

Chapter

**11**

**Rings**

Students viewing Saturn for the first time through a telescope often remark that “it looks like a picture,” because the majestic rings appear so crisp and perfect, like in the Voyager images. The other three giant planets also have rings, although not as grand as Saturn’s. Jupiter has a faint ring that contains intriguing resonances with its magnetic field. Uranus and Neptune each have several narrow rings that reflect ongoing resonant interactions with nearby satellites. Neptune’s three ring arcs are especially puzzling because they are confined in both the radial and azimuthal directions. Saturn’s ring system contains the full range of possibilities, from the massive B ring to the narrow F ring, to spiral ring waves and the completely unexpected ring spokes discovered by Voyager. It is interesting that there are rings around all of the giant planets but not around any of the terrestrial planets.

A large amount of information on rings has been amassed from both ground-based observatories and spacecraft. Ring positions are precisely determined by ground-based stellar occultation observations. The Voyager spacecraft encounters provided close-up photography that revolutionized the field of ring dynamics. Saturn’s rings serve as a proving ground for astrophysical theories of stellar and galactic disks. For example, the spiral arms in galaxies are thought to be analogous to the spiral density waves in Saturn’s rings.

Even though we have good data on planetary rings, many details remain unknown

because no one has actually seen an individual ring particle. Instead, our information comes from observing ensembles of particles, with the smaller particles generally accounting for the optical properties of the rings and the larger particles generally accounting for the dynamics. We begin with a survey of the different kinds of ring phenomena seen in the solar system, and then develop the basic ideas of ring-wave dynamics that will allow us to infer important properties like the total mass of Saturn's rings.

### 11.1 The Roche Limit

The study of planetary rings raises the question: under what conditions will a moon in orbit about its primary break apart due to tidal forces? This question was first addressed in a physical sense for a strengthless orbiting body by the French mathematician R.A. Roche in 1847. A moon of radius  $r$  and mass  $m$  located a distance  $R$  from a much larger body of mass  $M$  is accelerated to  $M$  by  $-GM/R^2$ . But the point on the moon nearest  $M$  feels a slightly higher acceleration  $-GM/(R-r)^2$ . So the net acceleration of that point relative to the rest of the moon is just the difference of those two accelerations. The moon will stay intact if its gravitational acceleration on its inner point is greater than the net acceleration

$$\frac{GM}{r^2} > GM \left[ \frac{1}{(R-r)^2} - \frac{1}{R^2} \right],$$

which to first order gives

$$\frac{r}{R} < \left( \frac{m}{3M} \right)^{(1/3)}.$$

If a strengthless moon is in synchronous rotation about a planet and subjected to both rotational (centrifugal) and tidal forces, the **Roche Limit** ( $r_{RL}$ ) is expressed

$$r_{RL} = 2.456 R_p \left( \frac{\rho_p}{\rho_o} \right)^{1/3}, \quad (11.1)$$

where  $R_p$  is the radius of the planet,  $\rho_p$  is the density of the planet, and  $\rho_o$  is the density of the orbiting object. It is observed that planetary rings are located within the Roche Limit, which defines the distance from a planet where tidal forces exceed the gravitational attraction that holds an orbiting body together. In practice, the roche Limit will vary somewhat depending on the tensile strength of material that makes up the moon or moonlets. A moon that strays inside the Roche limit will break up and moonlets within the Roche limit will not accrete into a moon.

### 11.2 Saturn's Rings

Saturn's rings are seen edge-on from Earth about every 15 years, but because of seeing limitations their thickness can not be determined from ground-based observations. Extrapolations to zero tilt angle of the photometric brightness yield an effective ring thickness of about 1 km. However, this is an overestimate because of the presence of large bending waves that warp the rings like the curving brim of a hat. The actual thickness of Saturn's

rings is now known to be only tens of meters – tiny compared to its 200,000 km spatial extent. Dynamical arguments tend to favor a ring system that is a monolayer of big boulders interspersed with a distribution of smaller particle sizes.

The particles that make up Saturn’s ring system are brightly reflective compared with the dark rings of Uranus and Neptune. Saturn’s rings are probably composed of meter-sized “dirty snowballs.” The Voyager radio occultation experiments probed Saturn’s rings with coherent 3.6 and 13 cm wavelength radio waves while the rings were between Earth and the spacecraft. If the typical particle size had been intermediate between these two wavelengths, then there would have been observable differences in the transmission properties of the rings at the two wavelengths. As it happens, both wavelengths yielded similar results, which implies that the typical ring particle is larger than 13 cm. It turned out that Saturn’s *B* ring is thick enough that the radio waves from Voyager were unable to penetrate much of it. The *A* ring exhibits a longitudinally asymmetric reflectivity in its inner regions, which appears to be unique. The Cassini Division, the large gap that separates the *A* and *B* rings, is in fact not clear but has an optical depth similar to the *C* ring.

### 11.2.1 Ring Spokes

When the Voyager spacecraft returned the first high-resolution photographs of Saturn’s rings no one was prepared for the amazing detail and sheer number of small ringlets and gaps discovered. But one of the most astonishing discoveries was the existence of **ring spokes** — dark, radial structures that seemed to defy Kepler’s Laws. Spokes have a characteristic length of 10,000 km and width of 2,000 km, and are observed only in the radii range from 103,900 km to 117,000 km as measured from the center of Saturn. This range of radii corresponds to the most optically thick region of the rings. The inner radial limit is particularly sharp. The observed range includes the synchronous-orbit distance, implying that spokes are probably formed by disturbances in Saturn’s magnetic field.

The spokes are about 10% darker than the surrounding regions when viewed in back-scattered light, but become 10–15% brighter than the surrounding regions in forward-scattered light. This implies that the spokes are composed of micron-sized particles. In the few cases when spokes have been observed as they formed, they appeared in less than five minutes (the shortest time between successive images) and started out in a nearly radial geometry. In fact, the spokes are observed to shear out with the local Keplerian motion once they are formed, but they are short-lived, with typical lifetimes between 0.5 and 4 hours.

### 11.3 Narrow Rings and Ring Arcs

Narrow rings were a surprising discovery when they were first detected at Uranus by stellar occultation in 1977. This is because there are several effects, like interparticle collisions, **Poynting-Robertson drag** (due to collisions with photons) and **plasma drag** (due to collisions with magnetospheric plasma), that act to gradually spread out and widen an unconstrained ring in only a few decades. Both processes are inversely proportional to particle size and so affect smaller particles much more than larger particles. Even the most conservative estimates of spreading indicate that the narrow rings of Uranus cannot

be more than ten million years old if they are unconstrained. Neptune's ring arcs are even more baffling, because they are constrained in both the radial and azimuthal directions. Narrow rings have a practical significance because their orbital positions can be precisely determined. The precession of the rings of Uranus has been used to probe the distribution of mass inside the planet by determining the gravitational zonal harmonics  $J_2$  and  $J_4$ .

One of the changes in perspective following the Voyager encounters is that rings are younger and more dynamic than originally thought. The leading theoretical model for the confinement of narrow rings uses **shepherd satellites**, one on each side of the ring, to provide gravitational confinement to the ring. Voyager discovered a shepherd-satellite pair associated with Saturn's narrow F ring and another associated with Uranus'  $\epsilon$  ring, but there have been fewer such pairs discovered than are called for by present theory. It may be that there are 18 shepherd satellites interspersed amongst the 9 rings of Uranus and Voyager was unable to spot them all. It is also conceivable that more complete models will show that the few observed satellites are sufficient to do the job, or that there are other processes besides the shepherd-satellite mechanism at work.

## 11.4 Dust in Rings

Only a spacecraft can view a planet from widely varying angles with respect to the Sun. Since a primary diagnostic of surface properties and of ring-particle size distributions is the variation in intensity of reflected sunlight with angle, called the **phase function**, spacecraft observations are essential for understanding many of the properties of surfaces and rings. Earth-based observers of the outer planets see only back-scattered sunlight. Anyone who has driven into a sunrise or sunset with a dirty windshield knows that dust particles scatter strongly in the forward direction, and so dust tends to be next to impossible to detect from Earth. Thus, each time a spacecraft has passed an outer planet for the first time and looked back, we have received a brand new view of that planet's rings in forward-scattered light. The first images of the rings of Uranus and Neptune in forward-scattered light were breathtaking, because they revealed previously unknown rings of dust that were much broader than the known narrow rings.

The existence of these wide lanes of micron-sized dust requires a present-day source for the dust, since effects like Poynting-Robertson (light) drag continuously cause the smallest particles to spiral into the planet. Over the age of the solar system, Poynting-Robertson drag can eliminate centimeter-sized pebbles from the orbit of Jupiter and millimeter-sized particles from the orbit of Saturn. It is significantly more efficient at removing micron-sized particles, and so a present-day source, most likely recent ring-particle collisions, is needed to explain the observations.

## 11.5 Ring Dynamics

Rotating disks are a central theme in astrophysics, and many of the dynamical theories developed to explain the structure of galaxies also have application with planetary rings. In fact, Saturn's rings can be used as a testing ground for theories on self-gravitating disks. The self gravity of the rings supports both spiral density waves and spiral bending waves, and examples of both have been identified in the Voyager images. The theory of ring waves

is sufficiently well understood that we can use the observed waves in Saturn’s rings as a diagnostic tool to measure the surface ring density.

There is an important difference between galaxies and the rings of Saturn when it comes to self-gravitating waves. In spiral galaxies, the natural scale of self-gravitational disturbances is the same order of the radius of the galaxy. However, in Saturn’s rings, the natural scale is much smaller than the radius of the rings. The reason is that the ratio of the mass of Saturn’s disk to the total mass, which includes Saturn itself, is much smaller than the same ratio for a spiral galaxy. Left to themselves, Saturn’s rings will not spontaneously produce waves that are large enough to be observable. The large waves we do see are excited by external forces, mostly from resonant interactions with external satellites.

### 11.5.1 Natural Frequencies

We will use cylindrical coordinates  $(r, \theta, z)$  to describe the three-dimensional location of a ring particle about a planet. The coordinate  $z$  measures the vertical displacement out of the plane of the unperturbed circular reference orbit. The excursions from the unperturbed orbit are assumed to be small. The particle executes harmonic motion with “epicyclic” frequency  $\kappa(r)$  in the  $r$  direction, orbital frequency  $\Omega(r)$  in the  $\theta$  direction, and vertical frequency  $\mu(r)$  in the  $z$  direction.

The multi-pole expression for a planet’s gravitational potential in the equatorial plane,  $\phi_p(r, 0)$ , is written:

$$\phi_p(r, 0) = -\frac{GM}{r} \left[ 1 - \sum_{n=1}^{\infty} J_{2n} \left( \frac{R}{r} \right)^{2n} P_{2n}(0) \right], \quad (11.2)$$

where  $M$  and  $R$  are the mass and equatorial radius of the planet, respectively, and

$$P_{2n}(0) = \left( -\frac{1}{4} \right)^n \frac{(2n)!}{(n!)^2} \quad (11.3)$$

is the value of the Legendre polynomial of order  $2n$  at the origin.

Now, consider the balance between the centripetal acceleration and the gravitational force on a ring particle in an unperturbed orbit:

$$r\Omega^2(r) = \left( \frac{\partial \phi_p}{\partial r} \right)_{z=0}. \quad (11.4)$$

In an analysis similar to a previous homework problem (3.2 *Motion near a Potential Minimum*), B. Lindblad developed a theory for small epicyclic motion about an unperturbed orbit, which yields the following expressions for the natural frequencies of an orbiting ring particle:

$$\kappa^2(r) = \frac{1}{r^3} \frac{\partial}{\partial r} (r^4 \Omega^2), \quad (11.5)$$

$$\mu^2(r) = \left( \frac{\partial^2 \phi_p}{\partial z^2} \right)_{z=0}. \quad (11.6)$$

A simple relationship between  $\kappa$ ,  $\Omega$ , and  $\mu$  may be found by combining (11.4)–(11.6) with Laplace’s equation for the potential at  $z = 0$ :

$$\nabla^2 \phi_p = \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial \phi_p}{\partial r} \right]_{z=0} + \left[ \frac{\partial^2 \phi_p}{\partial z^2} \right]_{z=0} = 0, \quad (11.7)$$

yielding:

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left[ r r \Omega^2 \right] + \mu^2 &= 0, \\ \frac{1}{r} \frac{\partial}{\partial r} \left[ (r^{-2}) (r^4 \Omega^2) \right] + \mu^2 &= 0, \\ \frac{1}{r} (-2r^{-3}) r^4 \Omega^2 + \frac{1}{r^3} \frac{\partial}{\partial r} (r^4 \Omega^2) + \mu^2 &= 0, \\ -2\Omega^2 + \kappa^2 + \mu^2 &= 0. \end{aligned}$$

The relationship is thus:

$$2\Omega^2 = \kappa^2 + \mu^2. \quad (11.8)$$

If the giant planets were perfectly spherical, then we would have  $\kappa = \Omega = \mu$ , and the orbit of a ring particle would be closed. But rotation makes all four giant planets oblate, and this splits the degeneracy between the three frequencies as follows:

$$\kappa < \Omega < \mu. \quad (11.9)$$

### 11.5.2 Forcing by a Satellite

We now introduce a forcing satellite of mass  $m_s$  with an orbit outside of the rings. The presence of the satellite alters the gravitational potential (ignoring the small effect of the satellite’s gravity acting on the planet’s center of mass) by an amount:

$$\begin{aligned} \phi_s(r, \theta, z, t) &= -\frac{Gm_s}{|\mathbf{r}_s - \mathbf{r}|} \\ &= -Gm_s \left\{ r_s^2(t) + r^2 - 2r_s r \cos[\theta_s(t) - \theta] + [z_s(t) - z]^2 \right\}^{-1/2}. \end{aligned} \quad (11.10)$$

The conventional notation for denoting **orbital commensurabilities** between the angular velocity (or mean motion) of a satellite  $\Omega_s$  and a ring particle  $\Omega$  is

$$l\Omega_s = (m - 1)\Omega, \quad \text{or} \quad l : (m - 1), \quad (11.11)$$

where  $l$  and  $m$  are small integers. Since we are only considering small perturbations of periodic variables, we can linearize the equations and express each perturbation with a Fourier series, with individual terms of the form:

$$\text{Re} \left\{ \Phi_s(r, z) e^{i(\omega t - m\theta)} \right\}, \quad (11.12)$$

where the disturbance frequency  $\omega$  is expressed as a sum of integer combinations of the forcing satellite's orbital frequencies:

$$\omega = m\Omega_s + n\mu_s + p\kappa_s. \quad (11.13)$$

The magnitude of the gravitational forcing can be shown to be proportional to:

$$\text{forcing} \propto (\epsilon_s)^{|p|} (\sin i_s)^{|n|}.$$

Since the satellite eccentricities ( $\epsilon$ ) and inclinations ( $i$ ) are small at Saturn, we need to only consider small values for the integers  $p$  and  $n$ . The highest-order resonances for which there are good observations satisfy  $(p + n) \leq 2$ .

The ring particle is orbiting at frequency  $\Omega$  and therefore feels the  $m$ -th component of the satellite's perturbing potential reduced by the Doppler effect caused by its orbital motion, such that the perturbation frequency felt by the ring particle is  $\omega - m\Omega$ . If this forcing frequency approaches one of the natural frequencies of the ring particle, then the particle will experience the effects of resonance. A particle will experience a horizontal, or **Lindblad**, resonance if it is placed at a radius  $r = r_L$ , such that:

$$\omega - m\Omega(r_L) = \pm \kappa(r_L). \quad (11.14)$$

The resonance is termed an **inner Lindblad resonance** if the right-hand side of (11.13) is negative, and an **outer Lindblad resonance** if the right-hand side is positive. Only inner Lindblad resonances are important for ring waves at Saturn, since the forcing comes only from satellites outside of the rings, such that  $(\omega - m\Omega) \approx (m\Omega_s - m\Omega) < 0$ .

If a ring particle happens to be at a radius  $r_V$ , such that

$$\omega - m\Omega(r_V) = \pm \mu(r_V), \quad (11.15)$$

then the particle will experience a vertical, or inclination, resonance. Saturn's satellite Mimas has sufficient mass and inclination to force observable vertical resonances in Saturn's rings. The oblateness of Saturn causes the Lindblad and vertical resonances of a given order to occur at different frequencies and therefore at different orbital distances from Saturn.

Gravitational resonances with external satellites do not represent the only important forcing mechanism acting on ring particles. In Jupiter's ring, resonances between the orbital elements of dust particles and Jupiter's magnetic field, called **Lorentz resonances**, are known to exist.

### 11.5.3 Spiral Density Waves

The study of density and bending waves in rings gives us our best handle on several important ring parameters, like the surface mass density and the effective viscosity of the rings. We will now follow the standard strategy for studying linear waves.

1. *Decide on a representative idealization of the physical system.* It has been demonstrated that a ring system or a galaxy behaves like a fluid that has no pressure or viscosity

if we are only interested in the large-scale, collective behavior. This greatly simplifies the analysis. We will idealize a ring system as a very thin fluid disk. The disk is self gravitating, which will modify the natural frequency for density waves. This theory was first developed to explain the spiral arms in galaxies. Poisson's equation for the disk potential  $\phi_d$  is:

$$\nabla^2 \phi_d = 4\pi G \sigma \delta(z), \quad (11.16)$$

where  $\sigma$  is the surface mass density and  $\delta(z)$  is the Dirac delta function. Written out explicitly, (11.15) becomes:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi_d}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi_d}{\partial \theta^2} + \frac{\partial^2 \phi_d}{\partial z^2} = 4\pi G \sigma \delta(z). \quad (11.17)$$

The conservation of mass is described by the continuity equation, which takes the form:

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{v}) = 0, \quad (11.18)$$

where  $\mathbf{v}$  is the two-dimensional fluid velocity. Written out explicitly, (11.18) becomes:

$$\frac{\partial \sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \sigma u) + \frac{1}{r} \frac{\partial}{\partial \theta} (\sigma v) = 0, \quad (11.19)$$

where  $u$  and  $v$  are the components of velocity in the  $r$  and  $\theta$  directions, respectively. The conservation of momentum is described by the equation of motion,  $\mathbf{a} = \mathbf{F}/m$ :

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -\nabla(\phi_p + \phi_d + \phi_s). \quad (11.20)$$

Written out explicitly, (11.19) becomes:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -\frac{\partial}{\partial r} (\phi_p + \phi_d + \phi_s), \quad (11.21)$$

$$\frac{\partial v}{\partial t} + \frac{u}{r} \frac{\partial}{\partial r} (rv) + \frac{v}{r} \frac{\partial v}{\partial \theta} = -\frac{1}{r} \frac{\partial}{\partial \theta} (\phi_d + \phi_s). \quad (11.22)$$

The left-hand-sides of (11.20) and (11.21) are just like the shallow-water equations used in meteorology and oceanography, but the right-hand-sides include the more complicated gravity field of the self-gravitating ring problem.

2. *Pick a basic state, and linearize the system.* We have idealized the ring system as a thin fluid disk, but the equations are still complicated from an analytical point of view because of the inherent nonlinearity of fluids and because of the non-Cartesian coordinate system. We now choose a simple basic state that satisfies (11.18), (11.20) and (11.21), and add to it a small harmonic perturbation:

$$\begin{aligned} u(r, \theta, t) &= 0 + \text{Re} \left[ U(r) e^{i(\omega t - m\theta)} \right], \\ v(r, \theta, t) &= r\Omega + \text{Re} \left[ V(r) e^{i(\omega t - m\theta)} \right], \\ \sigma(r, \theta, t) &= \sigma_0(r) + \text{Re} \left[ S(r) e^{i(\omega t - m\theta)} \right]. \end{aligned} \quad (11.23)$$

We get the linearized set of equations by substituting (11.22) into the nonlinear equations, and by retaining only terms linear in  $U(r)$ ,  $V(r)$ , and  $S(r)$ , with the result:

$$\begin{pmatrix} \frac{1}{r} \frac{d}{dr} \sigma_0 r & -i \frac{m}{r} \sigma_0 & i(\omega - m\Omega) \\ i(\omega - m\Omega) & -2\Omega & 0 \\ \frac{\kappa^2}{2\Omega} & i(\omega - m\Omega) & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ S \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\partial}{\partial r} [\Phi_d + \Phi_s] \\ i \frac{m}{r} [\Phi_d + \Phi_s] \end{pmatrix} \quad (11.24)$$

The terms  $\Phi_d$  and  $\Phi_s$  are the appropriate Fourier terms for  $\phi_d$  and  $\phi_s$ , and  $\Phi_d$  satisfies the linearized form of Poisson's equation for  $|z| > 0$ , in which case the right-hand side of (11.17) is zero.

Solving for  $U$ ,  $V$ , and  $S$  yields:

$$\begin{aligned} U &= i \Lambda_U [\Phi_d + \Phi_s], \\ V &= \Lambda_V [\Phi_d + \Phi_s], \\ S &= \Lambda_S [\Phi_d + \Phi_s], \end{aligned} \quad (11.25)$$

where the differential operators  $\Lambda_U$ ,  $\Lambda_V$ , and  $\Lambda_S$  are given by:

$$\begin{aligned} \Lambda_U &= \frac{1}{D} \left[ -(\omega - m\Omega) \frac{d}{dr} + \frac{m}{r} 2\Omega \right], \\ \Lambda_V &= \frac{1}{D} \left[ \frac{\kappa^2}{2\Omega} \frac{d}{dr} - \frac{m}{r} (\omega - m\Omega) \right], \\ \Lambda_S &= \left( \frac{1}{\omega - m\Omega} \right) \left[ -\frac{1}{r} \frac{d}{dr} r \sigma_0 \Lambda_U + \frac{m}{r} \sigma_0 \Lambda_V \right]. \end{aligned} \quad (11.26)$$

The term  $D$  is the determinant of the bottom-left  $2 \times 2$  matrix in the  $3 \times 3$  matrix on the left-hand-side of (11.24):

$$D \equiv \kappa^2 - (\omega - m\Omega)^2, \quad (11.27)$$

and measures how close the system is to a Lindblad (horizontal) resonance.

The free spiral density waves are found in the case where the satellite potential  $\Phi_s$  is ignored. By assuming that the disk potential  $\Phi_d(r, 0)$  in the ring plane is of the WKBJ form:

$$\Phi_d(r, 0) = A(r) e^{i \int k dr}, \quad (11.28)$$

a dispersion relation can be found for long spiral density waves:

$$D = \kappa^2 - (\omega - m\Omega)^2 = 2\pi G \sigma_0 |k|, \quad (11.29)$$

where  $|k|$  is the wavenumber, and  $2\pi/|k|$  is the wavelength of the density waves. The physical interpretation of the dispersion relation (11.28) is as follows. Without the effects of self-gravity, a disk containing a disturbance with periodicity  $m$  will oscillate in the radial direction at the natural frequency  $\kappa$ . However, the self-gravity in the compressed regions

slows down the re-expansion, as is described by (11.28) through in an effective reduction of  $\kappa$ :

$$\kappa^2 \rightarrow \kappa^2 - 2\pi G\sigma_0|k|.$$

The group velocity  $c_g$  of the waves may be calculated by differentiating (11.29) with respect to  $k$ , which yields:

$$c_g = \frac{\partial\omega}{\partial k} = -\frac{\pi G\sigma_0}{(\omega - m\Omega)}. \quad (11.30)$$

Since for inner Lindblad resonances the term  $(\omega - m\Omega)$  is negative, (11.30) implies that  $c_g > 0$ , such that spiral density waves propagate energy in the outward direction from a Lindblad resonance.

#### 11.5.4 Spiral Bending Waves

The linear analysis of spiral bending waves is analogous to the linear analysis of spiral density waves. For bending waves one obtains a dispersion relation analogous to (11.28) where  $\mu$  takes the place of  $\kappa$  and  $-G$  takes the place of  $G$ :

$$\mu^2 - (\omega - m\Omega)^2 = -2\pi G\sigma_0|k|. \quad (11.31)$$

Bending waves oscillate at a frequency that is faster than the natural frequency  $\mu$ . In other words,  $\mu^2$  is effectively increased by the amount:

$$\mu^2 \rightarrow \mu^2 + 2\pi G\sigma_0|k|.$$

This is because the disk's self-gravity helps pull back the vertical distortion. Because of the change in sign of the effect of self gravity, the group velocity is negative and bending waves propagate energy inward from a resonance instead of outward like density waves. Since Saturn is oblate, the waves caused by an external satellite are separated by a gap that has bending waves on its inner boundary moving inwards, and density waves on its outer boundary moving outwards.

#### 11.5.5 Application to Saturn's Rings

The Voyager images of Saturn's rings contain beautiful examples of spiral density waves and spiral bending waves. Equation (11.29) yields an expression for the wavelength  $\lambda = 2\pi/|k|$  of spiral density waves:

$$\lambda = \frac{2\pi}{|k|} = 2\pi \frac{2\pi G\sigma_0}{D} = \frac{4\pi^2 G\sigma_0}{D}. \quad (11.32)$$

Near the location  $r_L$  of a Lindblad resonance, we will approximate  $D(r)$  by a Taylor series:

$$D(r) \approx \left( \frac{dD}{dr} \right)_{r_L} (r - r_L) \equiv \mathcal{D}x, \quad (11.33)$$

where

$$\mathcal{D} \equiv \left( r \frac{dD}{dr} \right)_{r_L}, \quad \text{and} \quad x \equiv \left( \frac{r - r_L}{r_L} \right). \quad (11.34)$$

The expression for the wavelength becomes:

$$\lambda = \frac{4\pi^2 G \sigma_0}{\mathcal{D}} \frac{1}{x}. \quad (11.35)$$

Notice that the wavelength depends on the inverse of the normalized distance  $x$ . The observed waves nicely exhibit this  $1/x$  dependence.

The first density wave train discovered was associated with the  $m = 1$  (apsidal) inner Lindblad resonance of Iapetus, which generates density waves in the outer part of the Cassini Division. The quantity  $\mathcal{D}$  in (11.34) is especially small for the  $m = 1$  case, making the wavelengths large and easy to observe. Using (11.34),  $\sigma_0$  for the Iapetus density wave turns out to be  $16 \text{ g cm}^{-2}$ . The same analysis applied to stellar occultation observations of a set of density waves excited in the inner  $B$  ring by the 2:1 resonance with Saturn's co-orbital satellite 1980S1 yielded  $\sigma_0 = 60 \text{ g cm}^{-2}$ , consistent with the factor of four increase in optical depth in the  $B$  ring relative to the Cassini Division. About fifty wavetrain observations have been pieced together to form a surface-density model for Saturn's rings. The estimate for the total ring mass is about  $5 \times 10^{-8}$  Saturn masses, which is the mass of a satellite like Mimas that has about a 200 km radius.

## Problems

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## References

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