\[
\Gamma_a \phi_o = \sqrt{m} \left\{ \begin{array}{c} 1.207 \\ -0.207 \end{array} \right\} \frac{1}{\sqrt{m}} 0.6325 = \left\{ \begin{array}{c} 0.7634 \\ -0.1310 \end{array} \right\}
\]

and

\[
\omega_a^2 = \left\{ \begin{array}{c} \omega_{a1}^2 \\ \omega_{a2}^2 \end{array} \right\} = \left\{ \begin{array}{c} 58.579 \\ 341.42 \end{array} \right\}
\]

\[
\Omega_o^2 = \left\{ \begin{array}{c} \omega_{01}^2 \end{array} \right\} = 4 0.
\]
Hence
\[
\lambda^2 \begin{pmatrix}
1 & 0.7634 \\
0.7634 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-0.1310
\end{pmatrix} = \begin{pmatrix}
58.579 \\
341.42
\end{pmatrix}
\]
\[
\begin{pmatrix}
1.7071 \\
0.2929 \\
1.5772
\end{pmatrix} = \begin{pmatrix}
1.5772 \\
-0.1121 \\
2.50
\end{pmatrix} Z_n = \frac{100}{\lambda^2} Z_n = \varepsilon Z_n
\]

and from here

The above equation represents a special eigenvalue problem of third rank. An estimate of the fundamental frequency follows from the trace (1st Dunkerley approximation)

\[c_1 = \frac{100}{\lambda_1^2} = 1.7071 + 0.2929 + 2.50 = 4.5\]

that is,
\[\lambda_1^2 \approx \frac{100}{45} = 22.2, \quad \lambda_1 \approx 4.71.\]

This eigenvalue problem can be solved easily (but laboriously) by direct iteration. Here, we shall expand instead the determinant, and find the eigenvalues by trial and error:

\[D = (1.7071 - \varepsilon)(0.2929 - \varepsilon)(25 - \varepsilon) - 1.5772^2 (0.2929 - \varepsilon)
- 0.1121^2 (1.7071 - \varepsilon)
= 0.5 - 3.0 \varepsilon + 4.5 \varepsilon^2 - \varepsilon^3.\]

This equation has the three solutions
\[\varepsilon_1 = 3.7321 \Rightarrow \lambda_1^2 = \frac{100}{\varepsilon_1} = 22.22, \quad \lambda_1 = 4.714\]
\[\varepsilon_2 = 0.5000 \Rightarrow \lambda_2^2 = \frac{100}{\varepsilon_2} = 200.00, \quad \lambda_2 = 14.142\]
\[\varepsilon_3 = 0.268 \Rightarrow \lambda_3^2 = \frac{100}{\varepsilon_3} = 373.13, \quad \lambda_3 = 19.316\]
The eigenvectors are found by substitution of the eigenvalues into the characteristic equation. Calling $a,b$ the first two components, we assign the arbitrary value 1 to the third component:

First mode:

\[
(1.7071 - 3.7321)a + 1.5772\cdot1 = 0, \quad a = 0.7789 \\
(0.2924 - 3.7321)b - 0.1121\cdot1 = 0, \quad b = -0.0326
\]

Second mode:

\[
(1.7071 - 0.5000)a + 1.5772\cdot1 = 0 \quad a = -1.3066 \\
(0.2929 - 0.5000)b - 0.1121\cdot1 = 0 \quad b = -0.5413
\]

Third mode:

\[
(1.7071 - 0.2680)a + 1.5772\cdot1 = 0 \quad a = -1.0960 \\
(0.2929 - 0.2680)b - 0.1121\cdot1 = 0 \quad b = 4.5020
\]

The modal matrix is then

\[
Z' = \begin{bmatrix}
0.7789 & -1.3066 & -1.0960 \\
-0.0326 & -0.5413 & 4.5020 \\
1.000 & 1.000 & 1.000
\end{bmatrix}
\]

or renormalizing by the largest component in each column,

\[
Z = \begin{bmatrix}
0.7789 & 1.000 & -0.2434 \\
-0.0326 & 0.4143 & 1.000 \\
1.000 & -0.7653 & 0.2221
\end{bmatrix}
\]

The modal matrix $X$ is then

\[
X = \Omega^{-1}Z = \begin{bmatrix}
7.6357 & -1 \\
18.477 & \end{bmatrix}^{-1} \begin{bmatrix}
0.7789 & 1.000 & -0.2434 \\
-0.0326 & 0.4143 & 1.000 \\
1.000 & -0.7653 & 0.2221
\end{bmatrix}
\]
\[
X = \frac{1}{10} \begin{bmatrix}
1.0200 & 1.3096 & -0.3188 \\
-0.0176 & 0.2242 & 0.5412 \\
1.5811 & -1.2100 & 0.3512
\end{bmatrix}
\]

and the modal matrix \( Q \) (in relative coordinates) is

\[
Q = \frac{\phi X}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}} \begin{bmatrix}
1 & 1 \\
\frac{\sqrt{\lambda}}{2} & \frac{\sqrt{\lambda}}{2} \\
0.6325 & 0.6325
\end{bmatrix} \begin{bmatrix}
0.10200 & 0.13096 & -0.03188 \\
-0.00176 & 0.02242 & 0.05412 \\
0.15811 & -0.12100 & 0.03512
\end{bmatrix}
\]

\[
Q = \frac{1}{\sqrt{\lambda}} \begin{bmatrix}
0.10024 & 0.15338 & 0.02224 \\
0.07337 & 0.07675 & -0.06081 \\
0.10000 & -0.07653 & 0.02221
\end{bmatrix}
\]

(In practical cases, we would compute only \( Q_{on}^T = \phi_0X \), not \( Q \)). The modal damping ratios are obtained from eq. (6.134), specialized for the case at hand:

\[
\beta_n = \frac{\sum_{j=1}^{3} \frac{1}{2} \lambda_n Q_{on}^T C Q_{on} z_{nj}^2}{\sum_{j=1}^{3} z_{nj}^2}
\]

First mode

\[
z_{11}^2 + z_{21}^2 = (0.7789)^2 + (0.0376)^2 = 0.6077
\]

\[
z_{11}^2 + z_{21}^2 + z_{31}^2 = 0.6077 + (1.000)^2 = 1.6077
\]

Also,

\[
\frac{1}{2} \lambda_1 Q_{o1}^T C Q_{o1} = 0.5 \times 4.714 \times 0.1 \times 6 \times 0.1 = 0.1414
\]

\[
\beta_1 = \frac{0.05 \times 0.6077 + 0.1414}{1.6077} = 0.1069
\]

or \( \beta_1 = 10.7\% \)

Second mode

\[
z_{12}^2 + z_{22}^2 = (1.000)^2 + (0.4143)^2 = 1.1716
\]

\[
z_{12}^2 + z_{11}^2 + z_{32}^2 = 1.1716 + (0.7653)^2 = 1.7573
\]
\[
\frac{1}{2} \lambda_2 Q_{o2}^T C Q_{o2} = 0.5 \times 14.142 \cdot (-0.07653) \cdot 6 \cdot (-0.07653) = 0.2485
\]

\[
\beta_2 = \frac{0.05 \cdot 1.1716 + 0.2485}{1.7573} = 0.175
\]

or \( \beta_2 = 17.5\% \)

**Third mode**

\[
z_{13}^2 + z_{23}^2 = (0.2434)^2 + (1.000)^2 = 1.0592
\]

\[
z_{13}^2 + z_{23}^2 + z_{33}^2 = 1.9592 + (0.2221)^2 = 1.1086
\]

\[
\frac{1}{2} \lambda_3 Q_{o3}^T C Q_{o3} = 0.5 \times 19.316 \times (0.0222) \times 6 \times (0.0222) = 0.0286
\]

\[
\beta_3 = \frac{0.05 \times 1.0592 + 0.0286}{1.1086} = 0.0736
\]

or \( \beta_3 = 7.4\% \).