DYNAMIC ANALYSIS OF FOOTINGS
ON LAYERED MEDIA

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INTRODUCTION

A considerable amount of work has been done in recent years to obtain improved solutions for the dynamic response of a rigid circular plate resting on a stratum or an elastic half space. This problem is of particular interest in the seismic design of structures accounting for dynamic soil structure interaction effects.

Recent studies (9,17), have provided solutions for the case of a surface foundation on a homogeneous isotropic half space, with relaxed boundary conditions which seem to introduce very little error (smooth footing). In practice, however, soils are heterogeneous; their properties vary with depth, they may be stratified in layers, and underground water adds further complication to their physical behavior. Many foundations have, in addition, substantial embedment. To account more realistically for these effects, numerical solutions, based on finite element or finite difference techniques, are normally used. A main problem in this case is to account for the proper boundary conditions at the edges of a finite domain which will not introduce undesirable reflections of waves into the region of interest. A possible solution is to place the boundaries at a substantial distance from the footing if there is internal dissipation of energy in the soil. This approach requires a very large number of elements and is therefore expensive. Other forms of absorbing boundaries based on the propagation of specific types of waves have been suggested (1,8,10). They are all, however, of an approximate nature, and they may introduce substantial errors at high frequencies unless the boundary is again far away from the footing.

This paper presents an efficient numerical method for the dynamic analysis

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of circular foundations resting on, or embedded in, viscoelastic soil layers of infinite horizontal extent. The procedure is a generalization of a technique developed by Waas and Lysmer (11, 18), extended to the analysis of axisymmetric systems under arbitrary nonaxisymmetric loads or displacements.

The geometry is idealized by a finite irregular region joined to a semi-infinite far-field (Fig. 1). The irregular region is discretized by means of toroidal finite elements of arbitrary expansion order having three degrees-of-freedom per nodal ring. The far-field is represented by a semi-analytic energy transmitting boundary, based on the exact displacement functions in the horizontal direction, and an expansion in the vertical direction consistent with that used for the finite elements.

The procedure is restricted to harmonic loadings; therefore, the analysis must be performed in the frequency domain. General transient loadings can be handled using fast Fourier transformation techniques.

![Diagram](image)

**FIG. 1.—System Idealization**

**Displacement and Load Expansions**

For an axisymmetric geometry, the general three-dimensional problem can be divided into a number of uncoupled two-dimensional problems, by representing loads and displacements by a Fourier series of the form

\[
\begin{align*}
    u &= \sum_n \tilde{u} \left( \frac{\cos n\theta}{\sin n\theta} \right); \\
    w &= \sum_n \tilde{w} \left( \frac{\cos n\theta}{\sin n\theta} \right); \\
    v &= \sum_n \tilde{v} \left( \frac{-\sin n\theta}{\cos n\theta} \right)
\end{align*}
\]  

(1)

in which \( \tilde{u} \), \( \tilde{w} \), and \( \tilde{v} \) are components of the radial, vertical, and tangential displacements, respectively. The terms, \( \cos n\theta \) for \( \tilde{u} \) and \( \tilde{w} \) and \( -\sin n\theta \) for \( \tilde{v} \), are used for a symmetric situation, whereas an antisymmetric problem corresponds to the use of \( \sin n\theta \) for \( \tilde{u} \) and \( \tilde{w} \) and \( \cos n\theta \) for \( \tilde{v} \).

Similar expressions can be written for the radial, vertical, and tangential components of the loads. Due to the orthogonality of the Fourier series, there is a one to one correspondence between each term of the load and displacements expansions. This pseudo-tridimensional analysis offers considerable savings in storage and computation time as compared to a full three-dimensional analysis, and it has been used for many years in the solution of shells of revolution.
and other problems with axisymmetric geometry. For the particular problem considered here, an additional simplification is introduced by the fact that only the first two terms in the series are needed. Only the first component \( (n = 0) \) is necessary for the analysis of vertical vibrations (symmetric case) or torsional vibrations (antisymmetric case), and only the second term \( (n = 1) \) is needed for the study of horizontal or rocking vibrations. The method can be applied, however, to the solution of general cases where more terms of the series are necessary.

**Dynamic Equations**

Use of the displacement expansions, Eqs. 1, yields for each value of \( n \) a set of strain-displacement relations

\[
\bar{\varepsilon} = \mathbf{A} \bar{\mathbf{U}}
\]

and stress-strain relations

\[
\bar{\sigma} = \mathbf{D} \bar{\varepsilon}
\]

in which \( \bar{\mathbf{U}}, \bar{\varepsilon}, \) and \( \bar{\sigma} \) are the displacement, strain, and stress vectors corresponding to the value of \( n \) under consideration. Thus

\[
\bar{\mathbf{U}} = \begin{bmatrix} \bar{u} \\ \bar{\omega} \\ \bar{v} \end{bmatrix}^T; \quad \bar{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{r\theta} \\ \gamma_{rz} \end{bmatrix}^T; \quad \bar{\sigma} = \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{r\theta} \\ \tau_{rz} \end{bmatrix}^T
\]

Matrix \( \mathbf{A} \) is the operator matrix corresponding to the expression of strains as functions of the displacements, and \( \mathbf{D} \) is the constitutive matrix. Material properties may be a function of \( r \) and \( z \) only. Partitioning vectors and matrices so as to separate in-plane and out-of-plane components, it can be shown that these two matrices are of the form

\[
\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & n\mathbf{A}_{12} \\ n\mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}; \quad \mathbf{D} = \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{bmatrix}
\]

Substitution of the displacement expansion, Eqs. 1, into the dynamic equilibrium equations in cylindrical coordinates for harmonic loading yields

\[
(\lambda + 2G) \frac{\partial \bar{\Delta}}{\partial r} + 2G \frac{n}{r} \bar{\omega}_z + 2G \frac{\partial \bar{\omega}_\theta}{\partial z} + \rho \Omega^2 \bar{u} = 0;
\]

\[
(\lambda + 2G) \frac{\partial \bar{\Delta}}{\partial z} - 2G \frac{\partial (r \bar{\omega}_{\theta})}{\partial r} - 2G \frac{n}{r} \bar{\omega}_r + \rho \Omega^2 \bar{\omega}_r = 0;
\]

\[
(\lambda + 2G) \frac{n}{r} \bar{\Delta} - 2G \frac{\partial \bar{\omega}_r}{\partial z} + 2G \frac{\partial \bar{\omega}_z}{\partial r} + \rho \Omega^2 \bar{v} = 0
\]

which involve the independent variables, \( r \) and \( z \), and the order of the term of the Fourier series, \( n \). The terms, \( \bar{\Delta}, \bar{\omega}_r, \bar{\omega}_\theta, \) and \( \bar{\omega}_z \), are

\[
\bar{\Delta} = \frac{\bar{u}}{r} + \frac{\partial \bar{u}}{\partial r} - n \frac{\bar{v}}{r} + \frac{\partial \bar{w}}{\partial z} ; \quad \bar{\omega}_r = \frac{1}{2} \left( \frac{n}{r} \bar{\omega}_r - \frac{\partial \bar{\omega}_r}{\partial z} \right);
\]
\[ \tilde{\omega}_a = \frac{1}{2} \left( \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \tilde{v}}{\partial r} \right) ; \quad \tilde{\omega}_i = \frac{1}{2r} \left( \frac{\partial (r \tilde{v})}{\partial r} - \tilde{v} \right) \] \hspace{1cm} (7)

The general solution of this system of partial differential equations can be written (16) as
\[ \tilde{\mathbf{U}} = \tilde{\mathbf{U}}_1 + \tilde{\mathbf{U}}_2 + \tilde{\mathbf{U}}_3 \] \hspace{1cm} (8)
in which the particular solutions, \( \tilde{\mathbf{U}}_1, \tilde{\mathbf{U}}_2, \) and \( \tilde{\mathbf{U}}_3, \) and
\[ \tilde{\mathbf{U}}_1 = kA \begin{bmatrix} \frac{\partial H_n^{(2)}(kr)}{\partial r} \\ -lH_n^{(2)}(kr) \\ n \frac{H_n^{(2)}(kr)}{r} \end{bmatrix} e^{-iz} e^{i\Omega t} ; \quad \tilde{\mathbf{U}}_2 = B \begin{bmatrix} \frac{n}{r} H_n^{(2)}(kr) \\ 0 \\ \frac{\partial H_n^{(2)}(kr)}{\partial r} \end{bmatrix} e^{-mz} e^{i\Omega t} ; \]
\[ \tilde{\mathbf{U}}_3 = C \begin{bmatrix} m \frac{\partial H_n^{(2)}(kr)}{\partial r} \\ k^2 H_n^{(2)}(kr) \\ -mn \frac{H_n^{(2)}(kr)}{r} \end{bmatrix} e^{-mz} e^{i\Omega t} \] \hspace{1cm} (9)
in which \( H_n^{(2)}(kr) \) are second Hankel functions of order \( n; \) \( A, \) \( B, \) and \( C \) are constants of integration; \( k \) is an arbitrary parameter representing the wave number, and
\[ l^2 = k^2 - \alpha^2 ; \quad m^2 = k^2 - \beta^2 ; \quad \alpha = \frac{\Omega}{C_p} ; \quad \beta = \frac{\Omega}{C_s} ; \]
\[ C_p = \sqrt{\frac{\lambda + 2G}{\rho}} ; \] and \( C_s = \sqrt{\frac{G}{\rho}} \] \hspace{1cm} (10)

Analogous expressions containing first Hankel functions \( H_n^{(1)}(kr) \) have been omitted in the preceding particular solutions because they correspond to waves traveling towards the origin, which must be disregarded for the problem at hand. The index 2 and the argument \( kr \) will be omitted from here on, since there is no possibility of confusion.

Adding the three particular solutions and factoring out the Hankel functions, the general solution can be rewritten as
\[ \tilde{\mathbf{U}} = \mathbf{H} \mathbf{F} \] \hspace{1cm} (11)
in which \( \mathbf{H} = \begin{bmatrix} \frac{\partial H_n}{\partial r} & 0 & \frac{n}{r} H_n \\ 0 & kH_n & 0 \\ \frac{n}{r} H_n & 0 & \frac{\partial H_n}{\partial r} \end{bmatrix} ; \quad \mathbf{F} = \begin{bmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \end{bmatrix} \] \hspace{1cm} (12)
Correspondingly, the dynamic equilibrium equations for each term $n$ of the Fourier series (6) can be expressed as

\[
\tilde{\mathbf{W}} = \mathbf{HL} = \mathbf{O}
\]

with

\[
\mathbf{L} = [l_1(z), l_2(z), l_3(z)]^T
\]

and

\[
l_1(z) = k(\lambda + 2G)(f_3'' - kf_3) + G(f_1'' - kf_1) + \rho \Omega^2 f_1;
\]

\[
l_2(z) = (\lambda + 2G)(f_3'' - kf_3) + Gk(f_1' - kf_2) + \rho \Omega^2 f_2;
\]

\[
l_3(z) = G(f_3'' - k^2f_3) + \rho \Omega^2 f_3
\]

(14)

Stresses and strains can similarly be expressed for each $n$ in terms of the Hankel functions (function of $r$) and the $f_1, f_2, f_3$ (functions of $z$).

**Principle of Virtual Displacements**

Both the finite elements and the consistent energy absorbing boundary are based on a variational formulation which makes use of the principle of virtual displacements. Two equivalent forms of this principle are, for a given value of $n$ (term of the Fourier expansion):

\[
\int \int_V (\delta \tilde{\mathbf{e}}^T D \tilde{\mathbf{e}} - \rho \Omega^2 \delta \tilde{\mathbf{U}}^T \tilde{\mathbf{U}}) r dr dz - \int_{S_p} \delta \tilde{\mathbf{U}}^T \tilde{\mathbf{P}} r ds = 0 \quad \ldots \quad (15)
\]

and

\[
\int \int_V \delta \tilde{\mathbf{U}}^T \tilde{\mathbf{W}} r dr dz + \int_S \delta \tilde{\mathbf{U}}^T (\tilde{\mathbf{P}} - \tilde{\sigma}_v) r ds = 0 \quad \ldots \quad (16)
\]

in which $\tilde{\mathbf{W}} = \mathbf{HL}$ are the wave equations (12) for the value of $n$ under consideration; $\tilde{\sigma}_v$ are the projections of the stresses on the unit outward boundary normal; and $S_p$ is the portion of the boundary where stresses $\mathbf{P}$ are specified.

The first form of the principle of virtual displacements, Eq. 15, is the one normally encountered, and is used for the derivation of the finite element matrices in the core region. The second form, Eq. 16, is equivalent but it is particularly convenient for the formulation of the consistent boundary. Note that in this latter form, the second integral extends over the whole boundary if the displacement expansions do not satisfy all the boundary conditions.

**Finite Element Region**

In the present study, the core region was modeled by means of isoparametric quadrilateral elements, in which the displacement expansions are the same as the expressions for the coordinates. The details of this formulation are well known and need not be repeated here. It suffices to point out that separating again the in-plane and the out-of-plane degrees-of-freedom, the stiffness matrix of the element for a given value of $n$ is of the form

\[
\mathbf{K} = \begin{bmatrix}
K_1 + n^2 K_2 & n K_3 \\
K_3^T & K_4 + n^2 K_5
\end{bmatrix}
\]

(17)

in which $K_1, K_2, K_3, K_4,$ and $K_5$ are independent of $n$. The mass matrix is independent of $n$ as well.
Energy Transmitting Boundary

One of the basic problems in modeling a semi-infinite domain by a finite region is to provide boundary conditions that reproduce properly the behavior and interaction of the missing far-field. The main effect of the far-field is to cause a loss of energy due to the radiation of waves away from the core region. It is, therefore, often referred to as radiation or geometric damping.

The radiation will take place both vertically and laterally. When a stratum of soil is resting on a harder material (rock) with much stiffer properties, the vertical radiation is small and lateral radiation occurs only above the fundamental frequency of the stratum. Thus a special boundary is not of major concern below this frequency. Above the stratum's frequency, however, a considerable amount of radiation takes place, most of it laterally. The importance of this radiation will, of course, depend on the distance at which the boundary is located, the amount of internal damping in the soil (hysteretic dissipation of energy), and the contribution of these frequency components to the solution.

Several solutions have been used to reproduce this effect. When the amount of assumed internal damping is large and the boundary is sufficiently removed from the footing, elementary boundaries (such as roller boundaries which will reproduce exactly a one-dimensional shear type behavior) can be appropriate. The validity of these boundaries must be tested, however, above the resonant frequency of the stratum, and with appropriately small finite elements. Elements which are very elongated in the horizontal direction will automatically enforce a one-dimensional shear-type behavior and they will correspondingly mask out any lateral radiation.

Viscous type boundaries, as suggested by Lysmer and Kuhlemeyer (8,10) or Ang and Newmark (1), are based on a perfect absorption for specific types of waves and incidence angles. They are only approximate for a true two-dimensional situation.

A consistent boundary, which corresponds to the actual condition of finite element columns extending to infinity, has been recently developed by Waas and Lysmer (11,18) for the two-dimensional plane-strain case (strip-footing) and the $n = 0$ axisymmetric case (torsional or vertical vibrations of a circular footing). The foremost advantage of this boundary is that it can be located directly at the edge of the footing, with excellent results, leading thus to an economical and at the same time more accurate solution. The following presents a generalization of this boundary to the case of arbitrary Fourier number $n$ and expansion order in the finite elements.

If the core region is removed and substituted by equivalent distributed forces corresponding to the internal stresses, the dynamic equilibrium of the far-field will be preserved. Since no other prescribed forces act on the far-field, the displacements at the boundary (and any other point in the far-field) will be uniquely defined in terms of these boundary forces. The problem is thus to find the dynamic relation between these boundary forces and the corresponding boundary displacements.

It is always possible to express the displacements in the far-field in terms of eigenfunctions corresponding to the natural modes of wave propagation in the stratum. The general solution to the problem is given by Eqs. 9, in which $A$, $B$, and $C$ are constants of integration and $k$ is an undetermined parameter,
the wave number. In an unbounded medium, any value of \( k \), and thus any wave length, is admissible; for a layered stratum, however, only a numerable set of values of \( k \) (each one with a corresponding propagation mode) will satisfy the boundary conditions. There are thus, at a given frequency, \( \Omega \), an infinite but numerable set of propagation modes and wave numbers \( k \), which can be found by solving a transcendental eigenvalue problem. For each eigenfunction one can determine the distribution of stresses up to a multiplicative constant, the participation factor of the mode. Combining these modal stresses so as to match any given distribution of stresses at the boundary, one can compute the participation factors and, correspondingly, the dynamic stiffness function relating boundary stresses to boundary displacements.

The solution of the actual transcendental eigenvalue problem for the continuum problem is difficult and time consuming, requiring, in general, search procedures. A discrete eigenvalue problem can be obtained by substituting the actual dependence of the displacements on the \( z \) variable, as given by Eqs. 9 and 12, by an assumed expansion consistent with that used for the finite elements. The result is an algebraic (quadratic) eigenvalue problem with a finite number of eigenvectors and eigenvalues, for which efficient numerical solutions are available.

Consider a toroidal section of the far-field limited by two cylindrical surfaces of arbitrary radii, \( r \) and \( r + d \), as shown in Fig. 2. The stratum is discretized in horizontal layers, the interfaces of which match the nodal joints (circles) of the finite element mesh in the core region. The displacements for harmonic motion in each layer are given by Eqs. 11. Let \( x_{1i}, x_{2i}, x_{3i} \) represent the values of the functions \( f_1(z), f_2(z), f_3(z) \) at the \( i \)th interface. Rather than using the exact expressions for the functions, \( f(z) \), containing the integration constants and the wave number, \( k \), these functions are expanded in terms of their nodal values \( x_i \) using the same expansions of the finite element region (Fig. 3). Thus for any layer

\[
F = NX \hspace{1cm} \text{(18a)}
\]

with

\[
X = \begin{bmatrix} x_i \\ x_{i+m} \\ \vdots \end{bmatrix}; \quad X_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{bmatrix}; \quad F = \begin{bmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \end{bmatrix} \hspace{1cm} \text{(18b)}
\]

in which \( m \) is the order of the expansion for the finite elements; and

\[
N = [g_{11}, g_{21}, \ldots, g_{m+1,1}] = [g_{ij}] \hspace{1cm} \text{(18c)}
\]

in which \( I \) is a \( 3 \times 3 \) identity matrix; and \( g_i \) represents the expansion coefficients. For a linear expansion \( (m = 1) \), the coefficients are \( g_1 = 1/2 \) \( (1 - \eta) \), \( g_2 = 1/2 \) \( (1 + \eta) \), with \( -1 \leq \eta \leq 1 \). For a quadratic expansion, \( g_1 = 1/2 \eta \) \( (\eta - 1) \), \( g_2 = 1 - \eta^2 \), \( g_3 = 1/2 \eta \) \( (\eta + 1) \), etc.

The displacements across the layer are then given by Eqs. 11 and 18

\[
\bar{U} = HNX \hspace{1cm} \text{(19)}
\]

with nodal values \( \bar{U}_i = HX_i \hspace{1cm} \text{(20)} \)

These expressions will not satisfy identically the wave equations (12) because of the approximation introduced in the assumed functions of \( z \). Using the same
basic procedure of the finite element formulation, an approximate solution is obtained by substituting the displacement expansions into the expression of the principle of virtual displacements (16), integrating over the region, and requiring the result to vanish for an arbitrary $\delta U$. Furthermore, one requires that equilibrium be preserved on the average at any arbitrary vertical section by applying consistent nodal boundary forces $\bar{P}_o, \bar{P}_d$ at the two boundaries as predicted by the displacement expansions.

The result is a quadratic eigenvalue problem of the form

$$\begin{align*}
(A{k^2} + Bk + G - \Omega^2 M)\chi &= O \\
(21)
\end{align*}$$

and a boundary force vector

![Finite Element—Far Field Boundary](image1)

**FIG. 2.—Finite Element—Far Field Boundary**

![Discrete Layer](image2)

**FIG. 3.—Discrete Layer**

$$\bar{P}_o - r_o \left( A \Psi K^2 + (D - E + nN) \Phi K + n \left( \frac{n + 1}{2} L + Q \right) \Psi \right) \Gamma = 0 \quad (22)$$

in which $A, B, G, M, D, E, N, L,$ and $Q$ are matrices which depend only on the geometry and material properties of the discrete layers; $K$ is a diagonal matrix containing the admissible wave numbers $k$ (eigenvalues of Eq. 21); $\Psi$ and $\Phi$ are modified modal matrices computed from the eigenvectors $\chi$ and appropriate Hankel functions of argument $kr_0$; and $\Gamma$ is a vector containing the modal participation factors. The detailed form of these matrices can be found in Ref. 6.
Due to the special form of the matrices, $A$, $B$, $G$, and $M$, the eigenvalue problem can be broken into two uncoupled eigenvalue problems for $k$: (1) A quadratic problem involving the $x_1$, $x_2$ (nodal values of $f_1$, $f_2$); and (2) a linear problem involving the $x_3$ (nodal values of $f_3$). The former is referred to as a "generalized Rayleigh wave" problem, while the latter is a "generalized Love wave" eigenvalue problem. The reason for these names follows from the fact that $f_1$, $f_2$ define for $n = 0$ a vertically polarized wave; while $f_3$ defines a horizontally polarized wave. Since the eigenvalue problems are independent of $n$ (order of the Fourier expansion for the $\theta$ coordinate), it can be shown that any arbitrary three-dimensional harmonic displacement field in a layered stratum can be expressed as a superposition of generalized Rayleigh and Love waves, with the same wave numbers obtained for the two-dimensional case, and weighted by the appropriate participation factors, the terms $\cos n\theta$ or $\sin n\theta$, and Hankel functions.

The solution of the two eigenvalue problems yields a number of complex eigenvalues and eigenvectors equal to six times the number of layers. Half of them correspond to waves propagating away from the core region, and the other half to waves arriving from the far-field. The latter must be discarded. This is achieved by selecting only the eigenvalues that have a negative imaginary part (leading to an amplitude decaying with $r$) for complex wave numbers, or a positive real part for real wave numbers. A further analysis of the properties of these eigenvalue problems can be found in the original work by the third writer (18).

Expressing the vector of nodal displacements at the boundary interfaces in terms of the modal shapes and participation factors, it is possible to write

$$\bar{U}_o = W^* \Gamma \quad \cdots \quad (23)$$

in which $W^*$ is a matrix assembled with the modal displacements, $\chi$, weighted by Hankel functions of argument $kr_o$; and $\Gamma$ is the modal participation factors vector.

On the other hand, if $R$ is the dynamic stiffness matrix of the energy transmitting boundary, one would have

$$\bar{P}_o = R\bar{U}_o \quad \cdots \quad (24)$$

leading to the definition of $R$

$$R = r_o \left( A \Psi K^2 + (D - E + nN)\Phi K + n \left( \frac{n + 1}{2} L + Q \right) \Psi \right) W^* \quad \cdots \quad (25)$$

This matrix is symmetric, as required by the principle of reciprocity, relating nodal forces at the boundary to corresponding nodal displacements. It is a function of the frequency of vibration, $\Omega$.

The solution of the total problem can then be achieved by adding the boundary matrix, $R$, to the dynamic stiffness matrix of the complete finite element region. This implies operation in the frequency domain, obtaining the transfer functions of the desired effects and using Fourier transforms to obtain time histories.

Implementation of this formulation in a computer program is straightforward; neither the solution of the eigenvalue problems using the procedure described in Ref. 18 nor the formation of the boundary matrix are time consuming. Since
the boundary can be placed directly at the edge of the footing without any loss of accuracy, the finite element region is reduced to the minimum size. Not only are the results obtained over the complete range of frequencies superior to those provided by other schemes, but the total model is more efficient and less costly. The method has only two limitations:

1. It requires a solution in the frequency domain; therefore, it is only applicable to linear systems. Consideration of nonlinear material properties must be done in an approximate way by establishing cyclic linear analyses where values of moduli and internal damping are adjusted at the beginning of each cycle according to the levels of strain reached in the previous cycle. This procedure has been shown (3) to give reasonable results for one-dimensional problems in systems that are not too soft initially and are subjected to moderate levels of excitation. A proper justification for the two-dimensional or three-dimensional cases is still lacking. Note, however, that this is the procedure now commonly used, even with other boundaries.

2. In order to get the simple quadratic eigenvalue problem, it is necessary to consider under the far field a rigid base, where displacements are specified zero for the solution of the eigenvalue problem. Consequently, the formulation will ignore any possible vertical radiation. This limitation is not important if the results are properly interpreted. The effect of vertical radiation has been systematically ignored in many soil dynamics studies (4,15); however, its nature and possible magnitude have been evaluated for the one-dimensional amplification problem (12). In the case at hand, its effect is only important below and at the fundamental frequency of the total soil stratum considered in the analysis. In this range, no lateral radiation takes place, and if the soil had no internal damping, a resonance condition would occur, leading to an infinite peak in the flexibility (compliance) function (or a zero value of the dynamic stiffness). The peak will have a finite value if there is internal damping, and will be further reduced by the vertical radiation. The amount of this reduction and its importance is thus a function of the relative magnitude of the radiation damping versus the internal damping. When there is a clearly defined base rock with stiffness much larger than that of the overlying soil, or when internal damping is high (large intensities of motion), the vertical radiation can be neglected and the assumption of a rigid base is justified. When there is a deep layer of soil without a clear sharp change in elastic properties (elastic half space), and the internal damping is very small (very low intensities of motion), vertical radiation will be important in the range of very low frequencies (since the fundamental period of the deep stratum will be very long). It is then advisable to smooth out the peak occurring at the fundamental frequency of the stratum, since it does not have a true physical significance, and to extrapolate the radiational damping values into the low frequency range.

**EXAMPLE**

To illustrate the accuracy of the boundary, the horizontal and rocking stiffness functions for a circular plate on the surface of a deep homogeneous stratum were first obtained. The results were compared to available solutions for an elastic half space (17). While the half-space results have only limited application,
in practice, considering the true nature of soils and the variation of their properties with depth, they provide a convenient yardstick against which numerical solutions should be tested.

The case considered consisted of a circular plate of radius $R$ resting on a homogeneous viscoelastic stratum with a depth $H$ equal to eight times the radius (Fig. 4). The core region consisted of the vertical cylindrical volume under

![FIG. 4.—Mesh Used in Example: $R = 1$; $C_s = 1; \beta = 0.05$](image)

![FIG. 5.—Swaying Stiffness Coefficients: $\nu = 1/3$; $H/R = 8$](image)

![FIG. 6.—Rocking Stiffness Coefficients: $\nu = 1/3$ and $\nu = 0.45$; $H/R = 8$](image)

the footing, with the boundary placed directly at the edge of the foundation. A series of preliminary tests to investigate the effect of the mesh size in the finite element region (6) confirmed the results of other studies (10); in order to obtain accurate values of the transfer functions at a specified, $\Omega$, the maximum dimension of the finite elements should not exceed one-sixth to one-eighth of the corresponding wave length, $2\pi C_s / \Omega$, in which $C_s$ = the shear wave velocity of the stratum. Furthermore, this limitation applies over the whole region of
interest, the extent of which is a function of frequency, internal damping, and type of excitation.

The internal dissipation of energy in the soil was assumed to be of a hysteretic nature, with the energy loss per cycle independent of frequency. A realistic damping ratio of $\beta = 0.05$ was considered. The solution for the half space is based, however, on an elastic medium. In order to compare results, the dynamic stiffness were written in the form

$$k_D(\Omega) = k(1 + 2i\beta)(k_1 + ia_1c_1)$$  \hspace{1cm} (26)

in which $k$ = the static value; $\beta$ = the hysteretic damping ratio for the soil; $a_1 = \Omega R/C_1$ = the dimensionless frequency; and $k_1$ and $c_1$ are functions of frequency. It is important to note that this expression is only approximate. The ratio, $k_D/(1 + 2i\beta)$, is not independent of $\beta$ as the formula would suggest, and its variation increases with increasing $\beta$ and frequency.

Fig. 5 shows the values of $k$ and $c$ for the case of a horizontal motion of the foundation (swaying or horizontal stiffness), and Fig. 6 of the corresponding functions for the rocking case.

It can be seen that for a depth of the stratum equal to eight times the radius of the foundation, the results are already very close to the half-space solution. For a horizontal excitation, there is a clear waviness in the stiffness function, with valleys corresponding approximately to the fundamental frequencies of the stratum; without internal damping, the stiffness function would actually become zero at these points. In addition, the imaginary part, $c_1$, is essentially zero below the first fundamental frequency, confirming the lack of radiation in this range. Results for horizontal excitation were only obtained up to a dimensionless frequency of $\pi$ since their variation at higher frequencies is small.

For rocking excitation, the agreement is even better and the stiffness is a smoother curve. Only a limited area under the footing is significantly affected by the vibration. Results were extended up to a frequency of $2\pi$ in this case, since the solution is greatly dependent on Poisson’s ratio in the higher frequency range.

It must be concluded that the boundary accounts extremely well for the physical behavior of the far-field and it reproduces quite satisfactorily the radiation effects.

**Base Motion**

The preceding treatment has been intended primarily for the case of a surface excitation (determination of dynamic stiffnesses). For earthquake loadings when the structure is itself axisymmetric, it may be convenient to perform the analysis in a single step by modeling with finite elements both the soil and the structure, and subjecting the combined system to a compatible motion at bedrock. It is then necessary to introduce additional terms in the formulation.

Once more considering the Fourier transform of the specified motion, it is possible to express the displacements in the far-field region due to this motion in terms of a Fourier series in the angle, $\theta$. For each term of the series, one can again separate symmetric and antisymmetric components, and find for each of them the general solution of the homogeneous wave propagation problem, i.e., the displacements that would occur in the stratum if no structure were present. If $Y^*$ are the amplitude of these displacements at the far-field boundary
and $P^*$ are the corresponding nodal forces (resultants of the stresses), then

$$P^* - D^*Y^* \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (27)$$

The form of matrix $D^*$ depends on the assumed wave pattern (generally vertically propagating shear waves), and its terms are a function of $\Omega$ for a given $n$ (order of the Fourier expansion).

The deviation of the boundary displacements, $Y$, from the homogeneous solution, $Y^*$, caused by the presence of the structure requires the application of the boundary forces, $\Delta P$, computed with the boundary stiffness matrix previously described:

$$\Delta P = -R(Y - Y^*) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (28)$$

It is thus sufficient for any $n$ to add at the boundary a set of forces $(D^* + R)Y^*$, since the term, $-RY$, is already accounted for when adding $R$ to the dynamic stiffness matrix of the core region.

This formulation allows consideration of any two-dimensional earthquake motion. In practice, however, application is normally limited to the case of a uniform base motion. The far-field solution (represented by $P^*$, $Y^*$, and matrix $D^*$) is then simply the solution of the one-dimensional amplification problem.

**Conclusions**

An extension of the consistent boundary formulation derived by Lysmer and Waas for plane-strain problems or axisymmetric problems has been presented. The boundary can be placed directly at the edge of the foundation with considerable savings in memory requirements and time of execution for computer solutions. In addition, it provides very accurate results, properly reproducing the effect of waves radiating into the far-field.

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**Appendix I.—References**


**Appendix II.—Notation**

The following symbols are used in this paper:

\[ A, B, D, E, G, L, M, N, Q \] = layer matrices, defined in Ref. 6;
\[ C_p = \] pressure wave velocity;
\[ C_s = \] shear wave velocity;
\[ D^* = \] matrix relating forces \( F^* \) to displacements \( Y^* \) in homogeneous (seismic) wave propagation problem;
\[ G = \] shear modulus;
\[ H = 3 \times 3 \text{ matrix containing Hankel functions (Eq. 12);} \]
\[ H^{(2)}_n(kr) = \] second Hankel functions;
\[ k = \] wave number;
\( m \) = finite element expansion order (generally \( m = 1 \));
\( \mathbf{N} \) = matrix containing expansion coefficients;
\( n \) = Fourier number in expansion around axis;
\( \mathbf{R} \) = boundary stiffness matrix;
\( r_o \) = coordinate of boundary;
\( r, \theta, z \) = cylindrical coordinates;
\( \mathbf{U} \) = displacement vector at point;
\( \mathbf{U}_i \) = displacement vector at nodal point in boundary;
\( \mathbf{U}_o = \{ \mathbf{U}_j \} \) = boundary displacement vector;
\( \mathbf{W} \) = wave equations (Eq. 13);
\( \mathbf{X}, \mathbf{X}_i, \mathbf{F} \) = functions, as defined in Eq. 18;
\( \beta \) = hysteretic (material) damping ratio;
\( \Gamma \) = modal participation factors vector;
\( \Delta, \omega_r, \omega_\theta, \omega_z \) = terms of wave equations as defined by Eq. 7;
\( \mathbf{\epsilon} \) = strain vector;
\( \lambda \) = Lamé constant;
\( \rho \) = mass density;
\( \sigma \) = stress vector;
\( \Psi, \Phi, \mathbf{W}^* \) = modified modal matrices, defined in Ref. 6;
\( \chi \) = modal vector; and
\( \Omega \) = frequency of excitation.

Superscripts

\( T \) = transpose;
\( (') \) = derivatives w/r to \( z \); and
\( (') \) = Fourier expansion around axis.

11652 DYNAMIC ANALYSIS OF FOOTINGS ON LAYERED MEDIA

KEY WORDS: Dynamic structural analysis; Earthquakes; Engineering mechanics; Finite elements; Footings; Foundations; Soil dynamics; Soil-structure interaction; Vibration

ABSTRACT: A finite element formulation for the dynamic analysis of circular footings resting on or embedded in layered soil strata is presented. The formulation provides excellent results, accurately reproducing the lateral radiation effects through a consistent energy transmitting boundary, which is an extension of that suggested by Waas and Lysmer for two-dimensional problems. Because the boundary can be placed directly at the edge of the footing without loss of accuracy, it also provides savings in storage requirements and time of computation over other solutions. The analysis must be performed in the frequency domain; arbitrary transient loading conditions are then handled using fast Fourier transformation techniques.