BLAST LOADS versus POINT LOADS: THE MISSING FACTOR

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ABSTRACT: In the course of implementing and testing an efficient algorithm for the computation of the displacements (or Green's functions, or fundamental solutions) elicited by a pulsating blast source acting at some point in a laminated medium, the writer encountered what was at first an unexpected and puzzling anomaly: A benchmark test with the analytical solution for a blast source in a homogeneous, elastic full-space revealed a discrepancy by a constant factor. Since the solution for the blast source had been derived from available Green's functions for harmonic point sources, the first thought that came to mind was that there was a fault in the derivation. When a careful analysis demonstrated that the formulation was indeed correct, it became necessary to search for an alternative explanation. Finally, the cause for the mysterious discrepancy became clear: A homogeneous solid with an infinitesimally small cavity (i.e., a vanishingly small radius) is not the same as the continuous solid medium, at least not when a singularly large pressure acts within that point-like cavity. After this observation was accounted for, the differences between the numerical and analytical solutions could be fully resolved and a perfect match achieved. Since neither the problem nor its proof are obvious, the following note addresses both of these.

BLAST LOADS AND POINT LOADS

Consider an unbounded, homogeneous elastic solid containing a small spherical cavity of radius $a$. Within this cavity acts a harmonically oscillating pressure with amplitude $p$. Therefore, the net force acting on an elementary area $dA$ with position vector $\mathbf{a}$ on the cavity's wall, with unit outward normal $\mathbf{n}$, (see Fig. 1), is

$$d\mathbf{F} = pdA\mathbf{n} = dF_r\hat{\mathbf{r}} + dF_\theta\hat{\mathbf{\theta}} + dF_\phi\hat{\mathbf{\phi}} = pdA(n_r\hat{\mathbf{r}} + n_\theta\hat{\mathbf{\theta}} + n_\phi\hat{\mathbf{\phi}})$$

(1)

where $dF_r$, $dF_\theta$, $dF_\phi$ are the Cartesian components of the force.

Next, consider an external point in the medium with position vector $\mathbf{r}$ relative to the center of the cavity. The displacements caused by the elementary forces at that point are

$$d\mathbf{u} = pdA(n_\phi\hat{\mathbf{\phi}}(\mathbf{r} - \mathbf{a}) + n_\theta\hat{\mathbf{\theta}}(\mathbf{r} - \mathbf{a}) + n_r\hat{\mathbf{r}}(\mathbf{r} - \mathbf{a}))$$

(2)

in which $\hat{\mathbf{r}}$, $\hat{\mathbf{\theta}}$, $\hat{\mathbf{\phi}}$ are the Green's functions for unit loads in directions $x$, $y$, and $z$, respectively, whose argument $\mathbf{r} - \mathbf{a}$ is the distance from the point of application of the force to the observation point. Expansion of these functions in Taylor series yields

$$\hat{\mathbf{r}}(\mathbf{r} - \mathbf{a}) = \hat{\mathbf{r}}(\mathbf{a}) - \mathbf{r} \cdot \nabla \hat{\mathbf{r}} + (O(a^2))$$

(3)

and similar expressions for $\hat{\mathbf{\theta}}(\mathbf{r} - \mathbf{a})$ and $\hat{\mathbf{\phi}}(\mathbf{r} - \mathbf{a})$. Substituting (3) into (2), carrying out a straightforward integration over the entire cavity surface, defining a source strength $S = \frac{4\pi}{3}m\alpha p$, and considering the limit of an infinitesimal cavity $a \to 0$, one obtains an expression of the form

$$d\mathbf{u} = -S[\hat{\mathbf{r}} \cdot \nabla \hat{\mathbf{r}} + \hat{\mathbf{\theta}} \cdot \nabla \hat{\mathbf{r}} + \hat{\mathbf{\phi}} \cdot \nabla \hat{\mathbf{r}}]$$

(4)

which appears to relate the Green's functions for point loads with the Green's functions for blast loads having a source strength $S$. Eq. (2) contains, however, a seemingly small conceptual error: the Green's functions $\hat{\mathbf{r}}$, $\hat{\mathbf{\theta}}$, $\hat{\mathbf{\phi}}$ are those for the continuum—that is, they do not account for the presence of the infinitesimally small cavity. As it turns out, the contribution of the tiny particle filling the equally small cavity is important and leads to the missing factor, as will be shown. To obtain this factor, it is necessary to consider the contribution of both the cavity and the solid particle that completely fills it.

MOTION OF PULSATING CAVITY

The analytical solution for the displacements associated with a pulsating spherical cavity in a homogeneous medium is well-known, so its derivation need not be repeated here. It suffices to follow the method given in Graff (1975) while making appropriate modifications to take into account the positive sign of the harmonic factor $\exp(\mathrm{i}ot)$ implicit in the present formulation. One then finds that for a cavity of radius $a$ in an infinite, homogeneous medium with Lamé constants $\lambda$, $\mu$ and mass density $\rho$ that is subjected to a pulsating pressure of amplitude $p_0$, the exact displacements on the cavity wall are given by

$$u = \frac{p_0k^2h_0^2(ka)e^{\mathrm{i}ka}}{4\mu(ka - i) - (ka)^2(\lambda + 2\mu)i}$$

(5)

in which $i = \sqrt{-1}$, $h_0^2(z) = e^{-\mathrm{i}\pi(1 - z)/2}$ is the spherical Hankel function of first order and second kind; $k = \omega \alpha$ is the radial wave number; and $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ is the dilatational wave velocity. If one considers only small values of $a$, one approaches the expression

$$u = \frac{p_0a}{4\mu}$$

(6)

MOTION OF PULSATING SPHERICAL PARTICLE

Consider next an elastic, spherical particle of radius $a$ under the action of an external negative pressure (i.e., suction or

![Fig. 1](image_url)

FIG. 1

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tractions) \( p_r \). By using developments entirely analogous to the spherical cavity's, it can be shown that the exact motion on the outer surface of the particle is given by

\[
\frac{p_r a}{\lambda + 2\mu} \frac{j_0(ka)}{j_1(ka)} - 4\mu \tag{7}
\]

in which \( j_0 \) and \( j_1 \) are the spherical Bessel functions. When \( a \) is small, (7) approaches the expression

\[
u = \frac{p_r a}{3(\lambda + 2\mu) - 4\mu} \tag{8}
\]

**MOTION OF CONTINUUM**

The continuum is obtained by welding the spherical particle to the cavity and forcing the displacements of either wall to be the same. This is achieved by equating (6) and (8), which yields

\[
\frac{p_r a}{3(\lambda + 2\mu) - 4\mu} = \frac{p_e a}{4\mu} \tag{9}
\]

That is,

\[
p_r = \left[ \frac{3\lambda + 2\mu}{4\mu} - 1 \right] p_e \tag{10}
\]

After welding the particle to the cavity's wall (i.e., the solid), the total radial traction required to elicit in the continuum the same displacement as that caused by the blast load is

\[
p = p_r + p_e = \frac{3\lambda + 2\mu}{4\mu} p_e = \frac{3}{4} \left( \frac{\alpha}{\beta} \right) p_e \tag{11}
\]

with \( \beta = \sqrt{\mu/\rho} \) being the shear wave velocity. If \( S_s = (4/3)\pi a^3 p_e \) is defined as the strength of the blast source in the infinitesimal cavity, it follows that the total blast source required in the continuum to achieve the same displacements as for the solid with the cavity is

\[
S = \frac{3\lambda + 2\mu}{\mu} S_s = \frac{3}{4} \left( \frac{\alpha}{\beta} \right)^2 S_s \tag{12}
\]

Therefore, the true relationship between the Green's functions for point loads and the Green's functions for blast loads is

\[
\tilde{u} = \frac{3\lambda + 2\mu}{4\mu} [\hat{I} \cdot \nabla g_e + \hat{j} \cdot \nabla g_e + \hat{k} \cdot \nabla g_e] S_s \tag{13}
\]

The term in front of the braces—namely, \( 0.75(\lambda + 2\mu)/\mu \)—was exactly the factor missing in the original formulation (4), after which the numerical solution gave results that agreed with the theoretical solution.

**TWO-DIMENSIONAL BLAST LOADS**

Following an entirely analogous derivation, it can be shown that the appropriate factor needed to relate the Green's functions for two-dimensional blast loads (i.e., blasts in a cylindrical cavity) with the Green's functions for line loads is \( \lambda + 2\mu/\mu \).

**CONCLUSION**

The preceding developments show that an infinitesimally small cavity in a continuum can influence the behavior of the solid when singularly large forces are acting in its vicinity. Therefore, the a priori assumption that the vanishingly small particle at the center of the blast has no effect on the response functions was clearly naive. At the same time, the preceding developments show that the effect of the particle is a constant factor applied to the Green's functions. This factor depends only on the material properties near the source (i.e., the material properties of the particle).

**APPENDIX. REFERENCE**