Dynamics of piles and pile groups in layered soil media

Amir M. Kaynia

Department of Civil Engineering, Isfahan University of Technology, Isfahan, Iran

Eduardo Kausel

Department of Civil Engineering, M.I.T., Cambridge, Massachusetts, USA

A general formulation is presented for the dynamic response analysis of piles and pile groups in a layered halfspace. Green's functions for layered media, evaluated numerically by the application of Integral transform techniques, along with analytical solutions for the dynamic response of piles are the ingredients of this formulation. The analytical derivations are presented in this paper and the extension of the formulation to seismic analyses is described. In addition, a limited number of representative results on the dynamic stiffnesses and seismic response of pile groups are presented.

INTRODUCTION

Investigation into the behaviour of pile groups started with the pioneering work of Poulos in 1968. The early studies and results reported then, focussed on the static aspects of this problem. In these studies Mindlin's fundamental solution was used to obtain a flexibility matrix for the soil medium; the beam equations were then used to satisfy the compatibility requirements between the piles and the soil. These investigations highlighted the main characteristics of group behaviour, such as the reduction in the group stiffness as the pile spacing decreases and the unequal distribution of applied loads among the piles in a group.

The methodology proposed by Poulos was later extended to dynamic analysis of pile groups. The earliest contributions to this subject were due to Wolf and Von Arx and Nogami who showed that the dynamic behaviour of pile groups are strongly frequency-dependent. These observations along with the then raised interest in soil-structure interaction motivated more research on the subject. Some of the reported formulations were of a boundary integral nature, where the Green's functions were evaluated numerically, either by axisymmetric finite element methods or by the solutions of the wave propagation equations. In addition, a number of approximate methods have been developed for this problem. The results reported so far in the literature indicate that under steady-state vibrations one might obtain results that strongly contrast the well-established response characteristics learned from static analyses. For instance, over certain ranges of frequency, a pile group displays a large stiffness (much larger than the sum of stiffnesses of the individual piles) and the center piles take a larger share of the applied loads than edge piles. In addition to more research that remains to be done on certain fundamental aspects of the problem such as nonlinear effects, more parametric studies need to be conducted to enhance the understanding of pile group behaviour.

The objective of this paper is to present a general formulation along with the associated analytical derivations, that were previously reported in Reference 7, for the dynamic analysis of pile groups in semi-infinite layered soil media. In addition, a limited number of representative results are presented for the dynamic stiffnesses and seismic response of pile groups in homogeneous and nonhomogeneous media.

Figure 1 shows the problem under consideration. The

![Fig. 1. Pile groups in a layered semi-infinite soil medium](image)
piles and pile groups in layered soil media: A. M. Kaynia and E. Kausel

Fig. 2. Distribution of tractions on piles

FORMULATION

The surface tractions that develop at the pile-soil interface can be resolved into frictional (z-direction) and two lateral (x- and y-direction) components. Figure 2 shows the distribution of the lateral tractions in the x-direction on one of the piles, say the jth (other components of tractions can be portrayed by similar distributions).

The pile-soil interface is discretized into l arbitrary cylindrical segments along the pile and one circular segment at the pile-tip. The actual distribution of tractions is correspondingly replaced by a piecewise-constant distribution over segments (Fig. 2). In addition, each segment is identified by a node at its center.

If the vector of resultant forces acting on the segments of pile j is denoted by \( \mathbf{p}_j \) and the corresponding vector of nodal displacements is denoted by \( \mathbf{u}_j \), that is

\[
\mathbf{p}_j = \begin{bmatrix} p_{0x}^j & p_{0y}^j & p_{0z}^j & \cdots & p_{l+1x}^j & p_{l+1y}^j & p_{l+1z}^j \end{bmatrix}^T
\]

\[
\mathbf{u}_j = \begin{bmatrix} u_{0x}^j & u_{0y}^j & u_{0z}^j & \cdots & u_{l+1x}^j & u_{l+1y}^j & u_{l+1z}^j \end{bmatrix}^T
\]

then

\[
\mathbf{u}_j = \Psi^j \mathbf{u}_j + \mathbf{F}_j^p \mathbf{p}_j
\]

in which \( \mathbf{u}_j \) denotes the vector of displacements (translation and rotation) for the two ends of pile j, given by

\[
\mathbf{U}_j^e = \begin{bmatrix} u_{0x}^j & u_{0y}^j & u_{0z}^j & \phi_{0x}^j & \phi_{0y}^j & u_{l+1x}^j & \phi_{l+1x}^j & u_{l+1y}^j & \phi_{l+1y}^j \end{bmatrix}^T
\]

\( \Psi^j \) is a \((3(l + 1)) \times 10\) shape matrix defining displacements of the nodes (center of segments) in a fixed-end pile (more specifically, the ith column of \( \Psi^j \) defines the three components of translation of the nodes due to a unit harmonic pile end displacement corresponding to the ith component of \( \mathbf{U}_j^e \)), and \( \mathbf{F}_j^p \) is the flexibility matrix of pile j associated with nodes 1 through \( l+1 \), under fixed-end condition. (Note that, due to this condition, the entries in \( \mathbf{F}_j^p \) corresponding to node \( l+1 \) are zero).

If, in addition, one denotes the dynamic stiffness matrix of pile j by \( \mathbf{K}_j^p \) (relating end forces with end displacements) and the vector of reactions (forces and moments) at the two ends of this pile by \( \mathbf{P}_j^r \), that is

\[
\mathbf{P}_j^r = \begin{bmatrix} R_{0x}^j & R_{0y}^j & R_{0z}^j & M_{0x}^j & M_{0y}^j & M_{0z}^j \end{bmatrix}^T
\]

\( R_{0x}^j \) and \( R_{0y}^j \) and \( R_{0z}^j \) are the reactions at the pile-tip corresponding to unit horizontal, vertical, and rocking forces, respectively. \( \mathbf{F}_j^p \) is a (10 \times 10) flexibility matrix of pile j associated with nodes 1 through \( l+1 \). 

Then one can write

\[
\mathbf{P}_j^r = \mathbf{K}_j^p (\mathbf{U}_j^e) - \Psi^j \mathbf{P}_j^r
\]

(6)

The first term in Equation (6) corresponds to pile-end reactions in a fixed-end pile due to pile-end displacements \( \mathbf{U}_j^e \) when there are no loads on the pile and the second term corresponds to pile-end reactions due to loads on the pile, under fixed end conditions. (Since the reactions at the pile-tip are included in \( \mathbf{P}_j^r \) and matrices \( \mathbf{F}_j^p \) and \( \mathbf{K}_j^p \) are constructed such that they contain the effects of forces and displacements at this point, one has to set \( R_{0x}^j = 0, R_{0y}^j = 0 \) and \( R_{0z}^j = 0 \) equal to zero. Moreover, for floating piles \( M_{0x}^j \) and \( M_{0y}^j \) and \( M_{0z}^j \) are taken to be zero as well).

Defining now the global load and displacement vectors for the N piles in the group

\[
\mathbf{P} = \begin{bmatrix} \mathbf{P}_1^r & \mathbf{P}_2^r & \cdots & \mathbf{P}_N^r \end{bmatrix}
\]

\[
\mathbf{U} = \begin{bmatrix} \mathbf{U}_1^e & \mathbf{U}_2^e & \cdots & \mathbf{U}_N^e \end{bmatrix}
\]

\[
\mathbf{P}_e = \begin{bmatrix} \mathbf{P}_1^r & \mathbf{P}_2^r & \cdots & \mathbf{P}_N^r \end{bmatrix}
\]

\[
\mathbf{U}_e = \begin{bmatrix} \mathbf{U}_1^e & \mathbf{U}_2^e & \cdots & \mathbf{U}_N^e \end{bmatrix}
\]

(7)

as well as the global matrices:

\[
\mathbf{K}_p = \begin{bmatrix} \mathbf{K}_1^p & 0 & \cdots & 0 \\
0 & \mathbf{K}_2^p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{K}_N^p \end{bmatrix}
\]

(8a)

\[
\mathbf{F}_p = \begin{bmatrix} \mathbf{F}_1^p & 0 & \cdots & 0 \\
0 & \mathbf{F}_2^p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{F}_N^p \end{bmatrix}
\]

(8b)

\[
\Psi^p = \begin{bmatrix} \Psi_1^p & 0 & \cdots & 0 \\
0 & \Psi_2^p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Psi_N^p \end{bmatrix}
\]

(8c)

One can then write the following equations for the ensemble of piles in the group

Soil Dynamics and Earthquake Engineering, 1991, Volume 10, Number 8, November 387
Consider next the equilibrium of the soil mass under forces $-\mathbf{P}$ (with a uniform distribution over each segment as in Fig. 2). If $\mathbf{F}_s$ denotes the flexibility matrix of the soil medium, relating piecewise-constant segmental loads to the average displacements along the segments, then

$$\mathbf{U} = -\mathbf{F}_s \mathbf{P}$$

(9)

Finally combining equations (9) and (10) one gets:

$$\mathbf{P}_e = [\mathbf{K}_r + \mathbf{F}_s (\mathbf{F}_s + \mathbf{F}_p)^{-1} \mathbf{K}_r] \mathbf{U}_e = \mathbf{K}_r \mathbf{U}_e$$

(11)

$\mathbf{K}_r$ is a $(10N \times 10N)$ matrix which relates only the five components of forces at each end of the piles to their corresponding displacements. That is, the degrees of freedom along the pile length have been condensed out without forming a complete stiffness matrix.

Matrix $\mathbf{K}_r$ relates forces and displacements at the pile ends in a group of unrestrained piles. In order to obtain dynamic stiffnesses of a pile group, in which the piles are connected to a rigid pile cap, one needs to impose the appropriate kinematic and force boundary conditions at the pile heads and pile tips. The boundary conditions at the pile tips were discussed earlier. At pile heads, the boundary conditions are in general a combination of kinematic and force conditions, unless all the piles are rigidly connected to the cap, in which case only kinematic conditions should be considered.

To extend the formulation to seismic analysis, one only needs to express the displacements $\mathbf{U}$ as the summation of seismic displacements in the medium when the piles are removed (i.e., Soil with cavities) $\mathbf{U}_e$, and the displacements caused by pile-soil interface forces $-\mathbf{P}$, that is

$$\mathbf{U} = \mathbf{U}_e - \mathbf{F}_s \mathbf{P}$$

(12)

Combination of Equations (9) and (12) results in

$$\mathbf{P}_e = [\mathbf{K}_r + \mathbf{F}_s (\mathbf{F}_s + \mathbf{F}_p)^{-1} \mathbf{K}_r] \mathbf{U}_e$$

$$-\mathbf{F}_s (\mathbf{F}_s + \mathbf{F}_p)^{-1} \mathbf{U}$$

(13)

or

$$\mathbf{P}_e = \mathbf{K}_r \mathbf{U}_e + \mathbf{P}_e$$

(14)

Where $\mathbf{k}_r$, as before, is the dynamic stiffness of the pile group associated with the degrees of freedom at pile heads and tips, and $\mathbf{P}_e = -\mathbf{F}_s (\mathbf{F}_s + \mathbf{F}_p)^{-1} \mathbf{U}$ defines consistent fictitious forces at these points which reproduce the
seismic effects. Again, one has to impose the appropriate boundary conditions at the pile ends to obtain the transfer functions from the ground motion to the pile cap.

From the development of the preceding formulation it is clear that $F_z$ is the flexibility of a soil mass which results from the removal of the piles; that is, the soil mass with $N$ cavities. Similarly, $\tilde{U}$ refers to the seismic displacements in the medium with the cavities. Since evaluation of the same quantities in the undisturbed soil mass (i.e., the medium with no cavities) requires much less computational effort, it is desirable to modify the formulation to incorporate this feature. It has been shown that, this can be achieved, approximately, by subtracting the mass density and elasticity modulus of the soil from those of the pile. Then, one could use the same formulation, as expressed by equations (11) and (13), with $F_z$ evaluated for the undisturbed soil mass and $K_p$ evaluated for piles with reduced mass and elasticity modulus. A number of reported research efforts that ignored this modification introduced erroneous inertia effects in their results. Clearly, as frequency increases, such results deviate, to a greater extent, from the true ones. In the seismic analysis, an additional modification is necessary to relate $\tilde{U}$ with the associated free-field (no cavity) values $\tilde{U}^*$. This is achieved by the substructuring theorem; that is, by the relation $\tilde{U} = \tilde{U}^* - F_zP^*$, where $P^*$ denotes the free-field forces corresponding to $\tilde{U}^*$.

GREEN'S FUNCTIONS IN VISCOELASTIC LAYERED SOIL MEDIA

The formulation presented in the previous section requires the evaluation of a dynamic flexibility matrix, $F_z$, for the soil medium. This matrix defines a relationship between piecewise uniform loads distributed over vertical cylindrical surfaces or horizontal circular surfaces and a representative displacement of these regions. Although there are a number of ways to obtain a value to represent the displacement of a region, the weighted averaging, originally proposed by Arnold, Bycroft and Warburton is believed to provide the most meaningful displacement value. The types of loads involved in the problem are shown in Fig. 3. The loads on cylindrical surfaces (barrel loads) are associated with tractions on pile shaft and those on circular surfaces (disk loads) correspond to pile-tip tractions.

The method presented here is an extension of that proposed by Apsel. In the present method, however, a layer stiffness approach, similar to that of Kausel and Roesset, is employed.

Transformation of the Equations of Motion

If $u_r$, $u_\theta$ and $u_z$ are the displacements in the radial, tangential and vertical directions and $f_r$, $f_\theta$ and $f_z$ are the associated external loads per unit volume, the equations of motion of an elastic body in cylindrical coordinates are:

$$\frac{\lambda + 2\mu}{r} \frac{\partial \Delta}{\partial r} - \frac{2\mu}{r} \frac{\partial \omega_r}{\partial \theta} + 2\mu \frac{\partial \omega_\theta}{\partial z} + \omega^2 \rho u_r + f_r = 0 \tag{15a}$$

$$\frac{\lambda + 2\mu}{r} \frac{1}{\partial \theta} - 2\mu \frac{\partial \omega_z}{\partial z} + 2\mu \frac{\partial \omega_r}{\partial \theta} + \omega^2 \rho u_z + f_z = 0 \tag{15b}$$

$$\left(\lambda + 2\mu\right) \frac{\partial \Delta}{\partial z} - \frac{2\mu}{r} \frac{\partial \omega_\theta}{\partial r} + \frac{2\mu}{r} \frac{\partial \omega_r}{\partial \theta} + \omega^2 \rho u_\theta + f_\theta = 0 \tag{15c}$$

$$\Delta = -\frac{1}{r} \frac{\partial}{\partial r} \left(\mu u_r + \frac{1}{r} \frac{\partial \omega_\theta}{\partial \theta} + \frac{\partial \omega_r}{\partial z}\right) \tag{16}$$

$$\omega_r = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{\partial \omega_\theta}{\partial z}\right) \tag{17a}$$

$$\omega_\theta = \frac{1}{2} \left(\frac{\partial u_r}{\partial \theta} - \frac{\partial \omega_r}{\partial z}\right) \tag{17b}$$

$$\omega_z = \frac{1}{2r} \left(\frac{\partial}{\partial r} \left(\mu u_\theta - \frac{\partial \omega_r}{\partial \theta}\right)\right) \tag{17c}$$

For a viscoelastic medium with a hysteretic type dissipative mechanism, one only needs to replace $\lambda$ and $\mu$ by the complex Lamé moduli $\lambda' = \lambda(1 + 2\beta)$ and $\mu' = \mu(1 + 2\beta)$, respectively, where $\beta$ is referred to as the hysteretic damping ratio.

Substitution of these expansions along with equations (16) and (17) into equations (15) leads to the following three conditions to be satisfied for any value of $n$:

$$\mu \left(\frac{\partial^2}{\partial r^2} \left(\mu u_r + \mu u_\theta\right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\mu u_\theta + \mu u_z\right) - \frac{(n + 1)^2}{r^2} \left(\mu u_r + \mu u_\theta\right)\right) + \omega^2 \rho \left(\mu u_r + \mu u_\theta\right) + \left(f_r + f_\theta\right) = 0 \tag{20a}$$

$$\mu \left(\frac{\partial^2}{\partial r^2} \left(\mu u_r - \mu u_\theta\right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\mu u_\theta - \mu u_z\right) - \frac{(n - 1)^2}{r^2} \left(\mu u_r - \mu u_\theta\right)\right) + \omega^2 \rho \left(\mu u_r - \mu u_\theta\right) + \left(f_r - f_\theta\right) = 0 \tag{20b}$$

$$\mu \left(\frac{\partial^2}{\partial r^2} \left(\mu u_z + \mu u_\theta\right) + \frac{1}{r} \frac{\partial}{\partial r} \left(\mu u_\theta + \mu u_z\right) - \frac{n^2}{r^2} \left(\mu u_z + \mu u_\theta\right) + \frac{\partial^2 u_\theta}{\partial z^2}\right) + \left(f_r + f_\theta\right) = 0 \tag{20c}$$

Defining now the following Hankel transforms

$$\left\{\pm u_{1r}(k, z) + u_{2r}(k, z)\right\} = \int_0^\infty \int_0^\infty \left\{u_{1r}(k, z) + u_{2r}(k, z)\right\} J_{+1}\left(kr\right) dr dk$$

$$\left\{\pm u_{1\theta}(k, z) + u_{2\theta}(k, z)\right\} = \int_0^\infty \int_0^\infty \left\{u_{1\theta}(k, z) + u_{2\theta}(k, z)\right\} J_{+1}\left(kr\right) dr dk$$

$$\left\{\pm f_{1r}(k, z) + f_{2r}(k, z)\right\} = \int_0^\infty \int_0^\infty \left\{f_{1r}(k, z) + f_{2r}(k, z)\right\} J_{+1}\left(kr\right) dr dk$$

$$\left\{\pm f_{1\theta}(k, z) + f_{2\theta}(k, z)\right\} = \int_0^\infty \int_0^\infty \left\{f_{1\theta}(k, z) + f_{2\theta}(k, z)\right\} J_{+1}\left(kr\right) dr dk$$

$$\left\{\pm f_{1z}(k, z) + f_{2z}(k, z)\right\} = \int_0^\infty \int_0^\infty \left\{f_{1z}(k, z) + f_{2z}(k, z)\right\} J_{+1}\left(kr\right) dr dk$$
Dynamics of piles and pile groups in layered soil media: A. M. Kaynia and E. Kausel

\[ \int_0^\infty \left( \frac{d^2}{dz^2} + k^2 \right) J_n(kr)rdr = \int_0^\infty \frac{d^2}{dz^2} J_n(kr)rdr = \frac{1}{\gamma k} \mu f_{2n} \]

Then one can show that application of Hankel transforms to Equations (20) leads to

\[ g - k^2 + \omega^2 (u_{1n} + u_{3n}) + (2 + \gamma)(-k \Delta_n) + f_{1n} + f_{3n} = 0 \]  

(25a)

\[ (d^2 - k^2 \rho \frac{d}{dz}) (u_{1n} + u_{3n}) + \mu (\lambda + \mu) (k \Delta_n) - f_{1n} + f_{3n} = 0 \]  

(25b)

\[ \mu \frac{d^2}{dz^2} (u_{1n} + u_{3n}) + (\lambda + \mu) (k \Delta_n) - f_{1n} + f_{3n} = 0 \]  

(25c)

where \( \Delta_n = \int_0^\infty J_n(kr)rdr \) is the \( n \)th order Hankel transform of \( \Delta_n \) which can be shown to be expressible as

\[ \Delta_n = k u_{1n} + \frac{d}{dz} u_{2n} \]  

(26)

Finally, introducing Equation (26) into Equations (25) and combining Equations (18a) and (18b) one obtains the following differential equations

\[ (\lambda + 2\mu) \left( \frac{\mu}{\lambda + 2n} \frac{d^2}{dz^2} - \gamma \right) u_{1n} - (\lambda + \mu) k \frac{d}{dz} u_{2n} + f_{1n} = 0 \]  

(27a)

\[ (\lambda + \mu) k \frac{d}{dz} u_{1n} + \mu \left( \frac{\lambda + 2\mu}{\mu} \frac{d^2}{dz^2} - \gamma \right) u_{2n} + f_{2n} = 0 \]  

(27b)

\[ \mu \left( \frac{d^2}{dz^2} - \gamma \right) u_{3n} + f_{3n} = 0 \]  

(28)

where the two parameters \( \alpha \) and \( \gamma \) are defined by

\[ \alpha = \sqrt{k^2 - \frac{\rho \omega^2}{\lambda + 2\mu}} = \sqrt{k^2 - \frac{\omega^2}{C_p}} \]  

(29)

\[ \gamma = \sqrt{k^2 - \frac{\rho \omega^2}{\mu}} = \sqrt{k^2 - \frac{\omega^2}{C_s}} \]  

(30)

and \( C_s \) and \( C_p \) are the velocities of shear and pressure waves, respectively.

The solution of coupled differential equations (27a,b) and equation (28) are given by

\[ \{ u_{1n}(k, z) \} = \{ \begin{array}{c} -k \gamma -k \gamma \\ -k \gamma -k \end{array} \} \]  

\[ \{ u_{2n}(k, z) \} = \{ \begin{array}{c} C_{1n}e^{-\alpha z} \\ C_{2n}e^{\alpha z} \end{array} \} \]  

\[ \{ f_{1n}/z^2(\lambda + 2\mu) \} \]  

\[ \{ f_{2n}/\gamma^2 \mu \} \]  

(31)

\[ u_{3n}(k, z) = \{ \begin{array}{c} 1 \\ 1 \end{array} \} \{ \begin{array}{c} C_{5n}e^{-\alpha z} \\ C_{6n}e^{\alpha z} \end{array} \} + \frac{1}{\gamma^2 \mu} f_{3n} \]  

(32)

where \( C_{1n}(k) - C_{6n}(k) \) are unknown constants.

The next step in the solution procedure is to derive expressions for transformed stresses. The three components of stress on a plane perpendicular to the \( z \)-axis in cylindrical coordinates are given by

\[ \sigma_{xz} = \mu \frac{\partial u_x}{\partial r} + \frac{\partial u_z}{\partial z} \]  

(33a)

\[ \sigma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial r} \]  

(33b)

\[ \sigma_{zz} = 2\mu \frac{\partial u_z}{\partial z} + \lambda \Delta \]  

(33c)

Using the Fourier expansions of \( u_x, u_y \) and \( u_z \) (Equations 18 and 19) in Equations (33) one gets

\[ \{ \sigma_{rz}(r, \theta, z) \} = \sum_{n=0}^{\infty} \{ \sigma_{rz}(r, z) \} \cos n\theta \]  

(34)

\[ \{ \sigma_{rz}(r, \theta, z) \} = \sum_{n=0}^{\infty} \{ \sigma_{rz}(r, z) \} \sin n\theta \]  

(35)

where \( \sigma_{rz}, \sigma_{rn} \) and \( \sigma_{zm} \) are given by

\[ \sigma_{rz} = \mu \frac{\partial u_r}{\partial \theta} + \frac{\partial u_r}{\partial z} \]  

(36a)

\[ \sigma_{rn} = \mu \frac{\partial u_r}{\partial \theta} - n \frac{\partial u_r}{\partial z} \]  

(36b)

\[ \sigma_{zm} = 2\mu \frac{\partial u_r}{\partial z} + \lambda \Delta \]  

(36c)

If now the following Hankel transforms are defined

\[ \pm \sigma_{21n}(k, z) + \sigma_{23n}(k, z) = \int_0^\infty \{ \sigma_{rn} \pm \sigma_{zm} \} J_{n+1}(kr)rdr \]  

(37)

\[ \pm \sigma_{21n} + \sigma_{23n} = \mu \left[ \pm ku_{2n} + \frac{d}{dz} (\pm u_{1n} + u_{3n}) \right] \]  

(39)

\[ \sigma_{22n} = (\lambda + 2\mu) \frac{du_{2n}}{dz} + \lambda ku_{2n} \]  

(40)

Finally, introducing the expressions obtained for \( u_{1n}, u_{2n} \) and \( u_{3n} \) (Equations 31 and 32) into Equations (39) and (40), one can express the transformed stresses as
For the layer shown in Fig. 4b, one can use Equation (31) to express the transformed displacements $u_{1n}$ and $u_{2n}$ of the two planes A and B, associated with local coordinates $z' = 0$, and $z' = h$, in terms of the four unknown constants $C_1, ..., C_4$. Similarly, Equation (41) can be used to express the transformed stresses $\sigma_{21n}$ and $\sigma_{22n}$ on the exterior side of planes A and B in terms of the same constants. Deleting the four unknown constants between these expressions, one obtains the following result

$$\begin{bmatrix} \sigma_{21n}(k, z) \\ \sigma_{22n}(k, z) \end{bmatrix} = \mu \begin{bmatrix} 2ak \\ (k^2 + \gamma^2) \end{bmatrix} - \begin{bmatrix} -2ak \\ (k^2 + \gamma^2) \end{bmatrix} \begin{bmatrix} \sigma_{21n}(z) \\ \sigma_{22n}(z) \end{bmatrix} + \begin{bmatrix} -k f_{2n}/\gamma^2 \\ \lambda k^2 f_{1n}/\alpha^2(\lambda + 2\mu) \end{bmatrix} \begin{bmatrix} C_{1n} e^{-\alpha z} \\ C_{3n} e^{-\beta z} \end{bmatrix} + \begin{bmatrix} C_{5n} e^{-\alpha z} \\ C_{6n} e^{-\beta z} \end{bmatrix}$$

For the remaining developments all quantities associated with $u_{2n}$ will be identified as 'SH-wave' quantities. In a similar manner, 'SV-P waves' will be used to refer to quantities associated with $u_{1n}$ and $u_{2n}$.

**Layer and Half-space Stiffness Matrices and Fixed-end Force Vectors**

Figure 4a shows a layered medium consisting of $M$ layers resting on a half-space. Figure 4b shows the $j$th layer confined between the two planes denoted by A and B and Fig. 4c shows the half space bounded by the plane C. The objective of this section is to obtain a relationship between the transformed stresses on the two planes A and B and transformed displacements of these planes. Such a relationship can be used to define layer 'stiffness matrices' as well as layer 'fixed-end stresses'. Similarly, a relationship between stresses and displacements for plane C in Fig. 4c results in half-space stiffness matrices. For a given value of $k$ the stiffness matrices of the layers and the half-space and the associated force vectors can be used to assemble the stiffness matrix and the load vector for the layered medium; the resulting system of equations then yield the transformed displacements at layer interfaces.

**SV-P waves**

For the layer shown in Fig. 4b, one can use Equation (31) to express the transformed displacements $u_{1n}$ and $u_{2n}$ of the two planes A and B, associated with local coordinates $z' = 0$, and $z' = h$, in terms of the four unknown constants $C_1, ..., C_4$. Similarly, Equation (41) can be used to express the transformed stresses $\sigma_{21n}$ and $\sigma_{22n}$ on the exterior side of planes A and B in terms of the same constants. Deleting the four unknown constants between these expressions, one obtains the following result

$$\begin{bmatrix} \sigma_{21n}(k, z) \\ \sigma_{22n}(k, z) \end{bmatrix} = \mu \begin{bmatrix} 2ak \\ (k^2 + \gamma^2) \end{bmatrix} - \begin{bmatrix} -2ak \\ (k^2 + \gamma^2) \end{bmatrix} \begin{bmatrix} \sigma_{21n}(z) \\ \sigma_{22n}(z) \end{bmatrix} + \begin{bmatrix} -k f_{2n}/\gamma^2 \\ \lambda k^2 f_{1n}/\alpha^2(\lambda + 2\mu) \end{bmatrix} \begin{bmatrix} C_{1n} e^{-\alpha z} \\ C_{3n} e^{-\beta z} \end{bmatrix} + \begin{bmatrix} C_{5n} e^{-\alpha z} \\ C_{6n} e^{-\beta z} \end{bmatrix}$$

where

$$\begin{bmatrix} \sigma_{21n} \\ \sigma_{22n} \end{bmatrix} = \begin{bmatrix} \sigma_{21n} \\ \sigma_{22n} \end{bmatrix} + \begin{bmatrix} u_{2n} \\ u_{2n} \end{bmatrix}$$

and the elements of the symmetric 4 x 4 layer stiffness matrix, $K_{SV-P}$, are given in appendix A.

To obtain the stiffness matrix for the half-space one can again use equations (31) and (41) provided that the radiation conditions are satisfied. This requires that the unknown constants $C_3$, and $C_5$ in Equations (31) and (41) be set equal to zero. Thus, for the half-space shown in Fig. 4c, one can use Equations (31) and (41) (with $f_{1n} = f_{2n} = 0$) to express transformed displacements and stresses at plane C, associated with the local coordinate.
Dynamics of piles and pile groups in layered soil media: A. M. Kaynia and E. Kausel

\[ z' = 0, \text{ and to delete the unknown constants } C_{1n} \text{ and } C_{2n} \]

to arrive at the following result

\[ \sigma_{SV-P}^C = [K_{SV-P}^C] u_{SV-P}^C \]  (48)

where \( \sigma_{SV-P}^C \text{ and } u_{SV-P}^C \) denote the transformed stress and displacement vectors at plane C, that is

\[ \sigma_{SV-P}^C = \begin{bmatrix} C_{21n} \\ C_{22n} \end{bmatrix}, \quad u_{SV-P}^C = \begin{bmatrix} u_{1n}^C \\ u_{2n}^C \end{bmatrix} \]  (49)

and the elements of the symmetric 2 x 2 half-space stiffness matrix \( K_{SV-P}^C \) are given in the appendix.

**SH-waves**

Following the procedure described for SV-P waves, one can use Equations (32) and (42) to obtain the following results for SH-waves

\[ \sigma_{SH}^{AB} = [K_{SH}^{AB}] u_{SH}^{AB} + \sigma_{SH}^{AB} \]  (50)

where \( \sigma_{SH}^{AB} \) and \( u_{SH}^{AB} \) denote the stress and displacement vectors at the two planes A and B (Fig. 4b), that is

\[ \sigma_{SH}^{AB} = \begin{bmatrix} \sigma_{23n}^A \\ \sigma_{23n}^B \end{bmatrix}, \quad u_{SH}^{AB} = \begin{bmatrix} u_{1n}^A \\ u_{2n}^B \end{bmatrix} \]  (51)

\( \sigma_{SH}^{AB} \) is the vector of 'fixed-end stresses' expressed as

\[ \sigma_{SH}^{AB} = - [K_{SH}^{AB}] \begin{bmatrix} \tilde{u}_{3n}^A \\ \tilde{u}_{3n}^B \end{bmatrix} \]  (52)

where

\[ \tilde{u}_{3n} = \frac{u_{3n}}{\mu} \]  (53)

and the 2 x 2 layer stiffness matrix is given by

\[ K_{SH}^{AB} = \begin{bmatrix} \frac{\gamma \mu}{\sin h(\gamma h)} & -1 \\ -1 & \cos h(\gamma h) \end{bmatrix} \]  (54)

Finally, using Equations (32) and (42) and imposing the radiation condition for the half space, as discussed for the SV-P waves, one obtains the following relation between the transformed stress and displacement of plane C (Fig. 4c)

\[ \sigma_{23n}^C = \gamma \mu u_{3n}^C \]  (55)

**Displacements Within a Layer**

In order to obtain the average displacement in a layer one needs to compute the displacements at a number of points within the layer; these displacement values along with those of the two confining planes (planes A and B in Fig. 4b) can be used to define a displacement pattern across the layer.

For a typical layer such as the one shown in Fig. 4b one can use Equations (31) and (32) to calculate transformed displacements \( u_{1n}^E, u_{2n}^E, \) and \( u_{3n}^E, \) as a function of displacement values of the two planes A and B (associated with local coordinates \( z' = 0 \) and \( z' = h \)). One can show that, for a plane denoted by E as in Fig. 4b, this procedure would lead to the following expression for the transformed displacements for SV-P waves:

\[ \begin{bmatrix} \tilde{u}_{1n}^E \\ \tilde{u}_{2n}^E \end{bmatrix} = [T_{SV-P}^E] \begin{bmatrix} u_{1n}^E - \tilde{u}_{1n} \\ u_{2n}^E - \tilde{u}_{2n} \end{bmatrix} + \begin{bmatrix} \tilde{u}_{1n} \\ \tilde{u}_{2n} \end{bmatrix} \]  (56)

where \( u_{1n}^E \) and \( u_{2n}^E \) denote the transformed displacements of plane E, \( \tilde{u}_{1n} \) and \( \tilde{u}_{2n} \) are defined by Equations (47) and the elements of the 2 x 4 transition matrix \([T_{SV-P}^E]\) for mid-layer plane \( z' = \frac{h}{2} \) in Fig. 4b are given in appendix A.

Following the same procedure described above, one can obtain the following expression for the mid-layer transformed displacement for SH waves

\[ u_{3n}^E = \frac{1}{2 \cosh \left( \frac{\gamma h}{2} \right)} (u_{3n}^A + u_{3n}^B - 2\tilde{u}_{3n}) + \tilde{u}_{3n} \]  (57)

where \( \tilde{u}_{3n} \) is given by Equation (53).

**Integral Representation and Numerical Evaluation of Displacements**

The preceding analytical procedure can be used to evaluate the displacements in layered soil media due to a body force distributed uniformly across a layer (barrel load) or due to an external force at the interface of two layers (disk load). For the problem under consideration, however, the type of external forces are limited to the four cases shown in Fig. 3. In this section integral representation for displacements due to these forces are obtained and their numerical evaluation is outlined. As shown in Fig. 2, \( R \) denotes the radius of piles (or equivalently the radius of Barrel and disk loads) and \( h \) denotes the thickness of the loaded layer.

The load distribution in Fig. 3a (lateral barrel load) can be expressed in cylindrical coordinates as

\[ f_1(r, \theta, z) = \frac{1}{2\pi Rh} \delta(r - R) \cos \theta \]  (58a)

\[ f_2(r, \theta, z) = -\frac{1}{2\pi Rh} \delta(r - R) \sin \theta \]  (58b)

\[ f_3(r, \theta, z) = 0 \]  (58c)

where \( \delta \) is the Kronecker delta.

Comparing Equations (58) with the expansion of loads in Equations (18) and (19) one obtains the following results

\[ f_{1n} = \frac{1}{2\pi Rh} \delta(r - R) \]  (59a)

\[ f_{2n} = -\frac{1}{2\pi Rh} \delta(r - R) \]  (59b)

\[ f_{3n} = f_{4n} = f_{5n} = 0; \quad \text{for } n \neq 1 \]  (60)

which indicate that the associated displacements are contributed only by the terms corresponding to \( n = 1 \); therefore the displacement expansions reduce to the following expressions

\[ u_{1n}(r, \theta, z) = u_{1n}(r, z) \cos \theta \]  (61a)

\[ u_{2n}(r, \theta, z) = u_{2n}(r, z) \sin \theta \]  (61b)
Dynamics of piles and pile groups in layered soil media. A. M. Kaynia and E. Kausel

\[ u(r, \theta, z) = u_1(r, z) \cos \theta \]  

On the other hand, applications of Hankel transforms, according to Equations (21) and (22), to \( f_{11}, f_{01} \) and \( f_{11} \) given by Equations (54) leads to

\[ f_{11} = \frac{J_0(kR)}{2\pi h} \]  

\[ f_{21} = 0 \]  

\[ f_{31} = \frac{J_0(kR)}{2\pi h} \]  

If \( u_{11}, u_{21}, \) and \( u_{31} \) are the transformed displacements corresponding to \( f_{11} = f_{31} = (1/2\pi h) \) and \( f_{21} = 0 \), then the Hankel transform of displacements in Equations (21) and (22) can be written as \( n = 1 \)

\[ -J_0(kR)u_{11} + J_0(kR)u_{31} = \int_0^\infty (u_{11} + u_{31})J_2(kr)rdr \]  

\[ J_0(kR)u_{11} + J_0(kR)u_{31} = \int_0^\infty (u_{11} - u_{31})J_2(kr)rdr \]  

\[ -J_0(kR)u_{21} = \int_0^\infty u_{21}J_2(kr)rdr \]  

The application of inverse Hankel transform to these equations and use of the recurrence relations for the Bessel functions leads to the following integral representations for displacements

\[ u_{11} = \int_0^\infty \left[ u_{11}J_0(kr)J_0(kR) + (u_{31} - u_{11}) \right] \frac{J_1(kr)}{kr} J_0(kR) dk \]  

\[ u_{01} = -\int_0^\infty \left[ u_{21}J_0(kr)J_0(kR) + (u_{11} - u_{31}) \right] \frac{J_1(kr)}{kr} J_0(kR) dk \]  

\[ u_{21} = -\int_0^\infty u_{21}J_1(kr)J_0(kR)dk \]  

A similar procedure can be followed to obtain the integral representation of displacements for the load distribution in Fig. 3b (Frictional barrel load). For this case \( f_z = f_0 = 0 \) and \( f_z(r, \theta, z) = (1/2\pi R h) \delta(r - R) \); therefore, in the displacement expansion, only the terms corresponding to \( n = 0 \) are non-zero; that is

\[ u_z(r, \theta, z) = u_0(r, z) \]  

\[ u_\theta(r, \theta, z) = 0 \]  

\[ u_\phi(r, \theta, z) = u_0(r, z) \]  

Following a procedure similar to the one described for lateral loading, one can show that if \( u_{10} \) and \( u_{20} \) are transformed displacements due to transformed loads \( f_{10} = 0 \) and \( f_{20} = (1/2\pi h) \), then

\[ u_{10} = \int_0^\infty u_{10}J_0(kr)J_0(kR)dk \]  

For the loads distributed over circular surfaces (disk loads in Figs 3c and 3d) it is necessary to evaluate the corresponding transformed forces directly. The horizontal disk load in Fig. 3c can be expressed as

\[ \sigma_{rz} = \frac{1}{\pi R^2} \cos \theta \]  

\[ \sigma_{\theta z} = -\frac{1}{\pi R^2} \sin \theta \]  

\[ \sigma_{zz} = 0 \]  

\[ \sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0; \quad r > R \]  

Comparing a Fourier expansion of these loads, similar to the expansion of stresses in Equations (34) and (35), with Equations (67), one gets

\[ \sigma_{rz1} = \frac{1}{\pi R^2} \]  

\[ \sigma_{\theta z1} = 0 \]  

\[ \sigma_{zz1} = 0 \]  

\[ \sigma_{rzn} = \sigma_{\theta zn} = \sigma_{znn} = 0; \quad \text{for} \quad n \neq 1 \]  

Therefore, only one needs to consider the terms associated with \( n = 1 \) in the expansion of displacements; that is, \( u_z, u_\theta \) and \( u_\phi \) are expressed by Equations (61). Application of Hankel transforms, according to Equations (37) and (38), then leads to the following transformed loads associated with \( \sigma_{rz1}, \sigma_{\theta z1} \) and \( \sigma_{zz1} \)

\[ \sigma_{211} = -\frac{J_1(kR)}{kR} \]  

\[ \sigma_{221} = 0 \]  

\[ \sigma_{231} = \frac{1}{\pi} \frac{J_1(kR)}{kR} \]  

A procedure similar to that described for the lateral barrel load leads to the integral representation of \( u_{11}, u_{11}, \) and \( u_{31} \) similar to those presented by Equations (64) except that the term \( J_0(kR) \) should be replaced by \( J_1(kR)/kR \). The transformed displacements \( u_{10}, u_{11}, \) and \( u_{31} \) in these equations then correspond to transformed applied stresses

\[ \sigma_{211} = \frac{1}{\pi}, \quad \sigma_{221} = 0 \text{ and } \sigma_{231} = \frac{1}{\pi}. \]  

Finally, for the load distribution in Fig. 3d (vertical disk load) forces and displacements are contributed only by the terms corresponding to \( n = 0 \). For this load one can show that displacements are given by Equations (66) except that the term \( J_0(kR) \) should be replaced by \( J_1(kR)/kR \). Transformed displacements \( u_{10} \) and \( u_{20} \) in these equations then correspond to transformed applied stresses \( \sigma_{210} = 0 \) and \( \sigma_{220} = (1/\pi) \).

The integral representation of displacements for the four loading conditions considered above are of the form

\[ I = \int_0^\infty fJ_0(kr)J_0(kR)dk \]
in which the kernel $f$ represents a function of $k$, and $n$ and $m$ are integers that can take on zero or one.

The general procedure to evaluate such an integral is to divide the integration domain into a number of intervals, and to approximate the integrand in each interval by a function which can be integrated analytically. However, in order to implement an efficient numerical scheme one has to take proper account of the behaviour of the integrand. Examination of the kernel $f$ reveals that for values of $k$ of the order of $(\omega/c_s)$ and smaller ($c_s$ is the shear wave velocity of the loaded layer) the kernel is characterized by an erratic variation and pronounced peaks. (These peaks are associated with surface wave modes; in addition, as the number of layers increases more peaks appear in the kernel). This suggests that for this integration region one has to select, in general, small intervals, so that the erratic nature of the integrand can be captured.

LATERAL AND AXIAL VIBRATION OF PRISMATIC MEMBERS

The formulation presented earlier for the dynamic analysis of pile groups is based on the evaluation of $K_p$, $F_p$ and $\Psi_p$. Expressions for the response quantities that are needed to construct these matrices are derived in this section.

Lateral Vibration

The differential equation for a beam in steady-state lateral vibration (Fig. 5a), including the effect of axial force, is given by

$$\frac{d^4u}{dz^4} + \left(\frac{H}{EI}\right) \frac{d^2u}{dz^2} - \left(\frac{m\omega^2}{E}\right) u = 0 \quad (72)$$

where $m$ denotes mass per unit length of the beam (pile), $H$ is the constant axial force and $EI$ is the flexural rigidity of the beam.

The solution of Equation (72) can be expressed as

$$u = C_1 \cos(\eta z) + C_2 \sin(\eta z) + C_3 \cosh(\xi z) + C_4 \sinh(\xi z) \quad (73)$$

in which

$$\eta = \left\{ \left[ \frac{H}{2EI} \right]^2 + \frac{m\omega^2}{E} \right\}^{1/2} + \frac{H}{2EI} \quad (74a)$$

$$\xi = \left\{ \left[ \frac{H}{2EI} \right]^2 + \frac{m\omega^2}{E} \right\}^{1/2} - \frac{H}{2EI} \quad (74b)$$

In order to evaluate the elements of $\Psi_p$ associated with the lateral degrees of freedom, one needs to derive the expression for the lateral vibration of a beam caused by the displacements (translation and rotation) of the two ends, $u_a$, $\phi_a$, $u_b$ and $\phi_b$. The values of the unknown constants $C_1$ to $C_4$ un Equation (73) corresponding to these end conditions are given in appendix B.

The dynamic stiffness matrix of the beam, which is obtained by expressing the forces at the two ends of the beam in terms of the associated displacements, can be represented as

$$\begin{bmatrix} V_A \\ M_A \\ V_B \\ M_B \end{bmatrix} = K_p \begin{bmatrix} u_A \\ \phi_A \\ u_B \\ \phi_B \end{bmatrix} \quad (75)$$

The elements of the symmetric $4 \times 4$ dynamic stiffness matrix are given in appendix B, too.

Finally, in order to evaluate the elements of $F_p$ associated with the lateral degrees of freedom, it is necessary to derive expressions for the displacement caused by a lateral point load in a fixed-end beam. For the beam shown in Fig. 5c, subjected to a point force at $z = a$, one can use Equation (73) to express the displacements of the beam as

$$u = A_1 \cos(\eta z) + A_2 \sin(\eta z) + A_3 \cosh(\xi z) + A_4 \sinh(\xi z) \quad 0 \leq z \leq a \quad (76a)$$

$$u = B_1 \cos(\eta z) + B_2 \sin(\eta z) + B_3 \cosh(\xi z) + B_4 \sinh(\xi z) \quad a \leq z \leq L \quad (76b)$$
Dynamics of piles and pile groups in layered soil media: A. M. Kaynia and E. Kausel

The unknown constants $A_1 - A_4$ and $B_1 - B_4$ can be determined by imposing the kinematic boundary conditions at the two ends along with the displacement compatibility and equilibrium conditions at the point of application of the load. The values of the constants are again given in the appendix.

Axial Vibration

The differential equation for the steady-state axial vibration of a beam (pile) is given by (Fig. 6b)

$$\frac{d^2v}{dz^2} + \frac{m}{EI} \omega^2 v = 0$$

(77)

The solution of this equation can be written as

$$v = C_1 \cos (\zeta z) + C_2 \sin (\zeta z)$$

(78)

in which

$$\zeta = \sqrt{\frac{m\omega^2}{EA}}$$

(79)

To obtain the elements of the $\Psi$ matrix associated with the axial degrees of freedom one needs to obtain the values of $C_1$ and $C_2$ in Equation (78) corresponding only to end displacements; the result is

$$C_1 = v_A$$

(80a)

$$C_2 = \frac{1}{\sin \zeta L} (- \cos (\zeta L)v_A + v_B)$$

(80b)

The axial dynamic stiffness matrix of the beam can be obtained following the procedure outlined in the previous case, one can show that

$$\begin{pmatrix} F_A \\ F_B \end{pmatrix} = \begin{pmatrix} E A \zeta^2 & \cos (\zeta L) \\ \sin (\zeta L) & -1 \end{pmatrix} \begin{pmatrix} v_A \\ v_B \end{pmatrix}$$

(81)

Finally, the axial displacement in a fixed-end beam caused by an axial point load $P$ (Fig. 6c) is given by

$$v = A_1 \cos (\zeta z) + A_2 \sin (\zeta z) \quad 0 \leq z \leq a$$

(82a)

$$v = B_1 \cos (\zeta z) + B_2 \sin (\zeta z) \quad a \leq z \leq L$$

(82b)

where the unknown constants are

$$A_1 = 0$$

(83a)

$$A_3 = \frac{P}{E A \zeta} [\cos (\zeta a) - \cotan (\zeta L) \sin (\zeta a)]$$

(83b)

$$B_1 = \frac{P}{E A \zeta} \sin (\zeta a)$$

(83c)

$$B_2 = \frac{P}{E A \zeta} \cotan (\zeta L) \sin (\zeta a)$$

(83d)

Equations (82) with $A_1, A_2, B_1$ and $B_2$ given by Equations (83) can then be used to obtain elements of $f_p$ associated with axial degrees of freedom.

NUMERICAL RESULTS

A limited number of results are presented here to demonstrate the applicability of the preceding formulation and to highlight certain characteristics of the dynamic behaviour of piles. In these results $E_p, \rho_p, \nu_p$ and $\beta_p$ are used to denote the elastic modulus, mass density, Poisson's ratio and material damping ratio of the soil. The corresponding quantities for the piles are denoted by $E_p, \rho_p, \nu_p$ and $\beta_p$. In addition, $L$ and $d$ define the length and diameter of piles and $s$ denotes the center-to-center distance between adjacent piles. Finally, $a_0 = \omega d/C_s$ is the nondimensional frequency, in which $C_s$ is the largest shear wave velocity of the soil profile.

In order to verify the numerical solution scheme reported in this study, the results for the vertical stiffness and damping of a $2 \times 2$ pile group are compared in Fig. 7 with those reported by Nogami. The soil medium in this case is a uniform viscoelastic stratum, with depth $H = 75d$, resting on a rigid bedrock, and the piles are characterized by $(L/d) = 37.5$, $(\pi GL^2/E_pA) = 1$ and $(s/d) = 5$ ($G$ denotes the shear modulus of the soil). The figure shows a fairly good agreement between the two solutions. The small discrepancies observed in the figure are believed to be partly due to the approximation made by Nogami for the analysis of friction piles. To obtain the results in Fig. 7, Nogami had to introduce a soil column beneath each pile so that he could use the formulation developed for end-bearing piles.

Those features of the dynamic behaviour that are presented here are the dynamic impedances and the seismic response of single piles and pile groups. The impedances are complex quantities which can be expressed as

$$K = k + ia_0c$$

(84)

where $k$ and $c$ are usually referred to as the stiffness and damping of the pile foundation. The quantities associated
Dynamics of piles and pile groups in layered soil media. A. M. Kaynia and E. Kausel

For the seismic analysis it is assumed that the ground motion is due to vertically propagating shear waves that produce a free-field ground surface displacement $u_0 = 1$. These waves induce both a translation and a rotation in the pile cap (or in the pile head for single piles). The transfer functions for these quantities are complex-valued functions that are presented in terms of their absolute values.

Two different soil media are considered in this study. The first is a homogeneous viscoelastic halfspace with $E_e = (E_p/100)$ and the second is a nonhomogeneous semi-infinite medium in which the elastic modulus of the soil increases linearly from zero at the ground surface to $E_s = (E_p/100)$ at the pile tip and remains constant throughout the underlying halfspace. Other parameters of the two soil media, which are assumed to be similar, are $\nu_0 = 0.05$, $\nu_s = 0.4$ and $\rho_s = 0.7\rho_p$. Moreover, for all the piles considered here, the following values are used: $v_p = 0.25$, $\beta_p = 0.0$ and $L/d = 20$.

Figures 8 and 9 show that variations of horizontal and vertical stiffnesses and dampings of a single pile as a function of $a_o$, in the two soil media. (The stiffnesses and dampings in each mode of vibration, shown in these figures, are normalized with respect to the corresponding static stiffness). For the horizontal mode the static stiffnesses, $k_{xx}^s (a_o = 0.0)$, in the homogeneous and nonhomogeneous media are $4600(E_pA/L)$ and $770(E_pA/L^2)$, respectively. For the vertical mode the static stiffnesses, $k_{zz}^s (a_o = 0.0)$, in the same media are $1.58(E_pA/L)$ and $0.96(E_pA/L)$, respectively. Comparison between the different modes of vibration in these figures suggests that the horizontal response is affected, to a considerably greater extent, by near-surface soil properties. Moreover, this mode of vibration displays a more pronounced frequency-dependent behaviour.

Figures 10 and 11 show that horizontal and vertical stiffnesses and dampings for a $3 \times 3$ pile group with $s/d = 5$ in the two soil media. (The stiffness and damping values shown in these figures are normalized with respect to the static stiffness of a single pile in the group in their respective media, given above. This normalization scheme then eliminates the effect of differences in the absolute value of the soil moduli and reflects only the effect of nonhomogeneity in the soil profile on the stiffnesses and dampings of pile groups.) The results presented in Figs 10 and 11 indicate that the interaction effects, which are characterized by sharp peaks in impedance functions, are more pronounced in the nonhomogeneous medium. The fact that the behaviour of piles is influenced by the
near-surface soil properties (especially for horizontal vibration) and that pile-soil-pile interaction effects magnify as the soil becomes softer\textsuperscript{7,9} help one to explain this observation.

Figures 12 and 13 display the variations of rocking and torsional stiffnesses and dampings as a function of $a_0$, for the same pile group studied in Figures 10 and 11. (Rocking and torsional impedances in these figures are normalized with respect to $\Sigma_{i=1}^9 x_i^2 k_{zz} (a_0 = 0)$ and $\Sigma_{i=1}^9 r_i^2 k_{xx} (a_0 = 0)$, respectively, where $x$ and $r$ refer to the Cartesian and polar coordinates of pile location). These impedances display similar characteristics to vertical and horizontal impedances.

Finally, with regard to seismic analyses, Fig. 14 displays the absolute value of the transfer function from ground surface displacement to the pile head displacement for single piles in the two media. Similarly, Fig. 15 shows the absolute value of the transfer functions of the pile cap displacement for the same 3 x 3 pile groups considered earlier. The results indicate that piles and pile groups in the non-homogeneous medium filter out, to a greater extent, the high frequency components of the ground motion.

REFERENCES
Fig. 11. Vertical stiffness and damping of 3 x 3 pile group in the homogeneous and nonhomogeneous media

Fig. 12. Rocking stiffness and damping of 3 x 3 pile group in the homogeneous and nonhomogeneous media

APPENDIX A

1 Elements of the symmetric 4 x 4 layer stiffness matrix for SV-P waves (Equation 43):

\[ K_{11} = \frac{1}{D} \mu a(k^2 - \gamma^2)(\alpha \gamma S^C - k^2 S^C) \]

\[ K_{21} = \frac{1}{D} \mu k [\alpha \gamma (3k^2 + \gamma^2)(C^C - C^C) - (k^4 + k^2 \gamma^2 + 2\alpha^2 \gamma^2)S^SS^S] \]

\[ K_{31} = \frac{1}{D} \mu a(k^2 - \gamma^2)(k^2 S^C - \alpha S^S) \]

\[ K_{41} = \frac{1}{D} \mu k a(k^2 - \gamma^2)(C^C - C^C) \]
Dynamics of piles and pile groups in layered soil media: A. M. Kaynia and E. Kausel

Fig. 13. Torsional stiffness and damping of 3 x 3 pile group in the homogeneous and nonhomogeneous media

\[ K_{22} = \frac{1}{D} \mu \gamma (k^2 - \gamma^2)(\alpha \gamma S' S'' - \kappa S' S'') \]
\[ K_{33} = -K_{44} \]
\[ K_{42} = \frac{1}{D} \mu \gamma (k^2 - \gamma^2)(S' S'' - \kappa \gamma S') \]
\[ K_{32} = K_{21} \]
\[ K_{44} = K_{22} \]

where
\[ D = \frac{\alpha \gamma}{2} \left( -2k^2 + 2k^2 C'' S' - \frac{\alpha^2 \gamma^2 + k^4}{\alpha \gamma} S' S'' \right) \]

and \( C', C'', S' \) and \( S'' \) are used to denote the following quantities:
\[ C' = \cosh (\gamma h); \quad S' = \sinh (\gamma h) \]
\[ C'' = \cosh (\alpha h); \quad S'' = \sinh (\alpha h) \]

Asymptotic value of the stiffness terms in (a-1) for \((\omega/kC) < 1:\)
\[ K_{11} \approx \frac{2}{D} \mu k [kh(1 - \varepsilon^2) - (1 + \varepsilon^2)S^4 C^2] \]
\[ K_{21} \approx \frac{2}{D} \mu k [k^2 h^2 (1 - \varepsilon^2)^2 - \varepsilon^2 (1 + \varepsilon^2)S''^2] \]
\[ K_{31} \approx \frac{2}{D} \mu k [(1 + \varepsilon^2)S'' - kh(1 - \varepsilon^2)C^2] \]
\[ K_{41} \approx \frac{2}{D} \mu k [kh(1 - \varepsilon^2)S''] \]
\[ K_{22} \approx \frac{2}{D} \mu k [kh(1 - \varepsilon^2) + (1 + \varepsilon^2)S^4 C^2] \]
\[ K_{42} \approx \frac{2}{D} \mu k [(1 + \varepsilon^2)S'' + kh(1 - \varepsilon^2)C^2] \]

Fig. 14. Transfer function for seismic response of single pile in the homogeneous and nonhomogeneous media

\[ \frac{|u|/u_g}{|u|^g} \]

Homogeneous
Nonhomogeneous

Fig. 15. Transfer function for seismic response of 3 x 3 pile group in the homogeneous and nonhomogeneous media
\[ D' = k^2 h^2 (1 - \varepsilon^2) - (1 + \varepsilon^2)^2 (S^2) \]
\[ \varepsilon = C'/C_p \]
and \( C' \) and \( S' \) denote the following quantities
\[ C'_\varepsilon = \cosh (k h) \quad S'_\varepsilon = \sinh (k h) \]

2. The symmetric 2 \times 2 halfspace stiffness matrix for SV-P waves (Equation 48):
\[ K_{SV-P}^C = \mu \left( \begin{array}{cc} k (k^2 + \gamma^2 - 2a) & \gamma (k^2 - \gamma^2) \\ k (k^2 + \gamma^2 - 2a) & k (k^2 + \gamma^2 - 2a) \end{array} \right) \]

where \( D', C', S' \) and \( \varepsilon \) are defined by Equations (A-5), (A-6) and (A-8) and \( C'_\varepsilon \) and \( S'_\varepsilon \) denote the following
\[ C'_\varepsilon = \cosh (k h/2) \quad S'_\varepsilon = \sinh (k h/2) \]
\[ K_{31} = -\frac{EI}{T_0} (\eta^2 + \xi^2)(\eta S^\eta + \xi S^\xi) \]

\[ K_{41} = \frac{EI}{T_0} (\eta^2 + \xi^2)(C^\xi - C^\eta) \]

\[ K_{22} = \frac{EI}{T_0} (\frac{\eta}{\xi} + \frac{\xi}{\eta}) (\xi S^\xi \eta C^\xi - \eta C^\eta S^\xi) \]

\[ K_{32} = -\frac{EI}{T_0} (\eta^2 + \xi^2)(C^\eta - C^\xi) \]

\[ K_{42} = \frac{EI}{T_0} (\frac{\eta}{\xi} + \frac{\xi}{\eta}) (\eta S^\eta - \xi S^\xi) \]

\[ K_{33} = K_{11} \]

\[ K_{43} = -K_{21} \]

\[ K_{44} = K_{22} \]

(B-4)

3 The constants \( a_1 - A_4 \), \( B_1 - B_4 \) in Equations (76):

\[ A_4 = \frac{P}{EI\eta(\eta^2 + \xi^2)} \left[ T_3(\cosh(\xi a) - \cos(\eta a)) \right. \]

\[ + \left. (T_1 - 1) \sin(\eta a) - \left( T_4 + \frac{\eta}{\xi} \right) \sin h(\xi a) \right] \]

\[ B_1 = \frac{P}{EI\eta(\eta^2 + \xi^2)} \left[ T_3(\cosh(\xi a) - \cos(\eta a)) \right. \]

\[ + \left. \left( T_4 + \frac{\eta}{\xi} \right) \left( \frac{\xi}{\eta} \sin(\eta a) - \sinh(\xi a) \right) \right] \]

\[ B_2 = \frac{P}{EI\eta(\eta^2 + \xi^2)} \left[ (T_1 - 1)(\cosh(\xi a) - \cos(\eta a)) \right. \]

\[ + \left. \left( T_4 + \frac{\eta}{\xi} \right) \left( \frac{\xi}{\eta} \sin(\eta a) - \sinh(\xi a) \right) \right] \]

\[ B_3 = -T_1B_1 - T_3B_2 \]

\[ B_4 = T_2B_1 + T_4B_2 \]