# Towards Resistance Sparsifiers

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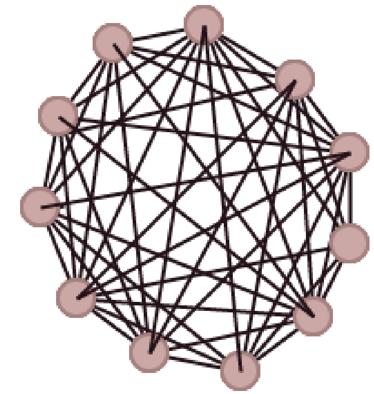
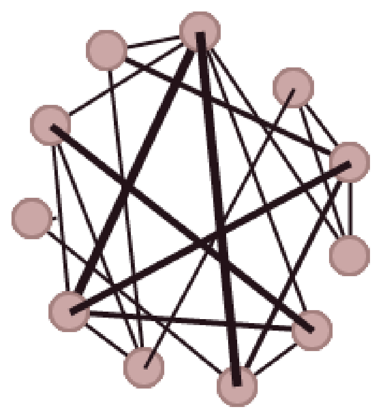
# Graph Sparsification

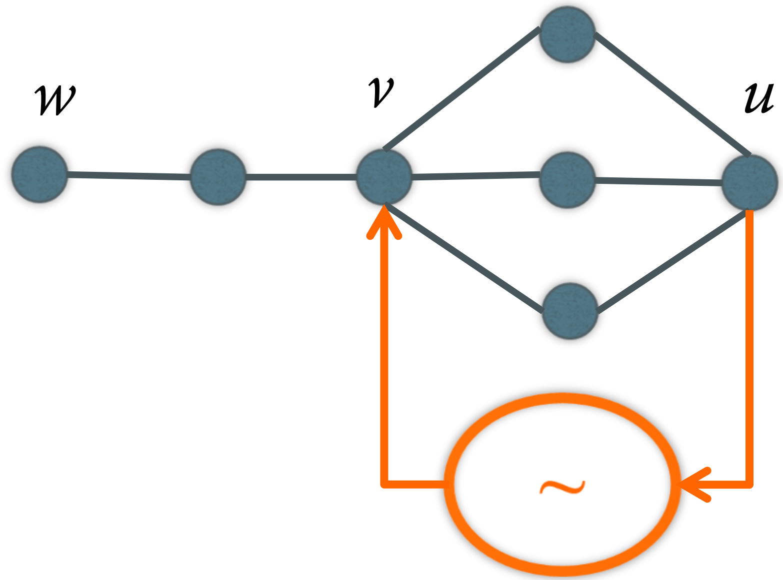
- Input: Dense graph **G**
- Goal: Sparse (weighted) subgraph **H** that approximately preserves some properties of **G**
- Examples:
  - Shortest paths ("spanners") [Peleg-Schäffer'89]
  - Cut values ("cut sparsifiers") [Benczúr-Karger'96]
  - Eigenvalues ("spectral sparsifiers") [Spielman-Teng'04]
  - Resistance distances





# Resistance Distance

- Fundamental graph metric
- Equivalent views:
  - Electric voltage difference
  - Random walk
  - Random spanning tree
- Widely used in applications




$$R_G(u,v) = (c_u - c_v)^T L_G^{-1} (c_u - c_v)$$

# Resistance Sparsifiers

- Can we construct efficient resistance sparsifiers?
- **Yes**: Spectral sparsifiers are resistance sparsifiers
  - Size (#edges): $O(n/\varepsilon^2)$ [Batson-Spielman-Srivastava'08]
- Can we do even better?
- **Yes** on the complete graph:
  - Spectral sparsifiers require size $\Omega(n/\varepsilon^2)$ [BSS'08]
  - Resistance sparsifiers have size $O(n/\varepsilon)$
    - Consequence of [von Luxburg-Radl-Hein'10]

# The vLRH Bound

[von Luxburg-Radl-Hein'10]:

- On expanders, resistance metric is essentially determined by vertex degrees:

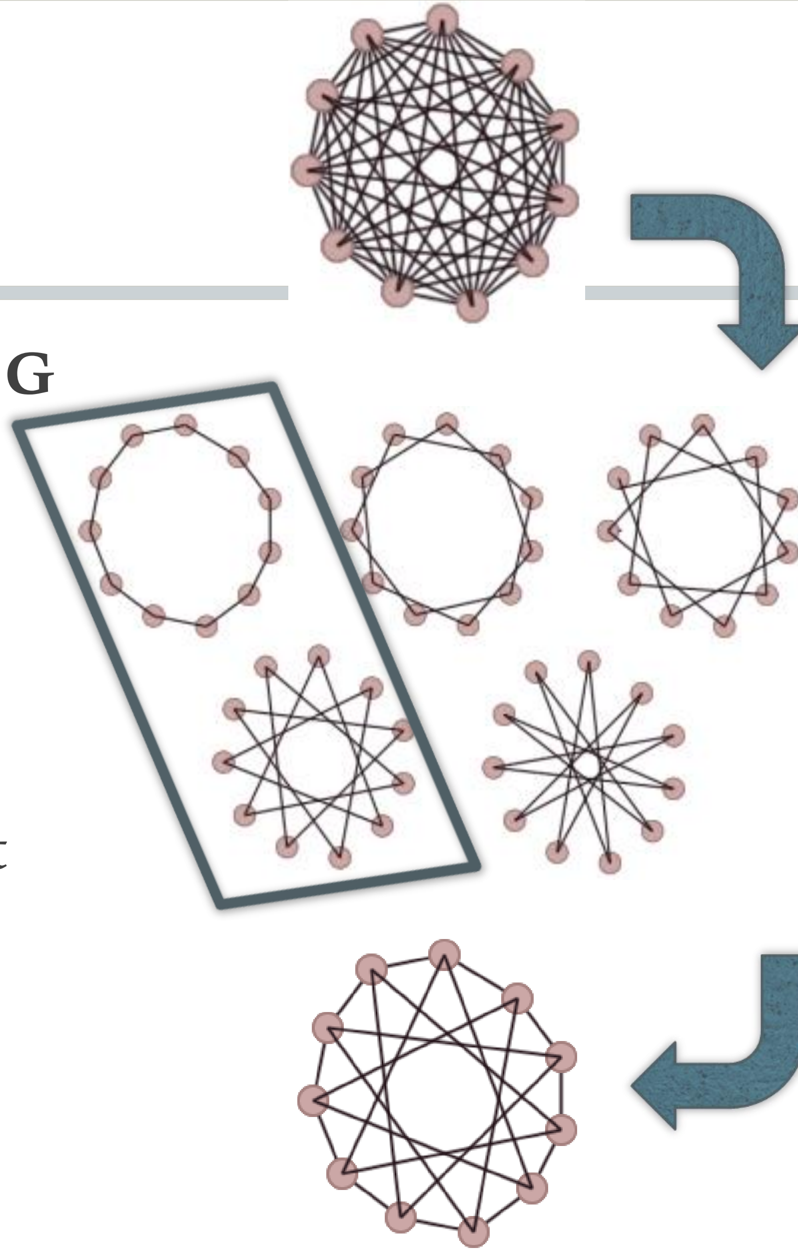
$$R(u,v) \approx \frac{1}{\deg(u)} + \frac{1}{\deg(v)}$$

- Hence: Need to preserve degree sequence

# Results

- Do more graphs have such efficient resistance sparsifiers?
- **Yes** for dense regular expanders:
- **Theorem 1**: Every $\Omega(n)$-regular expander has a $(1+\varepsilon)$-resistance sparsifier of size $\tilde{O}(n/\varepsilon)$.
- Underlying structural result:
- **Theorem 2**: Every $\Omega(n)$-regular expander contains a polylog$(n)$-regular expander as a subgraph.

# Algorithm



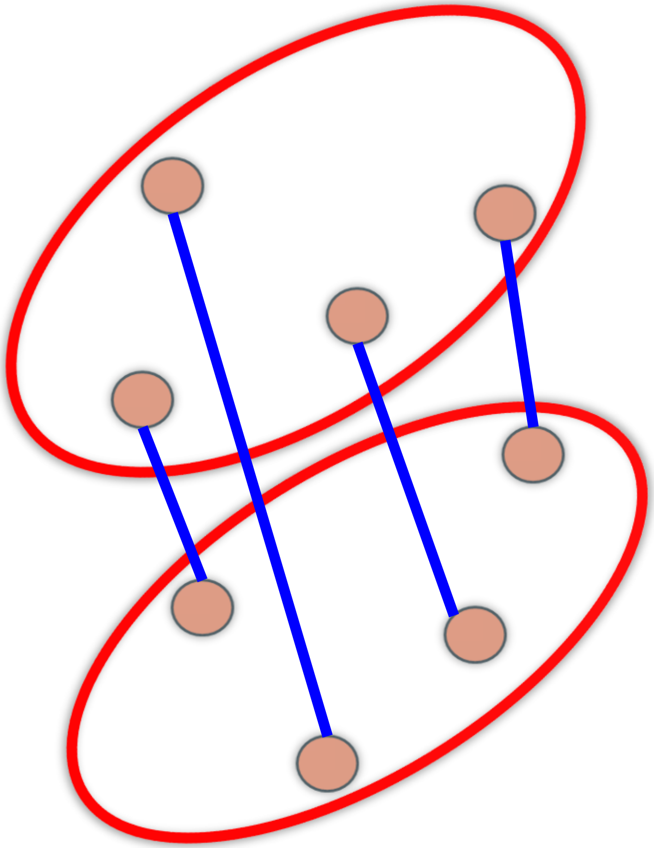
- **Input**: Dense regular expander **G**
- **Goal**: Find sparse regular expander subgraph **H**
- **Algorithm**:
  - Decompose **G** into disjoint Hamiltonian cycles or perfect matchings
  - Choose a uniformly random subset of them to form **H**
- **Analysis**: …

# Analysis

# The Cut-Matching Game

[Khandekar-Rao-Vazirani'06]

- Start with an empty graph on n vertices.
- In each turn,
  - The **Cut player** chooses a bisection.
  - The **Matching player** adds a perfect matching across the bisection.



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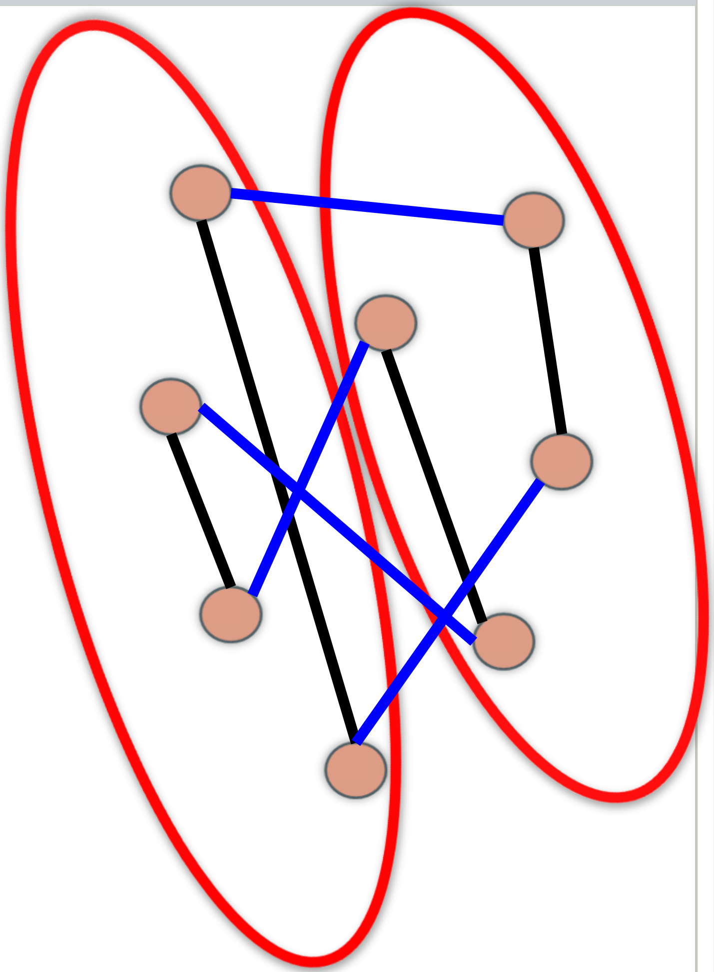
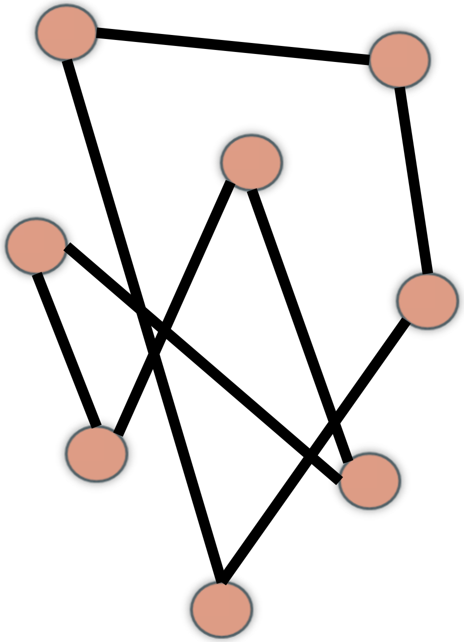


# The Cut-Matching Game

- **Cut player goal**: Construct an expander
- **Matching player goal**: Delay this



- **Theorem** [KRV'06]: Cut player can win within $O(\log^2 n)$ rounds.

# Warm up: Degree > (¾+δ)n

Suppose **G** is D-regular with D > (¾+δ)n.

Goal: Find sparse regular expander subgraph of **G**.

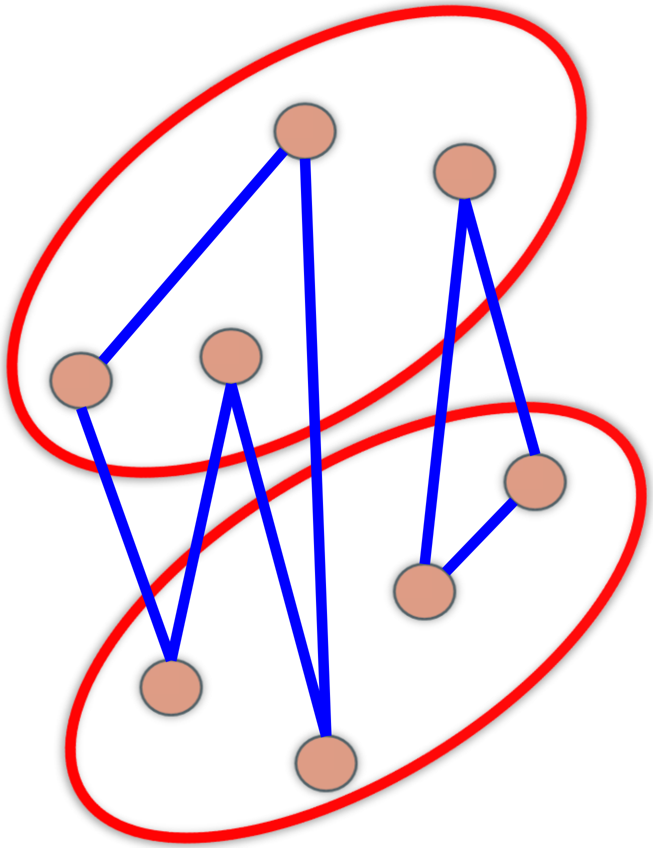
- **Claim**: **G** contains a perfect matching across any bisection.
- **Play the Cut-Matching game**:
  - For **Cut player**, use winning strategy
  - For **Matching player**, return a bisection given by claim
- Resulting **H** is an $O(\log^2 n)$-regular expander subgraph of **G**.

# The Cut-Weave Game

- **Definition**: Given a bisection of a vertex set, a **weave** is a graph in which every vertex has an incident edge across the bisection.

- In the **Cut-Weave** game,
  - Start with an empty graph.
  - The **Cut player** chooses a bisection.
  - The **Weave player** adds an **r-regular weave** across the bisection.

- **Theorem**: Cut player can win within $O(r \log^2 n)$ rounds.



# Step 2: Degree > (½+δ)n

Suppose **G** is D-regular with $D > (\frac{1}{2}+\delta)n$

- **Theorem**: **G** decomposes into disjoint Hamiltonian cycles.
  - [Perkovic-Reed'97, Csaba-Kühn-Lo-Osthus-Treglown'14]
- **Claim**: For any bisection in **G**, we get a weave by choosing $O(\log n)$ uniformly random cycles from the decomposition.
  - Proof: Set Cover
- **Play the Cut-Weave game** with $r = \log n$:
  - For **Cut player**, use winning strategy
  - For **Weave player**, sample random cycles to form a weave
- Resulting **H** is an $O(\log^4 n)$-regular expander subgraph of **G**.
- Extension to any $D=\Omega(n)$: No decomposition, no direct weaves…

# Conclusion

- Resistance sparsifiers of size $\tilde{O}(n/\varepsilon)$ for restricted family of graphs – dense regular expanders
- Gap between spectral and resistance sparsification
- Adaptive analysis for non-adaptive correlated sampling algorithm
- **Open questions:**
  - Improved resistance sparsifiers for more graphs?
  - Direct analysis for decompose-and-sample algorithm?

# Thank you