GREEN’S FUNCTIONS FOR A LAYERED HALF-SPACE IN THE WAVENUMBER-TIME DOMAIN

Joonsang Park¹, Member ASCE
and Eduardo Kausel², Member ASCE

ABSTRACT
A sub-structuring technique is used to obtain the Green’s functions in the wavenumber-time domain for a layered half-space subjected to $SH$ line sources. Two Green’s functions are needed for this purpose, namely one for the upper layered medium and another for the half-space. Thereafter, a time-marching scheme is used to evaluate the convolution required to combine the two media. Finally, simulations for wave motions in a two-layered half-space are presented and used to demonstrate the efficiency and accuracy of the proposed method.

KEYWORDS: Green’s function; fundamental solution; Thin-Layer Method; $SH$ line sources; wavenumber-time domain; layered half-space; wave propagation; elastodynamics; seismic sources; substructuring method; time-marching.

1. INTRODUCTION
Theoretical studies on wave propagation in layered media have found numerous applications in seismology, earthquake engineering, and geotechnical engineering. Examples are the problems of seismic source identification, soil-structure interaction, dynamic loads on pavements, non-destructive evaluation methods, and so forth. In the majority of cases, the formulation for sources in layered media has in the past been carried out in the frequency domain in the context of linear problems [Thomson, 1950; Haskell, 1953; Bouchon and Aki, 1977; Apsel and Luco, 1983; Luco and Apsel, 1983]. On the other hand, a formulation in the time domain is required for, but not restricted to, nonlinear problems. Indeed, a solution for the Green’s functions in the time domain allows circumventing many of the numerical difficulties inherent in a solution in the frequency domain, such as the waviness of the kernels when the Green's functions in the frequency domain are obtained by numerical integration over wavenumbers. As will be seen in this paper, a time domain formulation also allows making use of the highly efficient Thin Layer Method (TLM) [Kausel, 1994] together with rigorous representations for an underlying half-space. Hence, solutions can be effectively obtained directly in the time domain for transient sources in layered half-spaces.

In this paper, we present a wavenumber-time domain method for the Green’s functions in a layered medium underlain by a homogeneous half-space when the system is subjected to $SH$ line loads (or acoustic sources in a fluid). The analysis is performed in steps, separating the whole system into two substructures: 1) the upper layered medium and 2) the underlying homogeneous half-space. The analysis of each substructure is followed by a synthesis phase so as to ensure

¹ Post-Doctoral Associate, Department of Civil and Environmental Engineering, MIT, Room 1-330, Cambridge, MA, 02139, USA, E-mail: joonsang@mit.edu
² Professor, Department of Civil and Environmental Engineering, MIT, Room 1-271, Cambridge, MA, 02139, USA, E-mail: kausel@mit.edu
compatibility and equilibrium across the interface between the two substructures. For this purpose, we employ two closed-form Green’s functions, namely one for the upper layered medium obtained with the time-domain Thin-Layer Method (TLM), and another for the half-space, the exact expression of which is developed and presented herein.

The TLM is an effective numerical tool for the analysis of stresses, deformations, and wave motions in laminated media [Kausel, 1981, 1994]. In a nutshell, the TLM combines the Finite Element Method in the direction of layering together with analytical solutions in the remaining directions. We discuss the stability, accuracy, and efficiency of the proposed substructuring method and we define the criteria for choosing the associated parameters. To verify the proposed method, we analyze both a homogeneous and a layered half-space subjected to an $SH$ line source, and demonstrate the effectiveness of the method.

2. STATEMENT OF PROBLEM

Consider an isotropic elastic layered medium ($0 \leq z \leq H$: domain I) underlain by a homogeneous half-space ($z \leq 0$: domain II). This system is subjected to an anti-plane $SH$ transient line source $p_y(x, z, t)$ somewhere within the medium or on the surface, see figure 1. Of interest is the response $v(x, z, t)$ in direction $y$ at some arbitrary point in the medium. The upper medium is composed of $N$ locally homogeneous layers characterized by mass density $\rho_l$, shear velocity $C_{S,l}$, thickness $h_l$, shear modulus $G_l=\rho_lC_{S,l}^2$, and sub-layer index $l=1,\ldots,N$. The corresponding quantities for the half-space are denoted by an index $l=II$.

![FIG. 1. Semi-infinite medium subjected to a line load $p_y(x,y,z,t)$ at $z=z_n$ and the coordinate system in use.](image-url)
The governing equation for the whole domain and the boundary condition at the top surface are

\[ \rho \frac{\partial^2 v}{\partial t^2} - G \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} \right) = p_g(x, z, t) \quad \text{for } z \leq H \]  
(2.1)

\[ \tau_{zz} (= G \frac{\partial v}{\partial z}) = 0 \quad \text{at } z=H \]  
(2.2)

Consider next a source of the separable form

\[ p_g(x, z, t) = f(x) \delta(z - z_n) h(t) \]  
(2.3)

where \( f(x) \) and \( h(t) \) give the spatial and temporal variation of the source, respectively, \( \delta(x) \) is the Dirac delta function, and \( z_n \) is the \( z \)-coordinate of the source. A Fourier transform in the horizontal coordinate then leads to equations in the mixed horizontal wavenumber-time domain, i.e. \( k-t \).

We utilize the substructure method to solve the above problem, which can be summarized as follows. We begin by separating the whole domain into two substructures to which we impose the (unknown) internal stresses acting across the half-space interface as external tractions (see figure 2). We then enforce the displacement continuity condition and superimpose the combined response to the interface tractions and the external source so as to obtain the dynamic response anywhere in the medium. To accomplish this task, we resort to the use of two Green’s functions in the wavenumber-time domain, one for the upper layered medium and one for the half-space. Because of space considerations, however, we give these functions succinctly here without proof.

For the upper layered system, the Green’s function in the \( k-t \) domain can be obtained from a normal mode solution, an operation that can most efficiently be accomplished with the TLM, see Kausel [1994]. These Green's functions are of the form

\[ G_{\text{up}}(k, t) = \sum_{j=1}^{M} \phi_j^m \phi_j^n \frac{\sin \omega_j t}{\omega_j} \]  
(2.4)

where the superscripts \( m \) and \( n \) indicate the elevations of the receiver and source points. Also, \( \omega_j \) and \( \phi_j \) are the eigenvalues and eigenvectors for the \( j^{th} \) normal mode, and \( M \) is the number of modes included in the modal solution. As stated earlier, these modes are computed herein by means of the TLM.

For the homogeneous half-space, on the other hand, we have found that the exact Green’s function can conveniently be written in closed-form as [Park, 2002].

\[ G_{\text{h}}(k, z, t) = \begin{cases} \frac{1}{\rho hc_{S,\text{II}}} J_0 \left( k c_{S,\text{II}} \sqrt{t^2 - z^2} / c_{S,\text{II}}^2 \right) , & z / c_{S,\text{II}} \leq t \\ 0 , & z / c_{S,\text{II}} > t \end{cases} \]  
(2.5)

where \( J_0 \) is the first kind Bessel function of 0-th order.
3. SUBSTRUCTURE METHOD

3.1 Substructuring and superposition

We present next the Green’s functions in the wavenumber - time domain \((k-t)\), which we use in the context of the substructure method. Figure 2 shows the total medium separated into two substructures, namely I and II. Inasmuch as the problem is formulated in the \(k-t\) domain, the response variables have in turn arguments in this domain, i.e. \(p^n(k,t)\), \(\tau^0(k,t)\), \(v^{mn}(k,t)\), and so forth.

By virtue of the superposition principle of linear elasticity, the response for each substructure can be expressed as

\[
\begin{align*}
\nu_i^{mn}(k,t) &= G_i^{mn} \ast p^n - G_i^{m0} \ast \tau^0 \\
\nu_{II}^{mn}(k,t) &= G_{II}^{m0} \ast \tau^0
\end{align*}
\]

for \(0 \leq z \leq H\) \((3.1)\)

\[
\begin{align*}
\nu_i^{m0}(k,t) &= G_i^{m0} \ast \tau^0
\end{align*}
\]

for \(z \leq 0\) \((3.2)\)

where \(\nu_i^{mn}\), \(\nu_{II}^{mn}\) and \(G_i^{mn}\), \(G_{II}^{mn}\) are the displacements and Green’s functions for domains I and II, respectively. Also, \(\tau^0(k,t)\) is the a priori unknown internal stress at the interface between these two domains. The superscripts \(m, n\) (or 0) indicate the elevations of the receiver and the source points, respectively.

To calculate the displacements in equations (3.1-2), we need the traction \(\tau^0\). This can be obtained from the displacement-continuity condition at the interface, which is \(\nu_i^{m0} = \nu_{II}^{m0}\). Collecting the terms involving \(\tau^0\) on the left-hand side, we obtain an expression of the form

\[
F \ast \tau^0 = H \ast p^n
\]

\((3.3)\)
where \( F = G_{ii}^{00} + G_{ij}^{00} \) and \( H = G_{ii}^{0w} \). Since \( \tau^0 \) is a causal function of time, we can find it by marching forward in time as

\[
\sum_{i=0}^{i} \left[ F(t_i - \tau_j) \tau^0(\tau_j) \right] \Delta t = \sum_{i=0}^{i} \left[ H(t_i - \tau_j) p^0(\tau_j) \right] \Delta t \quad \text{with } i=0,1,2,\ldots,N_t \tag{3.4}
\]

where \( t_i=\Delta t \), \( \tau_j=\Delta t \), \( \Delta t=T/N_t \), and \( T \) is the total time window. The unknown traction \( \tau^0_i \) for each time step \( i \) can be then determined iteratively.

Now, if we choose an impulsive load \( h(t)=\delta(t) \) in equation (2.3), we obtain the impulse response functions \( g_{im}^{00} \) and \( g_{im}^{00} \) in \( k-t \) as:

\[
g_{im}^{00}(k,t) = G_{im}^{00} * p^0 - G_{im}^{00} * \tau^0 \quad \text{for } 0 \leq z \leq H \tag{3.5}
\]

\[
g_{im}^{00}(k,t) = G_{im}^{00} * \tau^0 \quad \text{for } z \geq 0 \tag{3.6}
\]

where \( \tau^0 \) follows from equation (3.4) while imposing \( p_{ij} = f(k) \) and \( p_{ij} = 0 \) for \( i=1,2,\ldots,N_t \).

Finally, to obtain the final responses in the space-time domain due to a general source \( p^0(x,t) \), we perform the inverse Fourier transformations in \( k \) as

\[
v_{im}^{00}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v_{im}^{00}(k,t) e^{-ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{im}^{00} * h(t)e^{-ikx} dk \quad \text{for } 0 \leq z \leq H \tag{3.7}
\]

\[
v_{im}^{00}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} v_{im}^{00}(k,t) e^{-ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{im}^{00} * h(t)e^{-ikx} dk \quad \text{for } z \geq 0 \tag{3.8}
\]

These integrals can readily be evaluated, because the kernels of do not exhibit any singularities and decay rapidly with wavenumber.

3.2 Stability and accuracy

The stability and accuracy of this method depends on the size of the time step (\( \Delta t \)) chosen to solve eq. 3.3, the highest modal frequency (\( f_m \)) in comparison to the highest frequency in the excitation (\( f_{\max} \)), and the number of thin-layers per wavelength (\( N_{t} \)). We have also found that adding a buffer layer of adequate thickness \( H_{BL} \) in contact with the half-space that has the same material properties as the half-space helps considerably. This avoids wave complexities at the interface between regions I and II that can lead to instabilities. In a nutshell, the four requirements for stability and accuracy are as follows, see Park [2002]:

\[
\Delta t \leq \frac{1}{4f_m} \tag{3.9}
\]

\[
N_{t} \geq 12 \quad \text{(linear elements) and } N_{t} \geq 6 \quad \text{(quadratic elements)} \tag{3.10}
\]

\[
f_m \geq 2f_{\max} \tag{3.11}
\]

\[
H_{BL} \geq \lambda_{\min} \tag{3.12}
\]

where \( f_{\max} \) is the highest frequency of the external source and \( \lambda_{\min} \) is the shortest wavelength in half-space, i.e. \( \lambda_{\min} = C_{S,II}/f_{\max} \).
4. NUMERICAL EXAMPLE

To illustrate the proposed method, we examine a two-layered system subjected to a transient line source applied at the surface. The mass densities, shear wave velocities, and thicknesses of the upper two layers are $\rho_1=C_{S1}=H_1=1.0$, and $\rho_2=H_2=1.0$, $C_{S2}=2$, respectively, while those of the half-space are $\rho_1=1.0$ and $C_{SIII}=3.0$. The variation of the source in both $x$ and $t$ is taken to be a bell-shaped function of half-width and duration $a=t_d=0.2$ [Park, 2002], which decays rapidly in both frequency and wavenumber. For these parameters, the maximum wavenumber and frequency can be estimated as $k_{max}=2\pi/a=10\pi$ and $f_{max}=2/t_d=10$ Hz, respectively. For the discrete model, we apply the TLM using 160 and 80 quadratic elements for the two upper layers, respectively, (i.e. $N_x=8$), a time step $\Delta t=1/(4f_M)=1/(8f_{max})=0.0125$, and a buffer layer of thickness $H_{BL}=\lambda_{min}=0.3$.

Figure 3 displays displacement snapshots at $t=1$, 2, 3, and 4 for the upper two-layered system ($0 \leq z \leq 2$) in terms of a standard surface plot. In each plot, the dotted line at $z=1$ represents the interface between the first ($1 \leq z \leq 2$) and second ($0 \leq z \leq 1$) upper layers. The wave reflections and refractions cause the motions to be more complicated than for a homogeneous half-space, yet the method performs well, and captures these complexities without difficulty. For example, the head waves can clearly be seen in the third snapshot. Comparison with alternative methods of computation (not shown here) demonstrate the accuracy of these results.

FIG. 3. Snapshots of $v$ in a two-layers over a homogeneous half-space ($\rho_1=C_{S1}=H_1=1$; $\rho_2=H_2=1$, $C_{S2}=2$; $\rho_1=1$, $C_{SIII}=3$) due to a surface line load of $a=t_d=0.2$, obtained with quadratic elements of $N_x=8$, $\Delta t=1/(4f_M)$ and with a buffer layer of $H_{BL}=\lambda_{min}$.
5. CONCLUSIONS

In this paper, we presented the $k$-$t$ domain Green’s function associated with $SH$ transient line sources in a layered medium over a homogeneous half-space, which is based on a sub-structuring technique. For this purpose, we combined the Green’s function for the upper layered medium obtained with the Thin Layer Method, and the Green’s function for the half-space given in this paper. With the aid of the two Green’s functions, we superimpose the effects of the external source and the internal stress at the interface to obtain the $k$-$t$ domain Green’s functions for a layered half-space, and solve the required convolution equation by reduction to a discrete time summation. To validate the proposed method, we simulate wave motions in a two-layered half-spaces subjected to a $SH$ transient line source, and conclude that the proposed method is an effective tool to accurately and efficiently obtain the response functions in the space-time domain.

REFERENCES