Response of Layered Half-Space Obtained Directly in the Time Domain, Part II: SV-P and Three-Dimensional Sources

by Eduardo Kausel and Joonsang Park

Abstract  In Part I of this set of two companion articles we presented a new, rigorous method to obtain the seismic response of horizontally layered, viscoelastic (or elastic) half-spaces caused by antiplane (SH) sources anywhere. This method is formulated directly in the time domain, and can be applied even in the absence of attenuation. This article (Part II) generalizes the concept to SV-P line sources and to three-dimensional point sources, including seismic moments.

Introduction

This article describes a novel procedure to obtain the response of layered media to plane sources and/or point sources acting anywhere in the medium. The method is cast directly in the time domain, so it circumvents the usual difficulties associated with the waviness of the frequency-response functions due to resonances in the layers. To avoid duplication of material and descriptions, refer Part I (Park and Kausel, 2006) of this set of two companion articles for a succinct introduction to the method and for a review of related and past work. Thus, we assume in the ensuing text that the reader is already familiar with Part I.

Response of Homogeneous Half-Space in k-t

Designating the wavenumber-time domain with the abbreviation k-t, consider an elastic half-space \( z = 0 \) of mass density \( \rho_H \), dilatational wave velocity \( a_H \), and shear-wave velocity \( \beta_H \) subjected to an SV-P source (tractions) arbitrarily distributed over its surface. Thus, the source term (i.e., the body-force vector) is of the form

\[
\begin{bmatrix}
    b_x(x, z, t) \\
    b_z(x, z, t)
\end{bmatrix} = \begin{bmatrix}
    p_x(x, t) \\
    p_z(x, t)
\end{bmatrix} \delta(z),
\]

or more compactly

\[
b(x, z, t) = p(x, t) \delta(z),
\]

in which \( p(x, t) \) describes the temporal-spatial variation of the load (i.e., external traction) at the surface. A spatial Fourier transform of equation (2) gives

\[
\tilde{b}(k, z, t) = \int_{-\infty}^{+\infty} b(x, z, t) e^{ikx} dx = \tilde{p}(k, t) \delta(z),
\]

in which \( k \) is the horizontal wavenumber, and \( \tilde{p}(k, t) \) is the spatial Fourier transform of the source. As shown by Park and Kausel (2004a), the exact response in direction \( i \) at depth \( d = |z| \) in the half-space caused by an impulsive, spatially harmonic SV-P source in direction \( j \) at the surface \( z = 0 \) is of the form

\[
g_{Hij}(k, z, 0, t) = \frac{e^{-\gamma t}}{\rho_H \beta_H} (R_{ij} + B_{ij}) \mathcal{H}(\tau - \tau_p),
\]

where \( \gamma = C_H / \beta_H \) (the ratio of the Rayleigh and shear-wave velocities in the half-space). Also, \( \xi_H = 1/2Q_H \) is the fraction of damping, with \( Q_H \) being the quality factor of the half-space (which can, but need not depend on \( k \)). \( \mathcal{H} \) is the Heaviside function, and \( \tau_p \) is the dimensionless arrival time of \( P \) waves at depth \( d \) (given in Table 1). Table 3 summarizes the result for the special case \( d = 0 \) (i.e., surface of half-space).

From Tables 1 and 2 (or 3), we observe that \( g_{Hxx}, g_{Hzz} \) are purely real and symmetric with respect to positive/negative values of the horizontal wavenumber, whereas \( g_{Hxz}, g_{Hzx} \) are purely imaginary and antisymmetric with respect to this wavenumber; this ensures that the inverse Fourier transforms into space are purely real. Thus, it is convenient to write the Green’s functions matrix in terms of purely real functions \( \tilde{g}_{Hij}(k, d, 0, t) \) as (see also Appendix I)

\[
\tilde{g}_{Hij}(k, d, 0, t) = \begin{bmatrix}
    \tilde{g}_{Hxx} & \tilde{g}_{Hzz} \\
    \tilde{g}_{Hxz} & \tilde{g}_{Hzx}
\end{bmatrix} = \begin{bmatrix}
    \tilde{g}_{Hxx} - i \tilde{g}_{Hzz} \\
    i \tilde{g}_{Hxz} & \tilde{g}_{Hzx}
\end{bmatrix}.
\]

Also, on account of the principle of reciprocity, the response at the surface due to an impulsive, spatially harmonic source at depth \( d \) is
Rayleigh Pole Term

\[ R_{ij} = A_{ij}(\gamma, d) \sin \gamma \tau \]

\[ \gamma = \frac{C_R}{\beta_H} \approx 0.874 + 0.197v - 0.056v^2 - 0.0276v^3, \quad (v = \text{Poisson's ratio}) \]

\[ a = \frac{\beta_H}{\alpha_H}, \quad s_R = \sqrt{1 - a^2}, \quad p_R = \sqrt{1 - q^2}, \quad q_R = 1 - \frac{1}{2}q^2 \]

\[ \tau = \frac{k\beta_H t}{c_H} = \text{dimensionless time} \]

\[ \tau_p = \frac{akd}{v} = \text{dimensionless arrival time of P waves at depth } d \]

\[ \tau_s = \frac{kd}{v} = \text{dimensionless arrival time of S waves at depth } d \]

\[ A_{ij}(\gamma, d) = \frac{N_{ij}(\gamma, d)}{D(\gamma)} \]

\[ N_{xx}(\gamma, d) = s_R \left[ \exp(-\tau_s p_R) - q_R \exp(-\tau_s s_R) \right] \]

\[ N_{xz}(\gamma, d) = i[p_R s_R \exp(-\tau_s p_R) - q_R \exp(-\tau_s s_R)] \]

\[ N_{zz}(\gamma, d) = \left| q_R \exp(-\tau_s p_R) - p_R s_R \exp(-\tau_s s_R) \right| \]

\[ N_{zz}(\tau, d) = p_R \left[ -q_R \exp(-\tau_s p_R) + \exp(-\tau_s s_R) \right] \]

\[ D(\gamma) = \gamma \left[ \frac{1 + a^2 - 2a^2\gamma^2}{\sqrt{(1 - \gamma^2)(1 - a^2\gamma^2)}} - (2 - \gamma^2) \right] \]

\[ g_{Hxx}(k, 0, d, t) = g_{Hxx}(k, d, 0, t) \quad (6a) \]

\[ g_{Hxz}(k, 0, d, t) = -g_{Hxz}(k, d, 0, t) \quad (6b) \]

\[ g_{Hzx}(k, 0, d, t) = -g_{Hzx}(k, d, 0, t) \quad (6c) \]

\[ g_{Hzz}(k, 0, d, t) = g_{Hzz}(k, d, 0, t) \quad (6d) \]

Observe that the coupling terms exhibit not only a reversal in the signs, but also an exchange of the subindices.

At first glance, it would seem that the evaluation of the impulse-response functions in equation (4) entails a significant computational cost because of the necessity to determine numerically the branch integrals. However, this is not so. First, for any given Poisson’s ratio, the branch integrals need be computed only once and then solely for a unit wavenumber \( k = 1 \) and unit shear-wave velocity \( \beta_H = 1 \), and in most cases, only for \( d = 0 \) (Table 3). Second, these integrals can be precomputed efficiently once-and-for-all for a sufficiently dense set of dimensionless times, and then placed into an appropriately indexed lookup table for later use. These functions exhibit decaying oscillations with a dominant period of about \( \tau = \frac{\pi}{2} \), so the required timestep for the table must be appropriately smaller, say \( \Delta \tau < \pi/18 \). A fast and accurate method for integrals of this type was presented by Xu and Mal (1987), which is based on a Clenshaw-Curtis approach with Chebyshev polynomials. Also, the tails of the improper integrals can be expressed analytically in terms of simple asymptotic expressions that lead to the sine integral function, for which efficient expressions are readily available (see Park and Kausel, 2004a). Once tabulated, function values for actual dimensionless times are obtained by fast access to the lookup table followed by, say, cubic spline or four-point Lagrange interpolation. Thus, once the table for any given Poisson’s ratio has been assembled and stored in the computer, the computation of the branch integrals is virtually instantaneous, whatever the actual problem being considered.

Response of Layered Half-Space in \( k-t \)

Consider next a layered system consisting of \( L \) material layers underlain by an elastic half-space \( H \). Without loss of generality, we place the origin of coordinates at the interface of the layers and the half-space so that the layers have a positive vertical coordinate. To obtain the response in \( k-t \), we resort once more to the principle of superposition and proceed to separate for this purpose the layered system from the half-space, thereby creating two distinct free bodies, one of which contains the source (figure 2 in Part I [Park and Kausel, 2006]). We preserve dynamic equilibrium by applying appropriate, at first unknown horizontal and vertical trac-
Table 2
Branch Integrals

\[ \Omega = \frac{\omega}{k\beta_H}, \quad q = 1 - \frac{1}{2} \Omega^2 \quad a = \frac{\beta_H}{\alpha_H} = \sqrt{\frac{1 - 2\nu}{2 - 2\nu}} \]

\[ S = \sqrt{1 - \Omega^2} = i\tilde{s}, \quad p = \sqrt{1 - a^2\Omega^2} = i\tilde{p} \]

\[ \tilde{s} = \sqrt{\Omega^2 - 1}, \quad \tilde{p} = \sqrt{a^2\Omega^2 - 1} \]

\[ \Delta = sp - q^2 = \text{Rayleigh function} \]
\[ \Delta_1 = -q^2 + isp = \text{Rayleigh function for } 1 \leq \Omega \leq 1/a \]
\[ \Delta_2 = -q^2 - \tilde{s}\tilde{p} = \text{Rayleigh function for } 1/a \leq \Omega \]

\[ B_{xx} = \frac{1}{\pi} \int_{1/a}^{1} \frac{\tilde{s}}{1\Delta_1} \text{ Re } \left[ \Delta_1 (q \exp(ik\tilde{s}d) - \exp(-kp\tilde{p}d)) \sin \Omega \tau d\Omega \right] \]

\[ \quad - \frac{1}{\pi} \int_{1/a}^{\infty} \frac{\tilde{s}}{\Delta_2} [\cos kp\tilde{p}d - q \cos k\tilde{s}d] \sin \Omega \tau d\Omega \]

\[ B_{xz} = i \frac{1}{\pi} \int_{1/a}^{1} \frac{p\tilde{s}}{\Delta_1} \left[ q \exp(-kp\tilde{p}d) + \text{ Re } (\Delta_1 \exp(i k\tilde{s}d)) \right] \sin \Omega \tau d\Omega \]

\[ \quad + i \frac{1}{\pi} \int_{1/a}^{\infty} \frac{1}{\Delta_2} \left[ q \sin kp\tilde{p}d + \tilde{s}\tilde{p} \sin k\tilde{s}d \right] \sin \Omega \tau d\Omega \]

\[ B_{zx} = -i \frac{1}{\pi} \int_{1/a}^{1} \frac{q}{\Delta_1} \left[ \text{ Im } (\Delta_1 \exp(ik\tilde{s}d)) - \tilde{s}\tilde{p}q \exp(-kp\tilde{p}d) \right] \sin \Omega \tau d\Omega \]

\[ \quad - i \frac{1}{\pi} \int_{1/a}^{\infty} \frac{1}{\Delta_2} \left[ \tilde{s}\tilde{p} \sin kp\tilde{p}d + q \sin k\tilde{s}d \right] \sin \Omega \tau d\Omega \]

\[ B_{zz} = \frac{1}{\pi} \int_{1/a}^{1} \frac{p}{\Delta_1} \text{ Im } [\Delta_1 (\exp(ik\tilde{s}d) - q \exp(-kp\tilde{p}d))] \sin \Omega \tau d\Omega \]

\[ \quad - \frac{1}{\pi} \int_{1/a}^{\infty} \frac{\tilde{p}}{\Delta_2} [\cos k\tilde{s}d - q \cos kp\tilde{p}d] \sin \Omega \tau d\Omega \]
Green’s Functions at Surface of Half-Space (\(d = 0\)) (Definitions as in Tables 1 and 2)

| \(N_{xx}(\gamma, 0)\) | \(\frac{1}{2}\gamma^2 s_R\) |
| \(N_{xx}(\gamma, 0)\) | \(i(p_{yR} s_R - q_{yR})\) |
| \(N_{xc}(\gamma, 0)\) | \(-i(p_{yR} s_R - q_{yR})\) |
| \(N_{zz}(\gamma, 0)\) | \(\frac{1}{2}\gamma^2 p_R\) |

Tight approximations to Rayleigh terms as functions of Poisson’s ratio:

\[
A_{xx}(\gamma, 0) = 0.308 - 0.718v + 0.499v^2,
\]

\[
A_{xz}(\gamma, 0) = i(0.392 - 0.738v + 0.357v^2),
\]

\[
A_{zz}(\gamma, 0) = 0.499 - 0.706v + 0.246v^2
\]

Branch integrals:

\[
B_{xx} = \frac{1}{2\pi} \int_{\Delta_1}^{1/\alpha} \frac{s_q^2 \Omega^2}{|\Delta|} \sin \Omega \, d\Omega - \frac{1}{2\pi} \int_{1/\alpha}^{\infty} \frac{s_\Omega^2}{\Delta_2} \sin \Omega \, d\Omega
\]

\[
B_{xc} = \frac{i}{2\pi} \int_{\Delta_1}^{1/\alpha} \frac{\rho s_q}{|\Delta|} \Omega^2 \sin \Omega \, d\Omega,
\]

\[
B_{cz} = \frac{1}{\pi} \int_{\Delta_1}^{1/\alpha} \frac{s_p^2 \Omega^2}{|\Delta|} \sin \Omega \, d\Omega - \frac{1}{2\pi} \int_{1/\alpha}^{\infty} \frac{s_\Omega^2}{\Delta_2} \sin \Omega \, d\Omega
\]

In analogy to the half-space case, we write the Green’s functions in matrix form as

\[
\mathbf{g}_L(k, z, z', t) = \left[\begin{array}{c}
\mathbf{g}_{Lxz} \\
\mathbf{g}_{Lzx}
\end{array}\right] = \left[\begin{array}{c}
\mathbf{g}_{Lxx} - i \mathbf{g}_{Lxz} \\
i \mathbf{g}_{Lxx} + \mathbf{g}_{Lzz}
\end{array}\right]
\]

\[
= \mathbf{Q}^{-1} \left[\begin{array}{c}
\mathbf{g}_{Lxz} \\
\mathbf{g}_{Lzx}
\end{array}\right] \mathbf{Q} = \mathbf{Q}^{-1} \mathbf{\tilde{g}}_L \mathbf{Q}
\] (7)

in which \(\mathbf{Q} = \text{diag} \{1 - i\}\).

In a later section, we provide a description of how to effectively obtain \(\mathbf{g}_L\) (or \(\mathbf{\tilde{g}}_L\)), but for now, we shall assume it to be known. Hence, by superposition the response anywhere in the layered system is given by the time convolution

\[
\mathbf{u}(k, z, t) = \mathbf{g}_L(k, z, z', t) * \mathbf{\tilde{p}}(k, t)
\]

\[
- \mathbf{g}_L(k, z, 0, t) * \mathbf{s}(k, t)
\] (8)

in which

\[
\mathbf{s}(k, t) = \left[\begin{array}{c}
r_{xx} \\
\sigma_{zz}
\end{array}\right]
\] (9)

are the unknown internal stresses at the interface of the layers with the half-space. In particular, at the bottom interface

\[
\mathbf{u}(k, 0, t) = \mathbf{g}_L(k, 0, z', t) * \mathbf{\tilde{p}}(k, t)
\]

\[
- \mathbf{g}_L(k, 0, 0, t) * \mathbf{s}(k, t).
\] (10)

On the other hand, from the previous section, the displacement caused by the internal stresses \(\mathbf{s}(k,t)\) at the surface of the half-space is

\[
\mathbf{u}(k, 0, t) = \mathbf{g}_H(k, 0, 0, t) * \mathbf{s}(k, t).
\] (11)

Compatibility requires the two displacement vectors in equations (10) and (11) to be equal at all times, which leads to...
As in Part I (Park and Kausel, 2006), we proceed at this point to discretize the convolutions in time by means of appropriate timesteps \( \Delta t \) such that \( t = j \Delta t, j = 0, 1, 2, \ldots \). Furthermore, we denote discrete values of the various functions in the convolution as

\[
G_j = g_{tt}(k, 0, z', j \Delta t) \quad (13a)
\]
\[
H_j = g_{lr}(k, 0, z', j \Delta t) \quad (13b)
\]
\[
F_j = g_{tt}(k, 0, 0, j \Delta t) + g_{lr}(k, 0, 0, j \Delta t) \quad (13c)
\]
\[
p_j = \tilde{p}(k, j \Delta t) \quad (13d)
\]
\[
s_j = s(k, j \Delta t) \quad (13e)
\]

The discrete version of equation (12) is then

\[
\sum_{j=0}^{i} F_{i-j}s_j = \sum_{j=0}^{i} G_{i-j}p_j. \quad (14)
\]

Evaluating equation (14) from time \( t = 0 \) on, we obtain the sequential system

\[
F_0s_0 = G_0p_0
\]
\[
F_1s_0 + F_0s_1 = G_1p_0 + G_0p_1
\]
\[
F_2s_0 + F_1s_1 + F_0s_2 = G_2p_0 + G_1p_1 + G_0p_2
\]
\[
\vdots
\]
\[
F_ns_0 + F_{n-1}s_1 + \cdots + F_1s_{n-1} + F_0s_n = G_np_0 + G_{n-1}p_1 + \cdots + G_1p_{n-1} + G_0p_n \quad (15)
\]

which leads immediately to the recursive system of equations for the unknown stresses

\[
s_0 = F_0^{-1}[G_0p_0]
\]
\[
s_1 = F_0^{-1}[G_1p_0 + G_0p_1 - F_1s_0]
\]
\[
s_2 = F_0^{-1}[G_2p_0 + G_1p_1 + G_0p_2 - (F_2s_0 + F_1s_1)]
\]
\[
\vdots
\]
\[
s_n = F_0^{-1}[G_np_0 + G_{n-1}p_1 + \cdots + G_1p_{n-1} + G_0p_n - (F_ns_0 + F_{n-1}s_1 + \cdots + F_1s_{n-1})]. \quad (16)
\]

As written, the solution to equation (16) involves complex algebra, but with a simple similarity transformation, all computations can be made real. This is accomplished by observing that the matrices \( F, G, H \) are all of the form (see equation 7)

\[
F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} \tilde{f}_{11} & -i \tilde{f}_{12} \\ i \tilde{f}_{21} & \tilde{f}_{22} \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \tilde{f}_{11} & \tilde{f}_{12} \\ \tilde{f}_{21} & \tilde{f}_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = Q^{-1} \tilde{F}Q. \quad (17)
\]

and similar expressions for \( G, H \). Hence, defining the real vectors \( \tilde{s}_n = Qs_n, \tilde{p}_n = Qp_n \), the complex-valued iteration of equation (16) can be changed into the purely real iteration

\[
\tilde{s}_n = \tilde{F}_0^{-1} [G_n \tilde{p}_0 + G_{n-1} \tilde{p}_1 + \cdots + G_1 \tilde{p}_{n-1} + G_0 \tilde{p}_n - (\tilde{F}_n \tilde{s}_0 + \tilde{F}_{n-1} \tilde{s}_1 + \cdots + \tilde{F}_1 \tilde{s}_{n-1})]. \quad (18)
\]

Having obtained the time history of the internal stresses at the interface of the layers and the half-space, we can proceed to compute the \( k-t \) response anywhere by means of equation (8), and thereafter the response in space-time by an appropriate inverse integral transform, as described in Appendix I.

Green’s Functions in \( k-t \) for Sources in the Layers

Proceeding as in Part I (Park and Kausel, 2006), we express the real form of the Green’s functions (see equation 7) for the upper layered system in terms of the normal modes (see also Appendix II):

\[
\tilde{g}_k(k, z', \tau) = \sum_{j=1}^{n} e^{-\zeta_j \omega_j \tau} \sin \omega_j \tau \frac{\phi_j(z, k)}{\phi_j^T(z', k)} \quad (19)
\]

\[
\phi_j(z, k) = \begin{bmatrix} \phi_j^z(z, k) \\ \phi_j^p(z, k) \end{bmatrix}. \quad (20)
\]

\[
\begin{array}{c}
G_0p_0, \quad i = 0 \\
G_1p_0 + G_0p_1, \quad i = 1 \\
G_2p_0 + G_1p_1 + G_0p_2, \quad i = 2 \\
\vdots
\end{array}
\]

where \( \omega_j, \phi_j(z, k) \) are the modal vectors for free waves with prescribed horizontal wavenumber \( k \) that are able to propagate in the layered plate with characteristic modal frequency \( \omega_j(k) \), and \( \zeta_j = 1/2Q_j \) is the fraction of modal damping, with \( Q_j = Q_j(k) \) being the modal quality factor. The latter may optionally depend on \( k \) and is typically assumed to be the same in all modes. Also,

\[
\omega_{dj} = \omega_j \sqrt{1 - \zeta_j^2} \quad (21)
\]

is the damped modal frequency. As in Part I (Park and Kau-
of the layers, which coincides with the top of the half-space. Both the layers and the half-space are assumed to be fully elastic (i.e., no attenuation!). This system, which is characterized by the material properties indicated in Table 4, is subjected to a plane, bell-shaped vertical source at an elevation \( z' = 5 \) km in the lower layer (i.e., a depth of 15 km below the surface). This source has the form of tractions in a horizontal plane at elevation \( z' \) (see Fig. 2a) of the form

\[
\mathbf{p} = \begin{pmatrix} p_0 \\ p_z \end{pmatrix} = \begin{pmatrix} \frac{1}{a} \cos^2 \frac{\pi x}{2a} \frac{t}{t_d} \\ \frac{1}{a} \sin^2 \frac{\pi t}{t_d} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (22a)
\]

in which \( t_d = 2 \) sec is the duration of source, and \( a = 1 \) km is the half-width of source, which is small enough for the source to be seen as an approximation to an ideal line load. The response is desired for a total time of \( t_{\text{max}} = 30 \) sec up to a maximum range \( x_{\text{max}} = 40 \) km.

Wavenumber Content of Source

Considering only the term that depends on \( x \), the spatial Fourier transform of the source is found to be

\[
\frac{1}{a} \int_{-a}^{a} \cos^2 \frac{\pi x}{2a} e^{ikx} dx = \frac{\pi^2}{\pi^2 - (ka)^2} \frac{\sin ka}{ka} \quad (23)
\]

The amplitude of the source drops off with wavenumber and can be neglected for, say, \( ka > 4\pi \), which means that the maximum wavenumber needed in the inverse Fourier transformation over wavenumbers can be taken as \( k_{\text{max}} = 4\pi/a = 4\pi \). Also, since the maximum time of interest is \( t_{\text{max}} =

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**Figure 1.** Two-layered system underlain by a homogeneous half-space.
Table 4
Material Properties

<table>
<thead>
<tr>
<th>Layer 1</th>
<th>Layer 2</th>
<th>Half-space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass density</td>
<td>$\rho_1 = 1.5 \text{ ton/m}^3$, $\rho_2 = 1.75 \text{ ton/m}^3$, $\rho_H = 2 \text{ ton/m}^3$</td>
<td></td>
</tr>
<tr>
<td>$S$-wave velocity</td>
<td>$\beta_1 = 1 \text{ km/sec}$, $\beta_2 = 1.5 \text{ km/sec}$, $\beta_H = 2 \text{ km/sec}$</td>
<td></td>
</tr>
<tr>
<td>$P$-wave velocity</td>
<td>$\alpha_1 = 2.08 \text{ km/sec}$, $\alpha_2 = 2.81 \text{ km/sec}$, $\alpha_H = 3.46 \text{ km/sec}$</td>
<td></td>
</tr>
<tr>
<td>Thickness</td>
<td>$H_1 = 10 \text{ km}$, $H_2 = 10 \text{ km}$</td>
<td></td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>0.35, 0.30, 0.25</td>
<td></td>
</tr>
</tbody>
</table>

30 sec, we can allow image sources at a distance of $\alpha_{\text{max}} t_{\text{max}} + a = 3.46 \times 30 + 1 \approx 105 \text{ km}$ from the most distant receiver at $x_{\text{max}} = 40 \text{ km}$, in which case the actual source could be replaced by a periodic array of sources placed at $105 + 40 = 145 \text{ km}$ from one another. This in turn means that the Fourier integral over wavenumbers can be replaced by a Fourier series with a wavenumber step.

$$\Delta k \leq \frac{2\pi}{x_{\text{max}} + \alpha_{\text{max}} t_{\text{max}} + a} \approx \frac{\pi}{73}$$ (24)

Thus, the discrete inverse Fourier transform over wavenumbers will have at most $N = 4\pi/\pi/73 = 292$ points (i.e., 512 if using an fast Fourier transform).

Frequency Content of Source

The Fourier transform of the temporal term in the source is

$$\frac{2}{t_d} \int_0^{t_d} \sin \frac{\pi}{t_d} \frac{t}{x} e^{-i\omega t} \, dt = \frac{\pi}{\omega_{\text{max}}^2} e^{-i\omega t_d} \sin \frac{\omega t_d}{2}$$ (25)

The source strength decreases with the cube of the frequency, and can be neglected above, say, $\omega_{\text{max}} t_d = 4\pi$. Thus, the modal summation will contain enough modes if the highest modal frequency included in the computation is not less than this threshold, that is,

$$\left(\omega_j\right)_{\text{max}} \approx \omega_{\text{max}} = \frac{4\pi}{t_d} = \frac{4\pi}{2} = 2\pi$$. (26)

Modal Solution

To find the modes of the two-layer upper part, we resort to a TLM model (Appendix II) with a sufficient number of thin layers; to decide on this number, we must establish lower bounds to the modal frequencies for zero horizontal wavenumber. Now, we know for a fact that these frequencies must be higher than the frequencies of a uniform plate with the same total thickness $H$ but possessing the lower of the two shear-wave velocities $\beta = \beta_1$; these frequencies are $\omega_j = j\pi\beta_1 H = j\pi/20$ for the shear modes, and $\omega_j = j\pi\alpha_1 H$ for the dilatational modes. Setting $j\pi/20 > 2\pi$, we obtain $j > 40$, that is, we need at least 40 accurate modes, which requires no less than 40 thin layers when using a TLM model with a linear expansion. To be on the safe side, we choose a TLM model with 80 thin layers of equal thickness (i.e., 40 in each physical layer). Using this model with $k = 0$ and $k = 4\pi$ (the maximum wavenumber), we obtain in each case 162 modes whose first few frequencies (in rad/sec) are shown in the following table.
We see that $\omega_{\text{max}} = 2\pi = 6.283$ is exceeded by the 52nd mode when $k = 0$, so that will be the maximum number of modes used in the model.

**Timestep**

The period that corresponds to the last mode used for $k = k_{\text{max}}$ is $T_{\text{min}} = \frac{2\pi}{\omega_{2}} = \frac{2\pi}{15.95} = 0.393$ sec, so $\Delta t < T_{\text{min}}/4 = 0.098$ sec. Also, the impulse-response functions of the half-space have oscillations with dominant dimensionless period on the order of $\tau = k_{\text{max}} \beta_{m} T = (4\pi) (2) T \approx \frac{2}{3} \pi$, that is, $T \approx 0.083$. Hence, $\Delta t = 0.02$ sec is chosen for the calculation.

**Results of Model**

Figure 3 shows a series of six snapshots of the displacement field $u, w$ in the upper two layers, obtained using equation (A5) in Appendix I, and calculated at times $t = 5.0, 10.0, 15.0, 20.0, 25.0, \text{ and } 30.0$ sec. The intensity of grey is chosen in proportion to the absolute amplitude $(u^2 + w^2)^{1/2}$ such that a light gray is for zero amplitude and a dark grey is for whatever the maximum amplitude in that snapshot is. Although this improves the visualization, it also exaggerates somewhat the contrast for snapshots at later times, inasmuch as displacements evanesce with time. The location of the wavefront in each snapshot is consistent with the position of the source and the magnitude of the wave velocities in the

![Figure 3](image-url)
two-layered system and half-space considered for this example. One can clearly observe the reflections, refractions, and the Rayleigh waves at the surface.

Example 2: Vertical (Three-Dimensional) Point Source

We consider again the problem of example 1, but this time we subject the system to bell-shaped, cylindrical tractions simulate a vertical point load.

\[ \mathbf{p} = \begin{pmatrix} p_r \\ p_\theta \\ p_z \end{pmatrix} = \begin{pmatrix} \frac{2}{t_d} \sin^2 \frac{\pi t}{t_d} \left( \frac{1}{\pi a^2} e^{-\left(\frac{\pi a^2}{2}\right)^2} \right) \\ 0 \\ 1 \end{pmatrix} \] (27a)

\[ 0 \leq r \leq \infty, \quad 0 \leq t \leq t_d \] (27b)

which has the same time variation and elevation as the example of a plane-strain load and is constant with the azimuth (i.e., \( n = 0 \)). For a sufficiently small radius \( a \), the bell-shaped tractions simulate a vertical point load. The zeroth-order Hankel transform of the spatial part of the load (see also Appendix I) is

\[ \frac{1}{\pi a^2} \int_0^\infty e^{-\left(\frac{\pi a^2}{2}\right)^2} J_0(kr) r \, dr = \frac{1}{2\pi} e^{-\left(\frac{ka}{2}\right)^2} \] (28)

so

\[ \tilde{\mathbf{p}}_0 = \begin{pmatrix} \tilde{p}_r \\ \tilde{p}_\theta \\ \tilde{p}_z \end{pmatrix}_0 = \frac{1}{2\pi} e^{-\left(\frac{ka}{2}\right)^2} \begin{pmatrix} 2 \frac{\sin^2 \frac{\pi t}{t_d}}{t_d} \left( \frac{1}{\pi a^2} \right) \\ 0 \\ 1 \end{pmatrix} \] (29)

which produces displacements (Appendix I, equation A17)

\[ \mathbf{u}(r, \theta,t) = \frac{1}{2\pi} \begin{pmatrix} 2 \frac{\sin^2 \frac{\pi t}{t_d}}{t_d} \end{pmatrix} \ast \int_0^\infty \begin{pmatrix} -J_1 \tilde{g}_{zz} \\ 0 \\ J_0 \tilde{g}_{zz} \end{pmatrix} e^{-\left(\frac{ka}{2}\right)^2} \, dk, \] (30)

in which \( \tilde{g}_{zz}, \tilde{g}_{zz} \) are the impulse-response functions at elevation \( z \) obtained as described in this article, which are identical with the real form of the impulse-response functions used earlier in the plane-strain problem. Also, the asterisk between the temporal term and the integral denotes a convolution, because the load is not impulsive.

Observe that the wavenumber spectrum of the load drops exponentially with the wavenumber squared, and it is negligible when the wavenumber exceeds, say, \( ka = 2\pi \), because \( \exp(-\pi^2) = 5.1 \times 10^{-5} \). Hence, for \( a = 1 \) km, the maximum wavenumber is \( k_{\text{max}} = 2\pi \), which is half as large as that used for the plane-strain load. The temporal variation is identical with that case, so the same considerations concerning frequency contents, maximum frequency, and timestep apply. Hence, the same number of modes can be used, and the same wavenumber step \( \Delta k \) applies, but the number of wavenumbers can be halved on account of the smaller \( k_{\text{max}} \).

Figure 4 shows a set of six snapshots of the displacement field \( u, w \), obtained via equation (30), and calculated at times \( t = 5.0, 10.0, 15.0, 20.0, 25.0, 30.0 \) sec. Inspection reveals that the snapshots for a point load in Figure 4 are similar, but not identical with those for a line load in Figure 3. Indeed, the wavefronts are sharper, which is consistent with the interpretation of a line source as the aggregate of infinitely many point sources placed along a straight line. Thus, the attenuated and retarded responses elicited by the more distant point sources broaden the wavefronts of the line load case.

Conclusions

An effective computational method for the simulation of synthetic seismograms due to line sources and point sources acting within layered half-spaces was presented in the context of a formulation in the wavenumber-time domain. The proposed algorithm has the following characteristics:

- The response is found directly in the time domain by Fourier synthesis over horizontal wavenumbers. Thus, typical problems encountered in other methods, such as resonances in the layers, are avoided, especially in systems with little or no attenuation.

- In principle, the method allows for sources with arbitrary spatial and temporal variation. Nonetheless, the numerical integrals over wavenumbers are most effective when the sources are band limited in the wavenumber domain. In the examples presented, this was the motivation for the simulation of line and point sources via distributed (bell-shaped) sources that are spatially smooth and narrow.

- The method relies on a combination of the impulse-response function for the layers with those of the half-space. An effective strategy for obtaining the former is by means of the TLM, whereas the latter emanate from branch integrals that are precomputed accurately once-and-for-all and then placed in a lookup table for further use with any arbitrary layered system.

- There is virtually no difference in the computational steps between two-dimensional line sources and three-dimensional point sources. Indeed, these two cases differ mainly in the inverse transforms used to revert from the wavenumber domain to the space domain: a Fourier transform for the former, and a Hankel (or Fourier-Bessel) transform for the latter. In most cases, these transformations can be accomplished “on the fly,” that is, accumulated within the loop over wavenumbers, a strategy that sidesteps the need to store the Green’s functions and modes for posterior transformations.
Figure 4. Normalized deformations due to vertical point load $p_z$. Snapshots at $t = 5.0, 10.0, 15.0, 20.0, 25.0, \text{ and } 30.0 \text{ sec.}$

References


Appendix I

Response to Line Sources versus Point Sources

In this article we consider both plane-strain sources and three-dimensional sources acting within horizontally layered media. As it turns out, these two problems are intimately related. When the equations of motion for impulsive, three-dimensional point sources acting within horizontally layered
Two-Dimensional Case

Consider a horizontally layered medium subjected to a plane source (or body load) of the form
\[
\mathbf{b}(x,z,z',t) = \mathbf{p}(x,t) \delta(z-z'),
\]
(A1)
in which \(z'\) is the elevation where the external tractions \(\mathbf{p}(x,t)\) are being applied. A spatial Fourier transform yields
\[
\tilde{\mathbf{b}}(k,z,z',t) = \tilde{\mathbf{p}}(k,t) \delta(z-z')
\]
\[
= \delta(z-z') \int_{-\infty}^{+\infty} \mathbf{p}(x,t) e^{ikx} dx.
\]
(A2)

In particular, a set of impulsive line loads in each of the three coordinate directions \(j = x, y, z\) is of the form \(\mathbf{b} = \delta(x) \delta(z-z') \delta(t) \mathbf{e}\), in which \(\mathbf{e}\) is the \(3 \times 3\) identity matrix that results from collecting the three unit load vectors in the three coordinate directions. This produces a displacement field or Green’s functions matrix that has the general structure
\[
\mathbf{g}(k,z,z',t) = \begin{bmatrix}
g_{sx} & 0 & g_{sz} \\
0 & g_{yy} & 0 \\
g_{zx} & 0 & g_{zz}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\tilde{g}_{sx} & 0 & -i \tilde{g}_{sz} \\
0 & \tilde{g}_{yy} & 0 \\
i \tilde{g}_{zx} & 0 & \tilde{g}_{zz}
\end{bmatrix}
\]
(A3)

It will also be found that the impulse-response functions \(g_{sx}\), \(g_{yy}\), \(g_{zz}\) are purely real and symmetric with respect to positive/negative values of the horizontal wavenumber \(k\), whereas \(\tilde{g}_{sx}, \tilde{g}_{zx}\) are purely imaginary and antisymmetric with respect to this wavenumber. This ensures that the inverse Fourier transforms into space are purely real, that is,
\[
\mathbf{u}(xp,z,z',t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\mathbf{u}}(k,z,z',t) e^{-ikx} dk
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{g}(k,z,z',t) * \mathbf{p}(k,t) e^{-ikx} dk.
\]
(A4)

In the last expression the convolution over time is needed because \(\mathbf{p}(x,t)\) is not necessarily impulsive. In particular, for a vertical impulsive line load (i.e., third columns of the identity matrix \(\mathbf{e}\) and of the Green’s functions matrix \(\mathbf{g}\), the formal inversion into space is
\[
\begin{bmatrix}
u_{xz} \\
u_{zz}
\end{bmatrix} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \begin{bmatrix}
g_{xz} \\
g_{zz}
\end{bmatrix} e^{-ikx} dk
\]
\[
= \frac{1}{\pi} \int_{0}^{\infty} \begin{bmatrix}
-g_{xz} \sin kx \\
g_{zz} \cos kx
\end{bmatrix} dk
\]
(A5)

with, of course, \(u_{zz} = 0\).

Three-Dimensional Case

We begin with the definitions
\[
c_n = 1/\pi \quad \text{if } n \neq 0
\]
\[
c_0 = 1/2\pi \quad \text{for } n = 0
\]
(A6a)

\[
\mathbf{T}_n = \text{diag} \left( \begin{matrix} \cos n\theta \\
\sin n\theta \\
-\sin n\theta \\
\cos n\theta \\
\sin n\theta \end{matrix} \right)
\]
(A6b)

\[
\mathbf{J}_n(kr) = \begin{bmatrix} J_n^x & 0 & 0 \\
0 & J_n^y & J_n^z \\
0 & 0 & J_n^z
\end{bmatrix},
\]
\[
J_n^z' = \frac{dJ_n(z)}{dz}, \quad z = kr. \quad (A6c)
\]

\(\mathbf{T}_n\) is a diagonal azimuthal matrix, in which either the upper or lower elements in parentheses are used, as may be appropriate. Also, \(\mathbf{J}_n\) is assembled with Bessel functions \(J_n(z)\) of order \(n\).

Consider now a horizontally layered system subjected to body loads \(\mathbf{b}\) that we choose to express in terms of external tractions in horizontal planes \(\mathbf{p}\) of the form
\[
\mathbf{b}(r,\theta,z,z',t) = \delta(z-z') \mathbf{p}(r,\theta,t),
\]
(A7)
in which \(r, \theta, z\) are cylindrical coordinates. Carrying out a Hankel transform over the radial coordinate, and expressing the load in a Fourier series in the azimuth, we obtain
\[
\tilde{\mathbf{p}}_n(k,t) = c_n \int_{0}^{2\pi} r J_n^x(\rho) T_n \mathbf{p}(r,\theta,t) d\theta dr
\]
(A8a)

whose formal inverse is
\[ p(r, \theta, t) = \sum_{n=0}^{\infty} T_n \int_{0}^{\infty} k J_n \tilde{p}_n(k, t) dk. \quad (A8b) \]

That equations (A8a) and (A8b) are indeed a pair of Fourier-Bessel transforms can be demonstrated as follows. Consider three sufficiently well-behaved functions \( f_j(r) \), \( j = 1, 2, 3 \), for which the transformation from radial distance to radial wavenumber \( F_j(k) \) is

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix} = \int_{0}^{\infty} r \begin{bmatrix}
J'_n \frac{n}{kr} J_n & 0 & f_1 \\
0 & 0 & 0 \\
0 & J_n & f_3
\end{bmatrix} dr. \quad (A9)
\]

which can be written in the fully diagonal form

\[
\begin{bmatrix}
\frac{1}{2}(F_1 + F_2) \\
\frac{1}{2}(F_2 - F_1) \\
F_3
\end{bmatrix} = \\
\begin{bmatrix}
J_{n-1} & 0 & 0 \\
0 & J_{n+1} & 0 \\
0 & 0 & J_n
\end{bmatrix} \begin{bmatrix}
\frac{1}{2}(f_1 + f_2) \\
\frac{1}{2}(f_2 - f_1) \\
f_3
\end{bmatrix} \int_{0}^{\infty} r \begin{bmatrix}
J'_n \frac{n}{kr} J_n & 0 & f_1 \\
0 & 0 & 0 \\
0 & J_n & f_3
\end{bmatrix} dr. \quad (A10)
\]

This is a set of three self-reciprocating Hankel transforms, that is,

\[
\begin{bmatrix}
\frac{1}{2}(f_1 + f_2) \\
\frac{1}{2}(f_2 - f_1)
\end{bmatrix} = \\
\begin{bmatrix}
J_{n-1} & 0 & 0 \\
0 & J_{n+1} & 0 \\
0 & 0 & J_n
\end{bmatrix} \begin{bmatrix}
\frac{1}{2}(f_1 + f_2) \\
\frac{1}{2}(f_2 - f_1) \\
f_3
\end{bmatrix} \int_{0}^{\infty} k \begin{bmatrix}
J'_n \frac{n}{kr} J_n & 0 & f_1 \\
0 & 0 & 0 \\
0 & J_n & f_3
\end{bmatrix} dk. \quad (A11)
\]

which after rearrangement attains the desired form

\[
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix} = \int_{0}^{\infty} k \begin{bmatrix}
J'_n \frac{n}{kr} J_n & 0 & f_1 \\
0 & 0 & 0 \\
0 & J_n & f_3
\end{bmatrix} dk. \quad (A12)
\]

To complete the proof, it suffices to consider the orthogonality of the trigonometric terms in \( T_n \), which are

\[
\begin{align*}
\int_{0}^{2\pi} \cos \theta \cos \theta d\theta &= \pi (1 + \delta_{0n} \delta_{0n}) \delta_{mn} \quad (A13a) \\
\int_{0}^{2\pi} \sin \theta \sin \theta d\theta &= \pi (1 - \delta_{0n} \delta_{0n}) \delta_{mn} \quad (A13b)
\end{align*}
\]

in which the \( \delta_{mn} \) are Kronecker delta functions. The result on the right-hand side is the reason for the scaling factor \( c_n \) used in the forward transform.

A pair of transforms similar to equations (A8a) and (A8b) also hold for the displacement vectors:

\[ \mathbf{u}_n(k, z, z', t) = c_n \int_{0}^{\infty} r J_n \mathbf{u}(r, \theta, z, z', t) d\theta dr \quad (A14a) \]

\[ \mathbf{u}(r, \theta, z, z', t) = \sum_{n=0}^{\infty} T_n k J_n \mathbf{u}_n(k, z, z', t) dk \quad (A14b) \]

\[ \mathbf{u}_n(k, z, z', t) = \tilde{g} \mathbf{p}_n \quad (A15) \]

in which \( \tilde{g} \) is the real form of the Green’s functions matrix obtained from equation (A3), which does not depend on the azimuthal index \( n \). It can also be shown that these equations do indeed satisfy the vector wave equation in cylindrical coordinates.

In the ensuing we consider impulsive horizontal and vertical tractions in horizontal planes of the form \( q_j(r, \theta, t) = q(r, \theta) \delta(t) \), with \( j = r, \theta, z \), in which case we obtain the following results:

**Vertical Load.**

\[ p(r, \theta, t) = \mathbf{T}_0 \begin{bmatrix} 0 \\ 0 \end{bmatrix} q(r) \delta(t) \quad (A16a) \]

\[ \tilde{p}_0(k, t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tilde{q}(k) \delta(t) \quad (A16b) \]

in which \( \tilde{q}(k) = \int_{0}^{\infty} r J_0(kr)q(r) dr \), and \( \tilde{p}_n = 0 \) for \( n \neq 0 \).

A vertical point load corresponds to \( q(r) = \delta(r)2\pi r \) \( \tilde{q}(k) = 1/2\pi \). Omitting the dependence on the vertical coordinate, the response in space-time in this case is obtained from the inverse Hankel transform

\[ \mathbf{u}(r, \theta, t) = \mathbf{T}_0 \int_{0}^{\infty} k \mathbf{J}_0 \mathbf{u}_0 dk 
\]

\[ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \int_{0}^{\infty} k \begin{bmatrix} J_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J_0 \end{bmatrix} \begin{bmatrix} \tilde{g}_{xx} & 0 & \tilde{g}_{zc} \\ 0 & \tilde{g}_{yy} & 0 \\ \tilde{g}_{xz} & 0 & \tilde{g}_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tilde{q}(k) dk 
\]

\[ = \int_{0}^{\infty} k \begin{bmatrix} -J_1 \tilde{g}_{zc} \\ J_0 \tilde{g}_{zc} \end{bmatrix} \tilde{q}(k) dk \quad (A17) \]
in which the \( g_{xx}, g_{zz} \) functions are identical with those of the plane-strain case, and both are real.

**Horizontal Load in Direction \( x \).**

\[
p(r, \theta, t) = \begin{bmatrix} \cos \theta & 0 & 0 \\ -\sin \theta & 0 & 0 \\ 0 & 0 & \cos \theta \end{bmatrix} \int_0^\infty \int_0^{2\pi} \begin{bmatrix} J_1(kr) + \frac{1}{kr} J_0(kr) \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{g}_{xx} & 0 & \tilde{g}_{xz} \\ 0 & \tilde{g}_{yy} & 0 \\ \tilde{g}_{zx} & 0 & \tilde{g}_{zz} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} q(k) \, dk \]

\[
\hat{\mathbf{p}}_n = \frac{1}{2\pi} \int_0^{2\pi} T_n \left[ \delta(\theta) - \delta(\theta - \pi) \right] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d\theta 
\]

in which now \( q(k) = \int_0^\infty r J_1(kr) q(r) \, dr \), and \( \hat{\mathbf{p}}_n = 0 \) for \( n \neq 1 \). Again, a horizontal point load corresponds to \( q(r) = \delta(r/2\pi) \), \( \tilde{q}(k) = 1/2\pi \). Omitting once more the dependence on the vertical coordinate, the response in space-time follows from the inverse Hankel transform

\[
\mathbf{u}(r, \theta, t) = \mathbf{T}_1 \int_0^\infty k \mathbf{J}_1 \mathbf{u}_1 \, dk
\]

in which the \( \tilde{g}_{xx}, \tilde{g}_{yy}, \tilde{g}_{zz} \) are once more the same as for the two plane-strain cases, and are all real.

**Single Seismic Couple.** To illustrate matters further, we consider next the case of one of the several single seismic point sources, namely an impulsive couple with moment \( M \), formed by a dipole with forces parallel to the \( z \) direction that turns around the \( y \) axis, that is, a dip-slip couple. This dipole can be expressed by the body-force equivalent

\[
b_c(r, \theta, z, t) = \lim_{a \to 0} \frac{M}{2a} \frac{\delta(r - a)}{r} \left[ \delta(\theta) - \delta(\theta - \pi) \delta(z - z') \delta(t) \right] 
\]

in which \( \delta_{1a} \) is the Kronecker delta. Thus, all components except \( n = 1 \) vanish identically. Although exact, this expression is not quite convenient in the context presented here, because its amplitude grows with the wavenumber. Instead, a better choice is to consider a dipole in the form of a bell of finite, but small width. For instance, we could simulate this couple by means of vertical tractions distributed over a small circular area with radius \( a \) of the form

\[
p_z = A J_1(\xi z) \cos \theta \quad 0 \leq r \leq a 
\]

in which \( \xi_1 = 3.83170597 \ldots \) is the first zero of the Bessel function of order 1, and \( A \) is an appropriate amplitude. This produces a moment

\[
\phi = A J_1(\xi_1 z) \cos \theta
\]
\[
M = \int_0^a \int_0^{2\pi} p_x r dr d\theta = A \int_0^a \int_0^{2\pi} J_1(\zeta_1 \frac{r}{a}) r^2 \cos^2 \theta dr d\theta \\
= \frac{\pi a^3 - J_0(\zeta_1)}{\zeta_1} = \frac{a^3 A}{3.0282841...} \Rightarrow A = \frac{M}{a^3} 3.0282841...
\]

(A23)

The Hankel transform of \( p_z \) is then (the dots in the matrix denote irrelevant integrals)

\[
\tilde{p}_1(k, t) = \delta(t) \frac{A}{\pi} \int_0^a r J_1(\zeta_1) \frac{r}{a} \int_0^{2\pi} \cos \theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d\theta dr \\
= \delta(t) A \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \delta(t) \tilde{q}(k) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

(A24)

with

\[
\tilde{q}(k) = A \begin{cases} J_1(ka) J_0(\zeta_1) & \text{if } ka \neq \zeta_1 \\
\frac{1}{2} J_0^2(\zeta_1) & \text{if } ka = \zeta_1 \end{cases}
\]

(A25)

which decays with the square of the wavenumber, \( k^2 \). Thus, for an impulsive seismic moment, the displacements in the space domain are obtained from the inverse transform

\[
u(r, \theta, t) = T_1 \int_0^\infty k J_1(\tilde{u}_1) dk
\]

\[
= A \zeta_1 J_0(\zeta_1) \begin{bmatrix} \cos \theta & 0 & 0 \\ 0 & -\sin \theta & 0 \\ 0 & 0 & \cos \theta \end{bmatrix} \int_0^\infty k \begin{bmatrix} J_1' \frac{1}{kr} J_1 & 0 & \tilde{g}_{xx} \\ 0 & 0 & \tilde{g}_{yy} \\ 0 & 0 & \tilde{g}_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tilde{q}(k) dk
\]

(A26)

Again, the \( \tilde{g}_{xx}, \tilde{g}_{zz} \) are identical with those of the two plane-strain cases, and both are real.

More generally, if the sources in the previous examples are not impulsive but change instead with time, then we must replace in equations (A17), (A19), and (A26) the products of the Green’s functions and \( \tilde{q}(k) \) by convolutions between these functions and \( \tilde{q}(k, t) \).
Appendix II

Normal Modes via Thin-Layer Method

Consider a viscoelastic system of finite thickness subject to SV-P waves (either a homogeneous or laminated, arbitrarily thick plate that is free in space). Subdivide this system into an adequate number of sublayers that are thin in the finite-element sense. The resulting layered system will consist of \( n \) thin layers \( l = 1, 2, \ldots, n \) numbered from the top down, and \( n + 1 \) interfaces. Let \( \lambda_l, \mu_l, \rho_l, h_l \) be the Lamé constant, shear modulus, mass density, and thickness of the \( l \)th sublayer, respectively (which need not change from layer to layer). Each thin layer is then characterized by a rigidity (impedance) matrix in frequency-wavenumber space of the form (see, for example, Kausel, 1994):

\[
K_l = A_l k^2 + B_l k + G_l - \omega^2 M_l \tag{A27}
\]

in which \( k \) is the horizontal wavenumber, \( \omega \) is the frequency, and the matrices \( A_l, B_l, G_l, M_l \) depend on the material properties of the layers only; in addition, their structure and size varies with the interpolation scheme used in the formulation. In a linear expansion, the thin-layer matrices are given by

\[
A_l = \frac{h_l}{12} \begin{bmatrix}
5(\lambda_l + 2\mu_l) & 0 & (\lambda_l + 2\mu_l) & 0 \\
0 & 5\mu_l & 0 & \mu_l \\
(\lambda_l + 2\mu_l) & 0 & 5(\lambda_l + 2\mu_l) & 0 \\
0 & \mu_l & 0 & 5\mu_l
\end{bmatrix}
\]

\[
B_l = \frac{1}{2} \begin{bmatrix}
0 & -(\lambda_l - \mu_l) & 0 & \lambda_l + \mu_l \\
-(\lambda_l - \mu_l) & 0 & -(\lambda_l + \mu_l) & 0 \\
0 & -(\lambda_l + \mu_l) & 0 & \lambda_l - \mu_l \\
\lambda_l + \mu_l & 0 & \lambda_l - \mu_l & 0
\end{bmatrix}
\]

\[
G_l = \frac{1}{h_l} \begin{bmatrix}
\mu_l & 0 & -\mu_l & 0 \\
0 & \lambda_l + 2\mu_l & 0 & -(\lambda_l + 2\mu_l) \\
-\mu_l & 0 & \mu_l & 0 \\
0 & -(\lambda_l + 2\mu_l) & 0 & \lambda_l + 2\mu_l
\end{bmatrix}
\]

\[
M_l = \frac{\rho_l h_l}{12} \begin{bmatrix}
5 & 0 & 1 & 0 \\
0 & 5 & 0 & 1 \\
1 & 0 & 5 & 0 \\
0 & 1 & 0 & 5
\end{bmatrix}
\]

which we have modified as described in Park and Kausel (2004b). Also, in comparison with the references on the TLM, we have changed for reasons of convenience the sign of the \( B \) matrix, to simplify the conversion between Cartesian and cylindrical coordinates. Using the same overlapping operand \( \oplus \) defined in Part I (Park and Kausel, 2006), the system matrices for the full stack of thin layers is found to be characterized by the real and symmetric block-tridiagonal rigidity matrix

\[
K = K_1 \oplus K_2 \ldots K_{n-1} \oplus K_n = A k^2 + B k + G - \omega^2 M \tag{A29}
\]

The material matrices \( A \) and \( M \) are positive definite, \( G \) is positive semidefinite, and \( B \) is indefinite. It can also be shown that \( A k^2 + B k + G \) is positive definite for \( k \neq 0 \), and semidefinite for \( k = 0 \). Now, for any fixed value of the horizontal wavenumber \( k \), the associated frequencies \( \omega \) and normal modes \( \phi \) are obtained by setting the determinant of the system rigidity matrix to zero, that is, they follow from the standard eigenvalue problem

\[
[\det(A k^2 + B k + G)] \phi_j = \omega^2 M \phi_j \tag{A30}
\]

When using linear elements in the TLM formulation, a model with \( n \) layers yields \( 2(n + 1) \) eigenvalues, that is, \( j = 1, 2, \ldots, 2n + 2 \). Because one of the matrices is positive definite and the other is semidefinite, all eigenvalues are real and nonnegative. Also, for \( k = 0 \), there are two rigid-body modes \( \omega_1 = \omega_2 = 0 \) whose modal vectors consist of all equal components, namely a horizontal and a vertical rigid body mode. A readily available and highly efficient subroutine package for the solution of the tridiagonal eigenvalue
After normalization, the eigenvalue problem satisfies the orthogonality conditions
\[
\Phi_j^T M \Phi_j = \omega_j^2 \quad \text{(A31a)}
\]
\[
\Phi_j^T [A k^2 + B k + G] \Phi_j = \omega_j^2 \quad \text{(A31b)}
\]  

A particularly effective technique for solving the eigenvalue problem consists in carrying out a determinant search with deflation by the eigenvalues already found, using Lancaster’s trace algorithm (Lancaster, 1966). Thereafter, the eigenvectors are found by one or two steps of inverse iteration in equation (A30), multiplying by the transposed eigenvalue of the eigenvector in terms of the known eigenvalues. Convergence can be greatly accelerated by projecting the eigenvalues from one wavenumber to the next by means of the eigenvalue perturbation formula
\[
\frac{\partial \omega_j}{\partial k} = \frac{1}{2 \omega_j} \Phi_j^T (2Ak + B) \Phi_j
\]  

which is obtained by taking the derivative of the eigenvalue in equation (A30), multiplying by the transposed eigenvector, expanding the derivative of the eigenvector in terms of the known eigenvectors, imposing the condition that the perturbed eigenvector has no component along the eigenvector’s own direction, and using the orthogonality conditions.

The solution to the eigenvalue problem (A30) yields the characteristic frequencies \(\omega_j\) and the normal modes of wave propagation \(\Phi_j\), which are real and of the form
\[
\Phi_j = \{ \Phi_{1,j}, \Phi_{2,j}, \Phi_{3,j} \ldots \Phi_{2n+1,j}, \Phi_{2n+2,j} \}^T
\]  

\[
\{ \Phi_{1,j}, \Phi_{2,j} \ldots \Phi_{n+1,j} \}^T
\]  

Carrying out a modal superposition along the lines presented in Part I, the impulse-response functions (in their purely real form) are then
\[
\bar{g}_{mn} = \sum_{j=1}^{2n+2} e^{-\zeta_j \omega_j t} \frac{\sin \omega_j y}{\omega_j} \Phi_j^T
\]  

(A34)

Thus, the actual Green’s functions at elevation \(m\) due to a source at elevation \(n\) are
\[
g_{mn} = \left\{ \bar{g}_{xx} - i \bar{g}_{xz} \right\}_{mn}
\]  

(A36)

**Note:** When the source and the receiver coincide in space, i.e., \(z = z'\), the initial values implied by equation A34 [or A35] are not zero. Instead, it can be shown that the appropriate initial values for \(z = z' = 0\) [i.e., half-space interface] should be
\[
\bar{g}_{xx} (k,0,0,0) = \frac{1}{\rho \beta}, \quad \bar{g}_{xz} (k,0,0,0) = 0,
\]
\[
\bar{g}_{zz} (k,0,0,0) = \frac{1}{\rho \alpha}
\]  

(A37)
in which \(\rho, \alpha, \beta\) are the properties of the medium where the source is acting. As in Part I, these values must be halved if the source acts within a layer instead of an external surface.

Again, it is important to use these limiting values, and not zero, when starting the equations given in the main body of this article.

**Appendix III**

**Attenuation**

The impulse-response function in \(k-t\) for an elastic half-space given by equation (4) was originally derived for an undamped medium (Park and Kausel, 2004a). We provide here a brief extension to mass-proportional damping. Consider the vector wave equation for a viscoelastic medium in plane strain
\[
\begin{pmatrix}
\alpha^2 & 0 \\
0 & \beta^2
\end{pmatrix} \frac{\partial^2 \mathbf{u}}{\partial z^2} + \left( \alpha^2 - \beta^2 \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial^2 \mathbf{u}}{\partial \alpha \partial z} \\
+ \begin{pmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{pmatrix} \frac{\partial^2 \mathbf{u}}{\partial \alpha^2} = \frac{\partial \mathbf{u}}{\partial t} + 2\eta \frac{\partial \mathbf{u}}{\partial t} + \eta' \mathbf{u}
\]  

(A38)
in which \(\alpha\) and \(\beta\) are the \(P\)- and \(S\)-wave velocities, and \(\eta\) is a small-magnitude constant with dimensions of frequency. The second term on the right-hand side represents viscous (mass-proportional) damping, and the third is analogous to an elastic foundation, although its effect on the total wave motion is virtually nil because \(\eta\) is small and \(\eta^2\) is negligible.
Assuming a plane wave of the form $u = a \exp(i(\omega t - kx - nz))$, we obtain, after carrying out differentiations and collecting terms,

$$\begin{align*}
\left\{ \begin{array}{l}
\omega^2 - 2i\eta\omega - \eta^2 - k^2\alpha^2 - n^2\beta^2 \\
(\alpha^2 - \beta^2)nk
\end{array} \right. \quad \left\{ \begin{array}{l}
(\alpha^2 - \beta^2)nk \\
\omega^2 - 2i\eta\omega - \eta^2 - k^2\beta^2 - n^2\alpha^2
\end{array} \right. \quad \mathbf{a} = 0 \quad (A39)
\end{align*}$$

which can be written as

$$\begin{align*}
\left\{ \begin{array}{l}
(\omega - i\eta)^2 - k^2\alpha^2 - n^2\beta^2 \\
(\alpha^2 - \beta^2)nk
\end{array} \right. \quad \left\{ \begin{array}{l}
(\omega - i\eta)^2 - k^2\beta^2 - n^2\alpha^2
\end{array} \right. \quad \mathbf{a} = 0 \quad (A40)
\end{align*}$$

This is an eigenvalue problem in the vertical wavenumber $n$. Other than the complex frequency $\bar{\omega} = \omega - i\eta$ on the diagonal, this equation is exactly the same as that for an undamped medium, so it leads to dispersion relations for the vertical wavenumbers for $P$ and $S$ waves in terms of the horizontal wavenumber and complex frequency that are identical in form with those of an undamped medium, that is,

$$n_P = \pm \sqrt{\frac{(\bar{\omega} - \alpha)^2}{\alpha} - k^2}, \quad n_S = \pm \sqrt{\frac{(\bar{\omega} - \beta)^2}{\beta} - k^2} \quad (A41)$$

Thus, when the contour integration in Park and Kausel (2004a) is repeated and carried out along a frequency axis that has been shifted down by a constant amount $i\eta$, and considers the exponential term in the Fourier inversion in frequency $\exp(i\bar{\omega}t) = \exp(i\bar{\omega}t)\exp(-\eta t)$ as well as $d\omega = d\bar{\omega}$, exactly the same results are obtained, except that one gains the additional exponential factor $\exp(-\eta t)$. Finally, the imaginary part of the frequency can be chosen as $\eta = k\beta\zeta$, where $\zeta = \frac{1}{2} Q^{-1}$ is the fraction of damping. Defining also the dimensionless time $\tau = k\beta t$, this leads to the formulas and tables presented earlier.

**Note:** The third term on the right-hand side of equation (A38) is necessary to complete the square of the complex frequency in the two vertical wavenumbers. By comparison, in the $SH$ case in Part I (Park and Kausel, 2001), we included no such term, because the $SV$-$P$ case has two wave velocities, and it is not possible to dispense of this term without arriving at different complex frequencies for $nP$ and $nS$. However, the $SH$ case also provides a gauge to assess the influence of this additional term; because of its absence there, we obtained a shifted horizontal wavenumber of the form $\bar{k} = k(1 - \zeta^2)$, where $\zeta = \frac{1}{2} Q^{-1}$, which is small compared with unity. Thus, for practical purposes, the square root term is unity, so there is only a negligible difference between $\bar{k}$ and $k$, which means in turn that the third term included here has negligible effect.

Concerning damping in the layers, exactly the same considerations apply for the $SV$-$P$ case as for the $SH$ case in Part I. Hence, choosing a constant fraction of damping (i.e., $Q$) is virtually the same as choosing frequency-independent anelastic (hysteretic) attenuation.

A more complete treatment of damping in $k$-$t$ is forthcoming by the authors.

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Manuscript received 7 December 2005.