SOME OBSERVATIONS ON TIME DOMAIN AND FREQUENCY DOMAIN BOUNDARY ELEMENTS

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SUMMARY

The numerical solution of problems in elastodynamics involving infinite media calls for the use of discrete techniques such as the boundary element method and the finite element method. These techniques can, in turn, be formulated in the time or frequency domains, and have each relative merits and drawbacks. This paper presents a comparative study of the accuracy and limitations of three different implementations of these methods.

The problem studied is that of transient loads on the surface of homogeneous elastic halfspaces, and of finite depth strata. In each case, the response is computed first for an uninterrupted (continuous) medium, and then for a medium that includes a trench (or cavity).

Three independent computer programs were used that incorporated the following methods: (i) frequency domain boundary element method (FD-BEM), using a discrete fundamental solution; (ii) time domain boundary element method (TD-BEM) using an analytical fundamental solution; and (iii) a coupled time domain boundary element–finite element model (BEM/FEM).

It is found that for convex domains (halfspace or stratum without a trench) the three independent implementations are in excellent agreement, while for non-convex domains (trench in the path of the waves), numerical errors associated with non-causal behaviour become evident in some cases.

INTRODUCTION

Accurate numerical models for the computation of wave propagation in unbounded media, such as those used in the solution of soil–structure interaction problems, are generally based on implementations of finite elements (FEM) with transmitting boundaries, and more recently, on boundary elements (BEM) in the time and frequency domains.

While in principle it is possible to model with these methods a given problem with any degree of accuracy, it is found in practical situations that it is difficult, if not impossible, to avoid errors completely. These errors are related to the refinement of the models used, and to the specific algorithms implemented.

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This paper explores certain errors associated with three very different implementations of these methods: (i) a (discrete) boundary element method in the frequency domain; (ii) a (continuous) boundary element method in the time domain; and (iii) a coupled finite element–boundary element formulation in the time domain. These implementations are compared in the context of dynamic loads acting on the surface of elastic halfspaces and strata. A brief description of the three methods used is given in the following paragraphs.

**Frequency domain BEM (FD-BEM)**

A two-dimensional BE formulation is first considered, which is able to take into account the variability of the soil material in the vertical direction. The approach is based on developments by Kausel and Peek,¹ ² who discretized a given soil profile into several thin sublayers and assumed that the displacements varied linearly across each sublayer (i.e. in the vertical direction), and continuously in the horizontal plane. Although their procedure is restricted to a stratum of finite depth, it is very efficient as the Green's functions (fundamental solutions) do not require the numerical evaluation of integral transforms. Seale³ extended this formulation to incorporate an underlying halfspace by using para-axial approximations for the dynamic impedance of the halfspace. These approximations work well if the waves approach the halfspace interface at angles not too distant from the vertical. To guarantee this requirement, a series of sublayers, having the same material properties as the halfspace, can be added underneath the zone of interest before the halfspace approximation is considered. A very efficient boundary element code was developed by Pais⁴ and Kausel and Pais⁵ using the procedure just described, and applied to the problems considered in this paper. The solution is expressed in the frequency domain but results in the time domain are easily obtained using the fast Fourier transform.

**Time domain BEM (TD-BEM)**

This implementation of the boundary integral method is carried out directly in the time domain using time dependent fundamental solutions and a numerical step-by-step integration scheme. It is based on a technique developed by Mansur⁶ and Mansur and Brebbia⁷ for general elastodynamic problems, assuming zero initial conditions. Antes⁸ extended the formulation for arbitrary initial conditions. The method has successfully been applied to a wide range of wave propagation problems in fluid dynamics and in soil dynamics by Antes and von Estorff.⁹ –¹¹ Its main advantage lies in the possibility of applying the formulation to nonlinear problems, as no transformations from frequency to time domain, which imply linearity, are needed.¹²

**Boundary element–finite element method (BEM/FEM)**

Based on formulations published by Spyrrakos and Beskos¹³ for flexible foundations coupled to an elastic halfspace, a BE–FE coupling procedure was developed by von Estorff and Prabucki.¹⁴ The method seems to be quite promising, as it combines the advantages of a finite element solution (treatment of inhomogeneities and non-linearities) with the advantages of a boundary element approach (inclusion of radiation conditions). The algorithm works also directly in the time domain and can readily be extended to problems involving material non-linearities in the region modelled with finite elements.
FUNDAMENTAL SOLUTIONS

The equations associated with the boundary element method for dynamic problems are well documented, and need not be repeated here. For example, an overview summarizing the applications of the BEM to soil-structure interaction problems is given by Beskos. The key in the implementation of boundary elements for a given problem lies in the choice made for the Green’s functions (or fundamental solutions). In many cases, the functions used are those for a dynamic point load in the interior of an infinite space (Stoke’s and Lamb’s problem), because they are known in closed form in the time and frequency domains and do not require an integral transform. The disadvantage of these (continuous) functions, however, is that material discontinuities such as layers—or even the free surface of the soil—need to be modelled by means of interface nodes. Kausel and Pais, on the other hand, overcome this problem by using appropriate (discrete) Green’s functions for a layered medium, which satisfy automatically the continuity conditions at the interface and the surface of the medium. As a result, when using such functions, nodes are required only at geometrical irregularities such as cavities, trenches or inclusions. Since these functions are, however, formulated only in the frequency domain, it is necessary to use the fast Fourier transform algorithm to obtain results in the time domain.

NUMERICAL STUDIES

All solutions discussed in this section, obtained by the methods referred to earlier, are transient displacements at different locations of the system under consideration. For reasons of conciseness only one dynamic load, namely a Ricker wavelet, will be considered.

The Ricker wavelet was chosen, since it decays very rapidly both in time and frequency, thus reducing the number of time and frequency steps that need to be analysed. Its equation in time is

\[ f(t) = a(1 - 2\tau^2)e^{-\tau^2} \]  

(1)

where \( \tau = (t - t_0)/t_o \); \( t_0 \) is the time at which the maximum occurs, \( a \) is the amplitude and \( t_o \) corresponds to the dominant period of the wavelet. The Fourier transform of \( f(t) \), \( F(\omega) \), is given by

\[ F(\omega) = a[2\sqrt{\pi t_o}(e^{-\omega t_o})] \Omega^2 e^{-\Omega^2} \]  

(2)

where \( \Omega = \omega t_o/2 \) and \( \omega \) is the angular frequency in rad/sec. For this study, \( t_o \) was set to 1/\( \pi \), which corresponds to a dominant frequency of the wavelet near 1 Hz. The time lag \( t_\tau \) was taken equal to 3\( t_o \) (\( t_o = 3/\pi \)) which means that it takes about 0.95 sec for the loading to attain its maximum. Figure 1 displays a graph of the applied load function \( P(t) = f(t)/a \).

The solutions in the time domain using the FD-BEM were obtained multiplying \( F(\omega) \) by the transfer functions obtained with the program, and Fourier-inverting the result (using 8192 points). These transfer functions were computed at intervals of 0.10 Hz in the range from 0 to 5 Hz; intermediate values were then computed by polynomial interpolation (using Newton’s quadrature). The solution in time was obtained at intervals of 0.035 sec.

Infinite halfspace

Figure 2 shows the infinite halfspace considered in these studies, its material properties and the discretizations used for the three different approaches:

In the frequency domain BEM (thin-layer method) model, the halfspace is modelled as a layer (100 m), divided into a number of thin sublayers (24), which rests on top of an elastic halfspace.
modelled by means of a para-axial approximation (Figure 2(a)). The displacements \( u \) vary linearly across each sublayer along the vertical direction.

In the time domain BEM model, the surface of the halfspace is discretized by a number of line elements (19), over which the displacements and tractions are assumed to be constant. The length of each element is 10 m (see Figure 2(b)).

The BEM/FEM coupling procedure was carried out by subdividing the halfspace into one FE and one BE subregion. The FE domain of 70 m \( \times \) 70 m is subdivided into 49 linear, isoparametric finite elements and coupled with 21 linear boundary elements along the BE–FE interface. The free surface on each side of the FE mesh is discretized by means of 2 boundary elements.

Figures 3–8 show the transient displacements on the surface of the halfspace, observed at various locations, due to a vertical and a horizontal point load with a Ricker wavelet time variation. Although the three approaches are very different, it can be seen that the solutions obtained agree remarkably well, both for vertical and horizontal loads and displacements. Small
differences arise only for horizontal displacements due to a vertical load, in particular after the maximum is attained. In all cases, however, it can be observed, that the main property of the halfspace, namely the radiation of the waves, is represented very well. This is particularly notable for the halfspace modelled by thin layers and para-axial approximation (frequency domain).
Figure 6. Vertical displacements at B due to a vertical load: comparison of the methods

Figure 7. Horizontal displacements at B due to a horizontal load: comparison of the methods

Figure 8. Horizontal displacements at B due to a vertical load: comparison of the methods
These comparisons, although corresponding to a very simple problem, serve to establish a common ground for the more complex situations of a stratum, and a halfspace with a trench.

**Layer over rigid bedrock**

Figure 9 shows the geometry and material properties used to model the finite depth stratum. In the case of the FD-BEM, the stratum is subdivided into 24 thin layers as described before. For the TD-BEM calculation the surface is modelled with 19 elements, while the interface between the stratum and the rigid bedrock (displacements $u = 0$) was modelled with 15 elements. All material properties are the same as for the halfspace.

Figures 10–12 show the displacements at point B on the free surface, obtained by the approaches described earlier. As can be seen, the results obtained with the two methods for the vertical deformations at point B, caused by a vertical load at 0 are in excellent agreement (Figure 10). The differences occurring during the first 0.5 sec probably arise due to aliasing and to truncation of the frequency results at 5 Hz, since the response at point B should be zero prior to $t = 0.12$ sec, which is the time required for the fastest wave (p-wave) to travel from 0 to B.
Figure 11. Horizontal displacements at B due to a horizontal load: comparison of the methods

Figure 12. Horizontal displacements at B due to a vertical load: comparison of the methods

On the other hand, the horizontal displacements at B due to a horizontal and a vertical load (Figures 11 and 12) show some differences at times that are greater than the travel time of waves from the surface to bedrock and back. These discrepancies are probably caused by the truncation of the bedrock boundary in the TD-BEM.

Halfspace vs. stratum. In order to compare the solution for the halfspace with the results for a single layer, the displacements at point A are displayed in Figures 13–15. The main differences in the behaviour of both systems can be described best by considering the vertical transient response due to a vertical load (Figure 13).

For the first few time steps, \(0 < t < 0.05 \text{ sec}\), no disturbances are observed at A, as none of the waves has reached that point yet. In the ensuing time interval, \(0.05 < t < 0.46 \text{ sec}\), the responses of the halfspace and the stratum are identical, because during this period the reflections emanating from the rigid boundary at a depth of 100 m could not have reached the surface. Only when \(t > 0.46 \text{ sec}\) can one observe differences between both soil profiles because of waves reflected at the soil–bedrock interface and returned back to the surface. As a result of these reflections, the deformations at point A increase significantly and the amplitudes of motion
decrease only very slowly with increasing $t$. The motion on the halfspace surface, on the other hand, vanishes after approximately 2 sec (point A). The horizontal displacements at A due to a vertical load at the origin of the stratum and the halfspace (Figure 15) behave, in general, similarly to the curves just discussed, except that the
effect of the first reflections is not as important, so that the motions have similar magnitude. Finally, considering the horizontal response due to a horizontal load (Figure 14), one finds that the reflections at bedrock do not affect significantly the displacements at the surface.

**Halfspace with a trench**

The model of an elastic halfspace with a trench is of relevance in connection with problems of vibration isolation. Such a system is intrinsically more complicated than the simple halfspace considered before, because of the non-convexity (or concavity) of the domain involved. In general, when the domains considered are non-convex, it is important to insure that direct waves travelling in a path passing outside the domain are not present in the solution, i.e. that causality is satisfied. Non-convex domains arise when considering open trenches, tunnels, embedded foundations or even hills. As will be seen, problems involving concave domains may lead in some cases to violations of the principle of causality.

**Causality of the response.** When analysing a dynamic problem in the time domain, one should verify that the response satisfies the causality principle. This principle stipulates that a response at an arbitrary point \( B \) due to an excitation at point \( A \) (assuming the system to be initially at rest) can be observed only after the shortest possible time interval required for the fastest waves to travel from \( A \) to \( B \) along a path contained by the domain considered. Since longitudinal waves have the highest speed, \( c_1 \), and assuming that the only loading is at point \( A \), it can be stated that 

\[
u_B = 0 \quad \text{for} \quad t < \frac{|B - A|}{c_1},
\]

where \( |B - A| \) represents the shortest distance between \( A \) and \( B \). This causality relation is very important because it follows directly from basic physical considerations, and indeed, any accurate solution should satisfy causality.

This problem has recently received some attention, as can be seen in the works of Groenenboom et al.\(^6\), Antes and coworkers\(^17,18\) and Triantafyllidis and Dasgupta.\(^19\) Antes and von Estorff\(^17\) used the time domain boundary element formulation as described earlier and showed that if the method is applied directly to non-convex domains, the causality condition is not well satisfied. If, however, the domain is subdivided into convex subdomains which satisfy compatibility and equilibrium at each new boundary, the results obtained improve substantially. Antes and Meise\(^18\) performed several studies using an integral formulation of the Helmholtz equation and pointed out that the error, which may occur in case of non-convex domains, depends only on the accuracy of the numerical approximation (i.e. on the number of elements). A comparison of how well causality is satisfied by the three numerical models referred to earlier is presented in the section that follows.

**Comparison of methods.** Consider an open trench that is 50 m deep and 10 m wide which has been excavated in a halfspace, as shown in Figure 16. The soil properties and the discretization scheme are similar to those of the halfspace problem without the trench considered before. In case of the frequency domain calculation, a small internal damping was prescribed (\( \beta = 0.005 \)) to prevent singularities at the natural frequencies of the system.

Figure 17 shows the vertical displacement at point \( A \) (on the far side of the trench), caused by a vertical load at point \( 0 \) (the near side of the trench). This figure includes four horizontal bars that indicate the shortest travel times between \( 0 \) and \( A \) for pressure and shear waves travelling, respectively, in the presence and absence of the trench. In addition, arrows mark the points on the time axis when the curves attain non-zero values. Figures 18–20, on the other hand, show the displacements at point \( B \) further away from the trench due to horizontal and vertical loads at \( 0 \).

Whereas the curves obtained with the FD-BEM and the convex time domain BEM using three subregions, TD-BEM(3R) (see also Reference 17), satisfy the causality constraint very well
Figure 16. Halfspace with an open trench

Figure 17. Causality constraint caused by the trench: comparison of the methods

(Figure 17), the results obtained using the TD-BEM with constant elements (marked by a (C)) obviously violate this condition. Substantial improvements can be observed, however, if the TD-BEM is applied using linear elements (marked by an (L)) instead of constant elements. This is also evident from the BEM/FEM model which uses linear expansions for the boundary elements. The foregoing demonstrates clearly the importance of the numerical approximation used, not only on causality, but on the accuracy of the solution as well. When the observation point is further removed from the boundary (Figures 18–20), however, the violations of causality is not so obvious, and all three approximations give similar results.

In general, the contribution of 'direct' (or phantom) waves in non-convex domains can be associated with the approximations made in modelling the exact boundary conditions for tractions and displacements in the boundary element formulation, since they are satisfied only in an integral sense. It should also be remembered that the fundamental solution used for the time
domain boundary element formulation is that of a continuous medium, and that cavities and trenches are modelled by mathematical deletion ('excavation') of material in such medium (via the boundary integral). Because of the discretization of the (singular) boundary integral, such deletion is accomplished only imperfectly, and residuals of this material remain implicit in the
discrete boundary. It is through this remnant that non-causal signals travel. The frequency domain solution, on the other hand, uses directly the Green's function for the discrete, layered medium, so that the additional discretization errors do not arise; as a result, the solutions obtained with the FD-BEM are causal. (This causality is not associated with the domain in which it is formulated—the frequency domain—but with the discrete nature of the Green's functions used). If, however, finer discretizations of the boundary elements are used in the time domain solution, then the errors on the boundary decrease, and consequently, so does the contribution of phantom waves.

It is important to realize that the BEM approximation errors are intrinsic to the method and exist whether or not the boundaries are concave. While these errors may be obvious for concave domains, because of violations of the causality constraint, they are not obvious but do exist in the form of inaccuracies for convex domains. Hence, a measure of the accuracy of a given implementation can be obtained by verifying the degree to which a solution to a non-convex problem satisfies causality.

Effect of a trench. Figures 21–23 show the effect of the trench on the displacements (obtained by the FD-BEM) at a distance of 20 m from the loading. These figures also indicate the shortest times that it would take shear and compressional waves to reach point A. In Figure 21, the load

![Figure 21](image1.png)

**Figure 21.** Vertical displacements at A due to a vertical load: comparison with/without trench

![Figure 22](image2.png)

**Figure 22.** Horizontal displacements at A due to a horizontal load: comparison with/without trench
Figure 23. Horizontal displacements at A due to a vertical load: comparison with/without trench

and displacement considered are vertical. It can be seen that, without the trench, the maximum displacement at A is delayed with respect to the maximum loading by about the time it takes a shear wave to travel from 0 to A, reflecting the fact that, for this direction of loading, the perturbation is transmitted along the surface essentially as surface (Rayleigh) waves with speeds close to the shear wave velocity. In the case with the trench, the maximum displacement is observed later and corresponds very closely to the extra time that it takes shear waves to contour the trench. Also, as expected, point A remains quiescent for a longer period of time, reflecting the impossibility for the waves to cross directly the trench.

Figure 22 corresponds to the case of a horizontal load at 0 and displacement at A. The displacements at A in the case with the trench are very different to the corresponding ones for the halfspace. Instead of the displacements varying in time like the loading function, they are affected by waves reflected at the trench's bottom; this also explains the change in sign of the solution during the initial period. Also, the difference in arrival time of the two peaks observed (one negative and the other positive) approaches the difference in travel time between p- and s-waves. Therefore, it seems that in this case the motion at A is first caused by p-waves and later by s-waves.

The horizontal displacement at A corresponding to a vertical load at 0 is shown in Figure 23. Again, there is a delay in the time at which the maximum response is observed when the trench is present, and this delay is somewhat longer than the difference in travelling times of the fastest shear waves. This may be caused by intermediate reflections at the walls of the trench. It should be noted that the presence of the trench induces a much more important horizontal displacement than would otherwise be observed (the trench leads to a reduction of the surface stiffness).

Another interesting observation concerns the comparison of the horizontal displacement at A due to a vertical load at 0 and the vertical displacement at 0 due to a horizontal load at A. According to the Maxwell–Betti reciprocity theorem, both should be equal; however, the BEM does not ensure this theorem because of the approximations used at the boundary. Only if the boundary conditions were exactly satisfied could the BE solution satisfy the reciprocity relationship. This implies that another test on the accuracy of BE results would be to check for reciprocity of loads and displacements. Since the domain and reference points considered are symmetric, the vertical displacement at 0 due to a horizontal load at A is equal to the negative of the vertical displacement at A due to a horizontal load at 0. Hence, comparing the results in Figure 23 with the ones corresponding to a horizontal load at 0 (in the negative direction) and a vertical
displacement at A, a good measure of the accuracy of the solution is obtained. For the case without trench, both results are identical since the Green's functions used satisfy reciprocity; in the case with trench, although some differences were found, they were minimal, less than 1 per cent.

CONCLUSIONS

A comparative study has been performed to show the applicability and accuracy of three different (and independent!) boundary element formulations, which were

(i) a frequency domain boundary element method using a thin layer approach;
(ii) a boundary element procedure formulated directly in the time domain;
(iii) a coupling approach combining boundary elements with finite elements in the time domain.

Considering simple, but representative problems, that included energy dissipation, wave reflection and vibration isolation effects, the following conclusions could be drawn:

1. The solutions obtained by the three approaches for simple semi-infinite domains (halfspace) were nearly identical, which demonstrates that each method works well and is able to take into account the radiation of energy. Reflections at the artificial interfaces between the thin layers (FD-BEM) or between the BE-FE subregions either did not occur, or were not detectable.

2. Good agreement between the three methods was also found for a problem involving a reflecting boundary, namely the stratum, which shows that the number of nodes used to model this interface in the time domain solutions is adequate.

3. Concerning the performance of the methods when calculating non-convex domains (vibration isolation by an open trench), all three approaches seem to give, in general, correct results. However, when using the TD-BEM, the solution can exhibit non-causal behaviour; this means that a dynamic excitation at a given point produces responses at other points before the time required for the fastest waves to travel thereto. Such observed responses are certainly not physically possible, and result from errors introduced by the discretization of the boundary integrals.

The advantages of one method over another depend on the particular problem being solved. Thus, the thin-layer method is very efficient (and causal), whenever a horizontally layered medium is under consideration and the problem is assumed to be linear. This method requires, however, that the bottom boundaries incorporating the para-axial approximation are placed at some distance from the points of interest to ensure an accurate solution. When dealing with arbitrarily shaped, infinite regions, on the other hand, the time domain BEM gains in importance, especially if a formulation is needed that can be used for non-linear problems. The use of the coupled BEM/FEM alternative is often the most advantageous, when inhomogeneities and/or material non-linearities occur, since these can be handled in a FE subdomain, while satisfying radiation conditions with the BEM.

All computations performed in connection with these studies were accomplished on a PC-AT compatible personal computer.

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