GREEN’S FUNCTIONS IN THE WAVENUMBER-TIME DOMAIN FOR LOADS ON THE SURFACE OF A HOMOGENEOUS HALF-SPACE

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ABSTRACT
We present herein expressions for the Green’s functions in the wavenumber-time domain for a homogeneous half-space subjected to either surface line loads or point loads. First, we derive the exact Green’s functions by means of direct integration for the anti-plane component, and by means of contour integration in the complex frequency plane for the in-plane components. For the in-plane case, we obtain the Green’s functions by adding the residues at the Rayleigh poles together with well-behaved improper branch-cut integrals that must be evaluated numerically. Through numerical examples, we verify the efficiency and accuracy of the solutions presented herein.

KEYWORDS: Green’s functions; fundamental solutions; wavenumber-time domain; elastic half-space; Lamb’s problem; seismic sources; elasto-dynamics.

INTRODUCTION
Numerous theoretical studies on Lamb’s problem in the last century have provided a solid foundation for the analysis of elastic wave propagation in semi-infinite media. In the majority of cases, the formulation for sources on a homogeneous half-space were carried out in the frequency domain in the context of linear problems. On the other hand, a formulation in the time domain is required for, but not restricted to, nonlinear problems. Indeed, a solution for the Green’s functions in the time domain allows circumventing many of the numerical difficulties inherent in solution in the frequency domain, such as the waviness of the kernels [Kausel, 1994].

In this paper, we present (partially) closed-form expressions for the Green’s function in the wavenumber-time domain for a homogeneous half-space subjected to either surface line loads or point loads. First, by means of direct integration, we obtain an exact formulation for the anti-plane component. Secondly, via contour integration in the complex frequency plane, we obtain exact expressions for the Green’s functions, which consist of free-vibration terms associated with the Rayleigh pole, and improper integrals without singularities associated with the branch-cuts. Then, we obtain make use of asymptotic expressions for the kernels, which allows us in turn to express the branch-cut integrals explicitly in terms of sine integral functions. Thereafter, we verify the efficiency and accuracy of the solutions developed herein through numerical examples. These Green’s functions are useful not only for wave propagation problems in a homogeneous half-space, but also for layered half-spaces formulated directly in the time domain. This entails

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coupling the Green’s function in this paper with discrete methods such as the Thin-Layer Method, as described in a companion paper by the writers [Park and Kausel, 2003].

GREEN’S FUNCTIONS IN THE WAVENUMBER-TIME DOMAIN

Consider a homogeneous elastic half-space in three dimensions subjected to loads on its surface, which may vary both in space and in time (Fig. 1). This problem is initially formulated in the frequency-wavenumber domain, or $k-\omega$ for short. Since the response functions in that domain are well known, we can exclude their derivation, and describe instead the derivation of the response functions in the wavenumber-time domain, or $k-t$ for short. Details can be found in Park [2002]. The half-space has mass density $\rho$, dilatational wave velocity $C_p$, shear wave velocity $C_s$, shear modulus $\mu$, Lamé constant $\lambda$, constrained modulus $\lambda+2\mu$, and Poisson's ratio $\nu$. The load and response functions are either of the anti-plane type, which involve only SH waves, or of the in-plane type, which involve SV-P waves. We begin with the former.

Green's function for SH loads

The Green’s function $G_{yy}$ in the $k-t$ domain for SH loads is derived by means of an inverse Fourier transform from $k-Z$ to $k-t$, of the antiplane Green's function in $k-Z$, which can be accomplished by means of available formulas (e.g. Spiegel, 1968). The result is:

$$G_{yy}(k,z,t) = \frac{1}{\rho C_s} J_0 \left( k C_s \sqrt{t^2 - (z/C_s)^2} \right)$$

in which $J_0$ is the Bessel function of the first kind and order zero, $t \geq z/C_s$, and the other parameters are listed in the appendix. This formula is exact.

Green's functions for SV-P loads

To evaluate the Green’s functions for SV-P loads, we carry out a contour integration whose path is deformed around the branch cuts and closed at infinity in the upper complex half-plane. The contour integral yields residues contributed by the Rayleigh poles together with improper branch-cut integrals without singularities, for details see Park [2002]. It is found that the Fourier
transformation into \( k-t \) gives impulse response functions \( G_y = G_y(k,z,t) \) that ultimately can be abbreviated as

\[
G_y(k,z,t) = \frac{1}{2\rho C_s} \left[ R_y + I_y \right] \quad \text{with } i,j = x,z
\]  
(2)

\[
I_y = \frac{1}{\pi} \int_\Omega f_y \sin \Omega \tau d\Omega
\]  
(3)

in which \( R_y \) contains the residues for the Rayleigh poles, \( I_y \) refers to the branch-cut integrals, and \( f_y \) are the kernels in these integrals. The residues due to the Rayleigh poles are:

\[
R_y = -\frac{1}{T_R} N_y(\Omega_R) \sin \Omega_R \tau
\]

\[
D_R = D(\Omega_R) = \frac{C_x}{\rho C_s} \left[ 2 - \left( \frac{C_x}{C_y} \right)^2 \right] \left[ 1 + \left( \frac{C_x}{C_y} \right)^2 - 2 \left( \frac{C_x}{C_y} \right)^2 \right] \sqrt{\left(1 - \left( \frac{C_x}{C_y} \right)^2 \right) \left(1 - \left( \frac{C_x}{C_y} \right)^2 \right)}
\]

\[
N_{xx}(\Omega_R) = s_x \left[ 2e^{i\beta z} - (1 + s_x^2) e^{i\beta z_0} \right]
\]

\[
N_{xz}(\Omega_R) = i \left[ 2r_x s_x e^{i\beta z} - (1 + s_x^2) e^{i\beta z_0} \right]
\]

\[
N_{zx}(\Omega_R) = i \left[ (1 + s_x^2) e^{i\beta z_0} - 2r_x s_x e^{i\beta z} \right]
\]

\[
N_{zz}(\Omega_R) = r_x \left[ (1 + s_x^2) e^{i\beta z_0} + 2 e^{i\beta z} \right]
\]

\[(4a-f)\]

and those for the branch integrals are:

\[
I_{xx} = -\frac{1}{\pi} \int_{\Delta_1} \frac{1}{\Delta_1 \Delta_2} \left[ 2\beta e^{ikz} \Re\left( \Delta_2 \right) - \beta (1 - \beta^2) \Re\left( e^{-ikz_0} \Delta_2 \right) \right] \sin \Omega \tau d\Omega
\]

\[-\frac{1}{\pi} \int_{\Delta_1} \frac{1}{\Delta_1 \Delta_2} \left[ 2\beta \cos k\alpha z - \beta (1 - \beta^2) \cos k\beta z \right] \sin \Omega \tau d\Omega
\]

\[(5a)\]

\[
I_{xz} = -\frac{1}{\pi} \int_{\Delta_1} \frac{1}{\Delta_1 \Delta_2} \left[ 2r_x e^{ikz} \Re\left( \Delta_2 \right) + (1 - \beta^2) \Im\left( e^{-ikz_0} \Delta_2 \right) \right] \sin \Omega \tau d\Omega
\]

\[-\frac{1}{\pi} \int_{\Delta_1} \frac{1}{\Delta_1 \Delta_2} \left[ -2\alpha \beta \sin k\alpha z - (1 - \beta^2) \sin k\beta z \right] \sin \Omega \tau d\Omega
\]

\[(5b)\]

\[
I_{zx} = -\frac{1}{\pi} \int_{\Delta_1} \frac{1}{\Delta_1 \Delta_2} \left[ -(1 - \beta^2) e^{ikz} \Im\left( \Delta_2 \right) - 2r_x \beta \Re\left( e^{-ikz_0} \Delta_2 \right) \right] \sin \Omega \tau d\Omega
\]

\[-\frac{1}{\pi} \int_{\Delta_1} \frac{1}{\Delta_1 \Delta_2} \left[ (1 - \beta^2) \sin k\alpha z + 2\alpha \beta \sin k\beta z \right] \sin \Omega \tau d\Omega
\]

\[(5c)\]

\[
I_{zz} = -\frac{1}{\pi} \int_{\Delta_1} \frac{1}{\Delta_1 \Delta_2} \left[ r(1 - \beta^2) e^{ikz} \Im\left( \Delta_2 \right) - 2r \Im\left( e^{-ikz_0} \Delta_2 \right) \right] \sin \Omega \tau d\Omega
\]

\[-\frac{1}{\pi} \int_{\Delta_1} \frac{1}{\Delta_1 \Delta_2} \left[ -\alpha (1 - \beta^2) \cos k\alpha z + 2\alpha \cos k\beta z \right] \sin \Omega \tau d\Omega
\]

\[(5d)\]
Again, all the parameters are defined in the appendix.

**Kernels in branch-cut integrals at z=0**

Figures 2a-c depict the kernels $f_{xx}$, $f_{zz}$, and $f_{xz}$ for Poisson’s ratio $\nu=0.30$ and $z=0$ (i.e. at the free surface). Note that $f_{xz}=f_{zx}$ at $z=0$. Solid lines are the actual kernels while dashed lines represent the hyperbolas $4/\Omega$ and $4/p\Omega$, which are, respectively, the asymptotes to which the kernels $f_{xx}$ and $f_{xz}$ rapidly converge. Circles in each figure represent the points of $\Omega = 1$, $\sqrt{2}$, $p$, and $5p$, respectively. The first and third correspond, respectively, to the S and P branch points ($s=0$ and $r=0$). The second, on the other hand, is a special point at which the kernels for both $f_{xx}$ and $f_{xz}$—for any Poisson’s ratio—go through zero. Finally, the fourth circle represents the point beyond which the kernels are, for all practical purposes, identical to their asymptote. Of interest is also the fact that the coupling term exists only in the interval of $1 \leq \Omega \leq p$, in which $p = p(\nu)$.

Other values of Poisson’s ratio yield similar plots, so they need not be shown here. Nonetheless, the full range of Poisson’s ratios from $0 \leq \nu \leq 0.5$ was considered by the writers. It should also be mentioned that the second interval vanishes when Poisson’s ratio equals zero, because $p = C_p / C_S = \sqrt{2}$. Conversely, in the case of an incompressible solid, $p = \infty$, for which $\nu=0.5$. In most cases of interest, however, $p$ lies somewhere between 1.5 and 10.

The kernel plots suggest a natural separation of the branch-cut integrals into three intervals: $1 \rightarrow \sqrt{2}$, $\sqrt{2} \rightarrow p$, and $p \rightarrow \infty$. Now, for large $\Omega$, say $\Omega \geq 5p$, the two kernels $f_{xx}$ and $f_{xz}$ tightly approximate the hyperbolas $4/\Omega$ and $4/p\Omega$, respectively. Thus, beyond this limit, the kernels can be replaced safely by their asymptotes. As a result, the branch-cut integrals can be written in terms of proper integrals as

$$I_{xx} \approx \frac{1}{4} \int_{2}^{\sqrt{2}} f_{xx} \sin \Omega \tau d\Omega + \frac{1}{4} \int_{2}^{\sqrt{2}} f_{xx} \sin \Omega \tau d\Omega + \frac{1}{4} \int_{p}^{4} f_{xx} \left[ -\frac{4}{\Omega} \right] \sin \Omega \tau d\Omega$$

$$+ \frac{1}{4} \int_{p}^{4} \frac{4}{\Omega} \sin \Omega \tau d\Omega = I_{xx}' + I_{xx}'' + I_{xx}''' + 2 \left[ \text{sgn}(r) - \frac{2}{\pi} \text{Si}(pr) \right]$$

**(6a)**
\[
I_{zz} \approx \frac{1}{\pi} \int f_{zz} \sin \Omega \tau d\Omega + \frac{1}{\pi} \int f_{zz} \sin \Omega \tau d\Omega + \frac{1}{\pi} \int f_{zz} - \frac{4}{p \Omega} \sin \Omega \tau d\Omega + \frac{4}{p \Omega} \sin \Omega \tau d\Omega
\]
\[
+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{4}{p \Omega} \sin \Omega \tau d\Omega = I_{zz}^{\frac{1}{1}} + I_{zz}^{\frac{2}{1}} + \frac{2}{p^2} \left[ \text{sgn}(\tau) - \frac{2}{\pi} \text{Si}(p \tau) \right]
\]
\[
I_{zz} = \frac{1}{\pi} \int f_{zz} \sin \Omega \tau d\Omega + \frac{1}{\pi} \int f_{zz} \sin \Omega \tau d\Omega = I_{zz}^{\frac{1}{1}} + I_{zz}^{\frac{2}{1}}
\]
with \( \text{Si} \) being the sine integral function (see Abramovitz and Stegun [1964]). In these equations, all integrals are well-behaved, have no singularities, and have finite integration limits. Therefore, any convenient numerical integration scheme is applicable. A convenient strategy is to pre-compute these branch integrals for a sufficiently dense set of wavenumbers, place these in a lookup table, and use them (via interpolation, if necessary), to obtain response functions for arbitrary loads.

**RESPONSE IN SPACE-TIME DOMAIN**

The applicability and accuracy of the proposed solution in obtaining synthetic seismograms is illustrated next by means of examples for classical problems, namely Lamb's problem for line loads and point loads (actually, the time derivative of these problems, because we consider impulse loads instead of step loads).

Consider first a transient, spatially local horizontal or vertical load of the form

\[
p(x,z,t) = f(x) \delta(z) \delta(t)
\]
acting on a homogeneous half-space with unit properties and Poisson's ratio \( \nu = 0.30 \), for which the P and Rayleigh wave velocities are 1.87 and 0.93, respectively. Because the response in space is obtained herein via numerical transform over wavenumbers, a load distribution is chosen for which its Fourier transform decays rapidly with wavenumber. A convenient choice is a bell shape of the form

\[
f(x) = \begin{cases} 
\frac{1}{a} \cos^2 \frac{\pi}{2a} x & |x| \leq a \\
0 & |x| > a 
\end{cases}
\]
in which \( a \) is the half width of the load. The total applied force equals the area under the load, which is unit in this case. Choosing reasonably small values of \( a \) allows simulating a line load. In the example that follows, \( a = 0.1 \) is selected. The displacements in the spatial domain can be evaluated through a numerical inverse Fourier transformation, for the Green's functions in \( k-t \) have no singularities. Figs. 3a-b show the horizontal and vertical displacements \( u, w \) at a receiver at a distance \( x = 5 \) for a vertical source. Indicated with the three capitals P,S,R, are the theoretical arrival times of P, S, and R waves, which are 2.67, 5.00, and 5.39, respectively. Clearly, the P, S, and R waves arrive exactly at the expected times.
FIG. 3. Surface displacements $u$ and $w$ at $x=5.0$ due to a vertical line load of bell-shaped spatial distribution.

Consider next a transient, spatially local vertical load of the form

$$p(r,z,t) = f(r) \delta(z) \delta(t)$$  \hspace{1cm} (9)$$

acting on the same half-space considered for the line load. The load distribution is a bell shape of the form

$$f(r) = \frac{a}{\pi} e^{-\omega \cdot r}$$  \hspace{1cm} (10)$$

Again, selecting small values of $a$ allows simulation of a point load. Then, the displacements in the spatial domain can be calculated by a numerical inverse Hankel transform. Figures 4a-b show

FIG. 4. Surface displacements $u$ and $w$ at $r=5.0$ due to a vertical point load of bell-shaped spatial distribution.
the horizontal and vertical displacements $u, w$ at a receiver at a distance $r=5$ due to a vertical source. Again, the three capitals $P, S, R$ indicate the theoretical arrival times of $P, S, \text{ and } R$ waves. The computed responses are in agreement with the known solutions for these canonical problems.

**CONCLUSIONS**

In this paper, we presented the Green’s functions in the $k$-$t$ domain for a homogeneous half-space subjected to either surface line loads or point loads. First, we derived the fully closed-form Green’s function for the SH component by means of direct integration. Secondly, via contour integration in the complex frequency plane, we derived exact expressions for the Green’s functions for the SV-P components, which consist of free-vibration terms associated with the Rayleigh pole, and improper integrals without singularities associated with the branch-cuts. Then, we integrate exactly the tails of the branch integrals by using the asymptotic values of the kernels, which allows us in turn to reduce the branch-cut integrals into a proper integrals plus a integral sine (for which efficient expressions are available). Thereafter, we presented numerical examples and verified the accuracy of the solutions presented by comparing calculated arrival times of the $P, S, \text{ and }$ Rayleigh waves with the theoretically estimated times. The Green’s functions presented are useful not only for wave propagation problems in a homogeneous half-space, but also for layered half-spaces formulated directly in the time domain. This entails coupling the Green’s functions presented in this paper with discrete methods such the Thin-Layer Method, as described in a companion paper by the writers [Park and Kausel, 2003].

**REFERENCES**


**APPENDIX: SYMBOLS, PARAMETERS AND DEFINITIONS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>Mass density</td>
</tr>
<tr>
<td>$\lambda, \mu$</td>
<td>Lamé constants</td>
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<tr>
<td>$\nu$</td>
<td>Poisson's ratio</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Frequency</td>
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<tr>
<td>$k$</td>
<td>Horizontal wavenumber</td>
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<tr>
<td>$C_s = \sqrt{\mu/\rho}$</td>
<td>S-wave velocity</td>
</tr>
<tr>
<td>$C_p = \sqrt{(\lambda + 2\mu)/\rho}$</td>
<td>P-wave velocity</td>
</tr>
<tr>
<td>$p = C_p/C_s = \frac{2 - 2\nu}{\sqrt{1 - 2\nu}}$</td>
<td>Ratio of $P$ to $S$ wave speeds</td>
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</tbody>
</table>

$k_p = \omega/C_p$
\[ k_s = \frac{\omega}{C_s} \]
\[ \Omega = \frac{k_s}{k} = \frac{\omega}{kC_s} \]
\[ \tau = kC_s t \]
\[ r = \sqrt{1 - \Omega^2 / \rho^2} = i\alpha \]
Vertical wavenumber for P waves, \( \Omega \leq \rho \)
\[ s = \sqrt{1 - \Omega^2} = i\beta \]
Vertical wavenumber for S waves, \( \Omega \leq 1 \)
\[ \alpha = \sqrt{\Omega^2 / \rho^2 - 1} \]
Vertical wavenumber for P waves, \( \Omega \geq \rho \)
\[ \beta = \sqrt{\Omega^2 - 1} \]
Vertical wavenumber for S waves, \( \Omega \geq 1 \)
\[ \Delta = rs - \frac{1}{4}(1 + s^2)^2 \]
Rayleigh function, \( \Omega \leq 1 \)
\[ \Delta_1 = -\frac{1}{4}(1 - \beta^2)^2 - a\beta \]
Rayleigh function, \( \Omega \geq \rho \)
\[ \Delta_2 = -\frac{1}{4}(1 - \beta^2)^2 + i\beta \]
Rayleigh function, \( 1 \leq \Omega \leq \rho \)
\[ \Delta_2^* = -\frac{1}{4}(1 - \beta^2)^2 - i\beta = \text{conj}(\Delta_2) \]
Conjugate of above
\[ \Omega_s = \frac{\omega_s}{kC_s} = \frac{kC_s}{kC_s} = 1 \]
Branch point for S waves
\[ \Omega_p = \frac{\omega_p}{kC_s} = \frac{C_p}{C_s} \ni p \geq \sqrt{2} \]
Branch-point for P waves
\[ \Omega_R = \frac{\omega_R}{kC_s} = \frac{C_R}{C_s} \]
Rayleigh pole
\[ r_R = \sqrt{1 - \left(\frac{C_R}{C_p}\right)^2} \]
\[ s_R = \sqrt{1 - \left(\frac{C_R}{C_s}\right)^2} \]
\[ D_R = \frac{d\Delta}{d\Omega_{\rho=\Omega_s}} = \Omega_R \left[ 2 - \Omega_R^2 + \frac{r_R}{s_R} \frac{s_R}{p^2 r_R} \right] \]