Graph structure in polynomial systems: chordal networks

Pablo A. Parrilo

Laboratory for Information and Decision Systems
Electrical Engineering and Computer Science
Massachusetts Institute of Technology

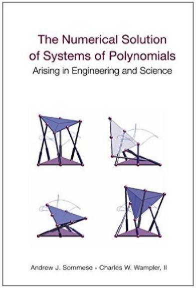
Based on joint work
with **Diego Cifuentes** (MIT)

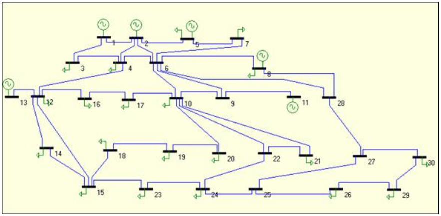
SIAM Annual Meeting - July 2016

Background: structured polynomial systems

Many application domains require the solution of large-scale systems of polynomial equations.

Among others: robotics, power systems, chemical engineering, cryptography, etc.





$$S + E \underset{k_{-1}}{\overset{k_{+1}}{\rightleftharpoons}} ES \underset{k_{-2}}{\overset{k_{+2}}{\rightleftharpoons}} P + E$$

$$\frac{d[S]}{dt} = -k_1[E][S] + k_{-1}[ES]$$

$$\frac{d[E]}{dt} = -k_1[E][S] + (k_{-1} + k_2)[ES] - k_{-2}[E][P]$$

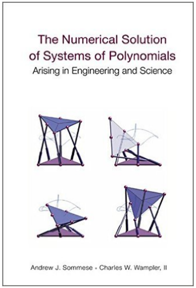
$$\frac{d[ES]}{dt} = k_1[E][S] - (k_{-1} + k_2)[ES] + k_{-2}[E][P]$$

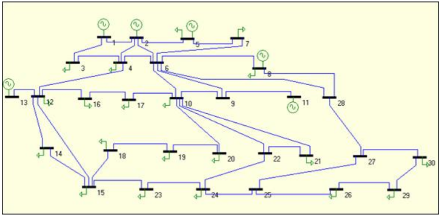
$$\frac{d[P]}{dt} = k_2[ES] - k_{-2}[E][P]$$

Background: structured polynomial systems

Many application domains require the solution of large-scale systems of polynomial equations.

Among others: robotics, power systems, chemical engineering, cryptography, etc.





$$S + E \underset{k_{-1}}{\overset{k_{+1}}{\rightleftarrows}} ES \underset{k_{-2}}{\overset{k_{+2}}{\rightleftarrows}} P + E$$

$$\frac{d[S]}{dt} = -k_1[E][S] + k_{-1}[ES]$$

$$\frac{d[E]}{dt} = -k_1[E][S] + (k_{-1} + k_2)[ES] - k_{-2}[E][P]$$

$$\frac{d[ES]}{dt} = k_1[E][S] - (k_{-1} + k_2)[ES] + k_{-2}[E][P]$$

$$\frac{d[P]}{dt} = k_2[ES] - k_{-2}[E][P]$$

Today: **MS111/MS126** - Structured Polynomial Equations and Applications

Polynomial systems and graphs

A polynomial system defined by m equations in n variables:

$$f_i(x_0, \ldots, x_{n-1}) = 0, \qquad i = 1, \ldots, m$$

Polynomial systems and graphs

A polynomial system defined by m equations in n variables:

$$f_i(x_0, \ldots, x_{n-1}) = 0, \qquad i = 1, \ldots, m$$

Construct a graph G ("primal graph") with n nodes:

- Nodes are variables $\{x_0, \ldots, x_{n-1}\}$.
- For each equation, add a clique connecting the variables appearing in that equation

Polynomial systems and graphs

A polynomial system defined by m equations in n variables:

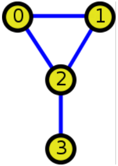
$$f_i(x_0, \ldots, x_{n-1}) = 0, \qquad i = 1, \ldots, m$$

Construct a graph G ("primal graph") with n nodes:

- Nodes are variables $\{x_0, \ldots, x_{n-1}\}$.
- For each equation, add a clique connecting the variables appearing in that equation

Example:

$$I = \langle x_0^2 x_1 x_2 + 2x_1 + 1, \quad x_1^2 + x_2, \quad x_1 + x_2, \quad x_2 x_3 \rangle$$



Questions

“Abstracted” the polynomial system to a (hyper)graph.

Questions

"Abstracted" the polynomial system to a (hyper)graph.

- Can the graph structure help *solve* this system?
- For instance, to optimize, or to compute Groebner bases?
- Or, perhaps we can do something *better*?
- Preserve graph (sparsity) structure?
- Complexity aspects?

(Hyper)Graphical modelling

Pervasive idea in many areas, in particular: numerical linear algebra, graphical models, constraint satisfaction, database theory, . . .

Key notions: chordality and treewidth.

Many names: Arnborg, Beeri/Fagin/Maier/Yannakakis, Blair/Peyton, Bodlaender, Courcelle, Dechter, Freuder, Lauritzen/Spiegelhalter, Pearl, Robertson/Seymour, . . .

(Hyper)Graphical modelling

Pervasive idea in many areas, in particular: numerical linear algebra, graphical models, constraint satisfaction, database theory, . . .

Key notions: chordality and treewidth.

Many names: Arnborg, Beeri/Fagin/Maier/Yannakakis, Blair/Peyton, Bodlaender, Courcelle, Dechter, Freuder, Lauritzen/Spiegelhalter, Pearl, Robertson/Seymour, . . .

Remarkably (AFAIK) almost no work in computational algebraic geometry exploits this structure.

(Hyper)Graphical modelling

Pervasive idea in many areas, in particular: numerical linear algebra, graphical models, constraint satisfaction, database theory, . . .

Key notions: chordality and treewidth.

Many names: Arnborg, Beeri/Fagin/Maier/Yannakakis, Blair/Peyton, Bodlaender, Courcelle, Dechter, Freuder, Lauritzen/Spiegelhalter, Pearl, Robertson/Seymour, . . .

Remarkably (AFAIK) almost no work in computational algebraic geometry exploits this structure.

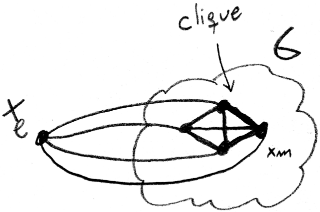
Reasonably well-known in discrete (0/1) optimization, what happens in the continuous side?
(e.g., Waki et al., Lasserre, Bienstock, Vandenberghe, Lavaei, etc)

Chordality

Let G be a graph with vertices $x_0, \ldots, x_{n-1}$.
A vertex ordering

$$x_0 > x_1 > \cdots > x_{n-1}$$

is a perfect elimination ordering if for all l, the set



$$X_l := \{x_l\} \cup \{x_m : x_m \text{ is adjacent to } x_l,\ \ x_l > x_m\}$$

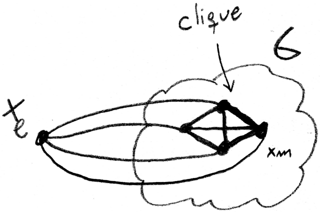
is such that the restriction $G|_{X_l}$ is a clique.

Chordality

Let G be a graph with vertices $x_0, \ldots, x_{n-1}$.
A vertex ordering

$$x_0 > x_1 > \cdots > x_{n-1}$$

is a perfect elimination ordering if for all l, the set



$$X_l := \{x_l\} \cup \{x_m : x_m \text{ is adjacent to } x_l,\ \ x_l > x_m\}$$

is such that the restriction $G|_{X_l}$ is a clique.

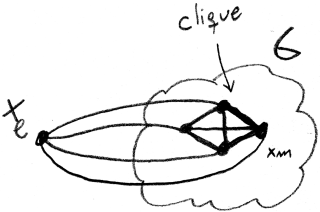
A graph is chordal if it has a perfect elimination ordering.

Chordality

Let G be a graph with vertices $x_0, \ldots, x_{n-1}$.
A vertex ordering

$$x_0 > x_1 > \cdots > x_{n-1}$$

is a perfect elimination ordering if for all l, the set

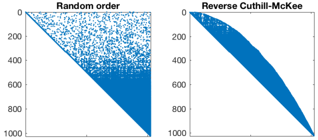


$$X_l := \{x_l\} \cup \{x_m : x_m \text{ is adjacent to } x_l,\ \ x_l > x_m\}$$

is such that the restriction $G|_{X_l}$ is a clique.

A graph is chordal if it has a perfect elimination ordering.

(Equivalently, in numerical linear algebra:
Cholesky factorization has no "fill-in")



Chordality, treewidth, and a meta-theorem

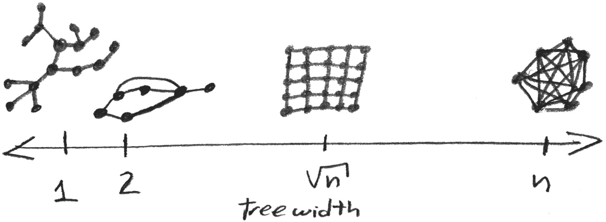
A chordal completion of G is a chordal graph with the same vertex set as G, and which contains all edges of G.

Chordality, treewidth, and a meta-theorem

A chordal completion of G is a chordal graph with the same vertex set as G, and which contains all edges of G.

The treewidth of a graph is the clique number (minus one) of its smallest chordal completion.

Informally, treewidth quantitatively measures how "tree-like" a graph is.

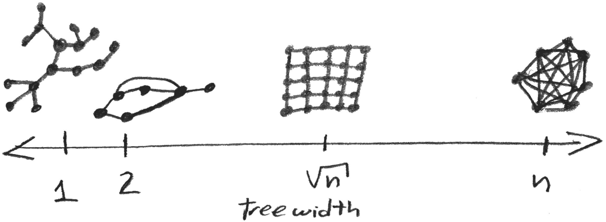


Chordality, treewidth, and a meta-theorem

A chordal completion of G is a chordal graph with the same vertex set as G, and which contains all edges of G.

The treewidth of a graph is the clique number (minus one) of its smallest chordal completion.

Informally, treewidth quantitatively measures how "tree-like" a graph is.



> Meta-theorem:
> NP-complete problems are "easy" on graphs of small treewidth.

Bad news? (I)

Recall the *subset sum* problem, with data $A = \{a_1, \dots, a_n\} \subset \mathbb{Z}$.
Is there a subset of A that adds up to 0?

Letting s_i be the partial sums, we can write a polynomial system:

$$\begin{aligned} 0 &= s_0 \\ 0 &= (s_i - s_{i-1})(s_i - s_{i-1} - a_i) \\ 0 &= s_n \end{aligned}$$

The graph associated with these equations is a path (treewidth=1)

$$s_0 - s_1 - s_2 - \cdots - s_n$$

But, subset sum is NP-complete... :(

Bad news? (II)

For *linear* equations, "good" elimination preserves graph structure (perfect!)

Bad news? (II)

For *linear* equations, “good” elimination preserves graph structure (perfect!)

For polynomials, however, Groebner bases can destroy chordality.

Ex: Consider

$$I = \langle x_0 x_2 - 1, x_1 x_2 - 1 \rangle,$$

whose associated graph is the path $x_0 - x_2 - x_1$.

Bad news? (II)

For *linear* equations, “good” elimination preserves graph structure (perfect!)

For polynomials, however, Groebner bases can destroy chordality.

Ex: Consider

$$I = \langle x_0 x_2 - 1, x_1 x_2 - 1 \rangle,$$

whose associated graph is the path $x_0 - x_2 - x_1$.

Every Groebner basis must contain the polynomial $x_0 - x_1$, breaking the sparsity structure.

Bad news? (II)

For *linear* equations, "good" elimination preserves graph structure (perfect!)

For polynomials, however, Groebner bases can destroy chordality.

Ex: Consider

$$I = \langle x_0 x_2 - 1, x_1 x_2 - 1 \rangle,$$

whose associated graph is the path $x_0 — x_2 — x_1$.

Every Groebner basis must contain the polynomial $x_0 - x_1$, breaking the sparsity structure.

Q: Are there alternative descriptions that "play nicely" with graphical structure?

How to resolve this (apparent) contradiction?

“Trees are good” $\overset{???}{\Longleftrightarrow}$ “Trees can be NP-hard”

How to resolve this (apparent) contradiction?

$$\text{“Trees are good”} \quad \overset{???}{\Longleftrightarrow} \quad \text{“Trees can be NP-hard”}$$

Underlying hero/culprit: dynamic programming (DP), and more refined cousins (nonserial DP, belief propagation, etc).

How to resolve this (apparent) contradiction?

"Trees are good" $\overset{???}{\Longleftrightarrow}$ "Trees can be NP-hard"

Underlying hero/culprit: dynamic programming (DP), and more refined cousins (nonserial DP, belief propagation, etc).

Key: "nice" graphical structure allows DP to work *in principle*. But, we also need to control the *complexity* of the objects DP is propagating. Without this, we're doomed!

[Ubiquitous theme: "complicated" value functions in optimal control, "message complexity" in statistical inference, . . .]

How to get around this?

Need to impose conditions on the geometry!

How to get around this?

Need to impose conditions on the geometry!

In the algebraic setting, a natural condition:
degree of *projections onto clique subspaces*.

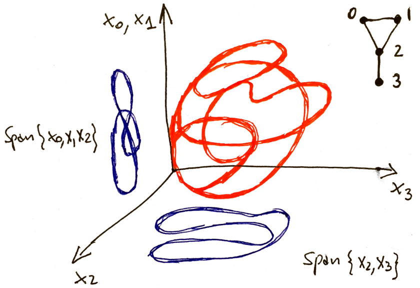
How to get around this?

Need to impose conditions on the geometry!

In the algebraic setting, a natural condition: degree of *projections onto clique subspaces*.

Consider the full solution set (an algebraic variety).

Require the projections onto the subspaces spanned by the maximal cliques to have bounded degree.



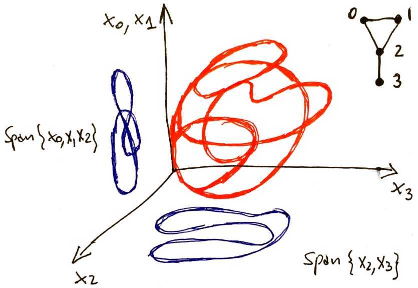
How to get around this?

Need to impose conditions on the geometry!

In the algebraic setting, a natural condition:
degree of *projections onto clique subspaces*.

Consider the full solution set
(an algebraic variety).

Require the projections onto the subspaces spanned by the maximal cliques to have bounded degree.



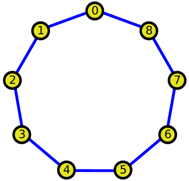
- For discrete domains (e.g., 0/1 problems), always satisfied.
- Holds in other cases, e.g., low-rank matrices (determinantal varieties).

Two approaches

- Chordal elimination and Groebner bases (arXiv:1411:1745)
  - New *chordal elimination* algorithm, to exploit graphical structure
  - Conditions under which chordal elimination succeeds
  - For a certain class, complexity is *linear* in number of variables! (exponential in treewidth)
  - Implementation and experimental results
- *Chordal networks* (arXiv:1604.02618)
  - New representation/decomposition for polynomial systems
  - Efficient algorithms to compute them. Can use them for root counting, dimension, radical ideal membership, etc.
  - Links to BDDs (binary decision diagrams) and extensions

Example 1: Coloring a cycle

Let $C_n = (V, E)$ be the cycle graph and consider the ideal I given by the equations

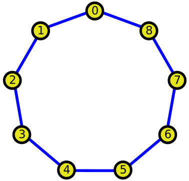


$$\begin{aligned} x_i^3 - 1 &= 0, \qquad && i \in V \\ x_i^2 + x_i x_j + x_j^2 &= 0, \qquad && ij \in E \end{aligned}$$

These equations encode the proper 3-colorings of the graph. Note that coloring the cycle graph is very easy!

Example 1: Coloring a cycle

Let $C_n = (V, E)$ be the cycle graph and consider the ideal I given by the equations



$$\begin{aligned} x_i^3 - 1 &= 0, & i &\in V \\ x_i^2 + x_i x_j + x_j^2 &= 0, & ij &\in E \end{aligned}$$

These equations encode the proper 3-colorings of the graph. Note that coloring the cycle graph is very easy!

However, a Gröbner basis is not so simple: one of its 13 elements is

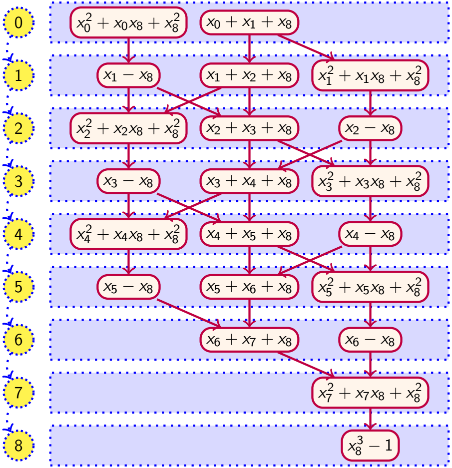
$$\begin{gathered}
x_0x_2x_4x_6 + x_0x_2x_4x_7 + x_0x_2x_4x_8 + x_0x_2x_5x_6 + x_0x_2x_5x_7 + x_0x_2x_5x_8 + x_0x_2x_6x_8 + x_0x_2x_7x_8 + x_0x_2x_8^2 + x_0x_3x_4x_6 + x_0x_3x_4x_7 \\
+x_0x_3x_4x_8 + x_0x_3x_5x_6 + x_0x_3x_5x_7 + x_0x_3x_5x_8 + x_0x_3x_6x_8 + x_0x_3x_7x_8 + x_0x_3x_8^2 + x_0x_4x_6x_8 + x_0x_4x_7x_8 + x_0x_4x_8^2 + x_0x_5x_6x_8 \\
+x_0x_5x_7x_8 + x_0x_5x_8^2 + x_0x_6x_8^2 + x_0x_7x_8^2 + x_0 + x_1x_2x_4x_6 + x_1x_2x_4x_7 + x_1x_2x_4x_8 + x_1x_2x_5x_6 + x_1x_2x_5x_7 + x_1x_2x_5x_8 \\
+x_1x_2x_6x_8 + x_1x_2x_7x_8 + x_1x_2x_8^2 + x_1x_3x_4x_6 + x_1x_3x_4x_7 + x_1x_3x_4x_8 + x_1x_3x_5x_6 + x_1x_3x_5x_7 + x_1x_3x_5x^8 + x_1x_3x_6x_8 + x_1x_3x_7x_8 \\
+x_1x_3x_8^2 + x_1x_4x_6x_8 + x_1x_4x_7x_8 + x_1x_4x_8^2 + x_1x_5x_6x_8 + x_1x_5x_7x_8 + x_1x_5x_8^2 + x_1x_6x_8^2 + x_1x_7x_8^2 + x_1 + x_2x_4x_6x_8 + x_2x_4x_7x_8 \\
+x_2x_4x_8^2 + x_2x_5x_6x_8 + x_2x_5x_7x_8 + x_2x_5x_8^2 + x_2x_6x_8^2 + x_2x_7x_8^2 + x_2 + x_3x_4x_6x_8 + x_3x_4x_7x_8 + x_3x_4x_8^2 + x_3x_5x_6x_8 + x_3x_5x_7x_8 \\
+x_3x_5x_8^2 + x_3x_6x_8^2 + x_3x_7x_8^2 + x_3 + x_4x_6x_8^2 + x_4x_7x_8^2 + x_4 + x_5x_6x_8^2 + x_5x_7x_8^2 + x_5 + x_6 + x_7 + x_8
\end{gathered}$$

Example 1: Coloring a cycle

There is a nicer representation, that respects its graphical structure.
The solution set can be *decomposed* into *triangular* sets:

$$\mathcal{V}(I) = \bigcup_{T} \mathcal{V}(T)$$

where the union is over all *maximal directed paths* in the figure.
The number of triangular sets is 21, which is the 8-th Fibonacci number.



Chordal networks

A new representation of structured polynomial systems!

- What do they look like?
  - "Enlarged" elimination tree, with polynomial sets as nodes.
  - Efficient encoding of components in paths/subtrees.

Chordal networks

A new representation of structured polynomial systems!

- What do they look like?
  - "Enlarged" elimination tree, with polynomial sets as nodes.
  - Efficient encoding of components in paths/subtrees.
- How can you compute them?
  - A nice algorithm to compute chordal networks.
  - Remarkably, many polynomial systems admit "small" chordal networks, even though the number of components may be exponentially large.

Chordal networks

A new representation of structured polynomial systems!

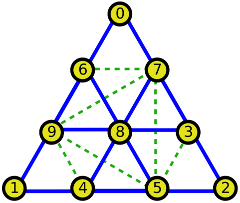
- What do they look like?
  - "Enlarged" elimination tree, with polynomial sets as nodes.
  - Efficient encoding of components in paths/subtrees.
- How can you compute them?
  - A nice algorithm to compute chordal networks.
  - Remarkably, many polynomial systems admit "small" chordal networks, even though the number of components may be exponentially large.
- What are they good for?
  - Can be effectively used to solve feasibility, counting, dimension, elimination, radical membership, ...
  - Linear time algorithms (exponential in treewidth)
  - Implementation and experimental results.

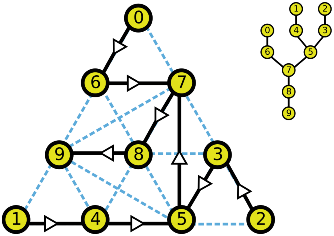
Elimination tree of a chordal graph

The elimination tree of a graph G is the following *directed spanning tree*:

For each ℓ there is an arc from x_ℓ towards the largest x_p that is adjacent to x_ℓ and $p > \ell$.

Note that the elimination tree is rooted at x_{n-1}.





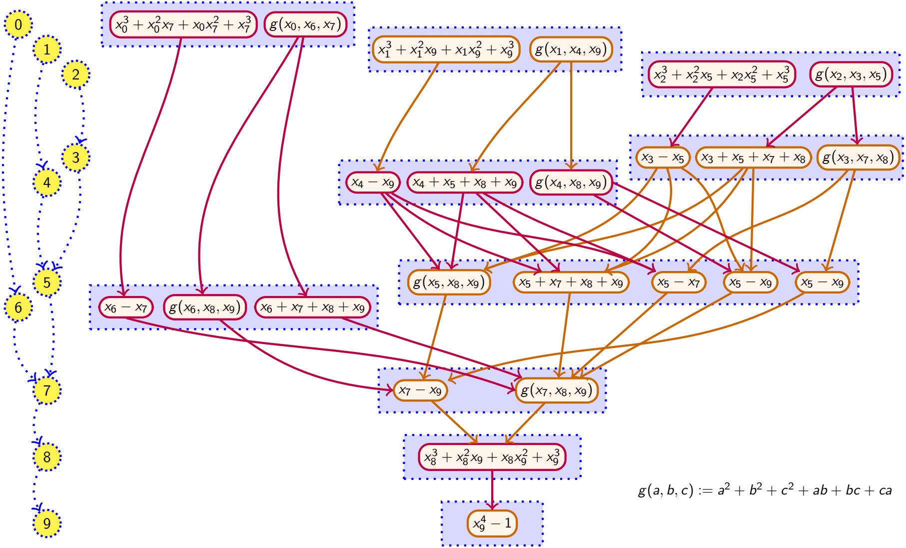
Chordal networks (definition)

A *G-chordal network* is a directed graph $\mathcal{N}$, whose nodes are polynomial sets in $\mathbb{K}[X]$, such that:

- each node F is given a $\text{rank}(F) \in \{0, \ldots, n-1\}$, s.t. $F \subset \mathbb{K}[X_{\text{rank}(F)}]$.
- for any arc (F_ℓ, F_p) we have that x_p is the parent of x_ℓ in the elimination tree of G, where $\ell = \text{rank}(F_\ell), p = \text{rank}(F_p)$.

A chordal network is *triangular* if each node consists of a single polynomial f, and either $f = 0$ or its largest variable is $x_{\text{rank}(f)}$.

Chordal networks (Example)

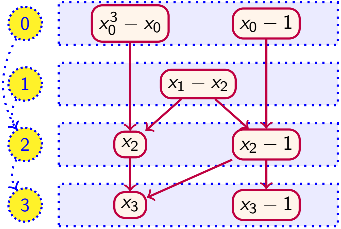


$$g(a, b, c) := a^2 + b^2 + c^2 + ab + bc + ca$$

Computing chordal networks (Example)

$$I = \langle x_2 - x_3, x_1 - x_2, x_1^2 - x_1, x_0x_2 - x_2, x_0^3 - x_0 \rangle$$

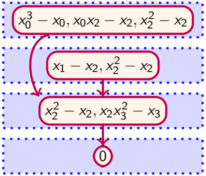
The output of the algorithm will be



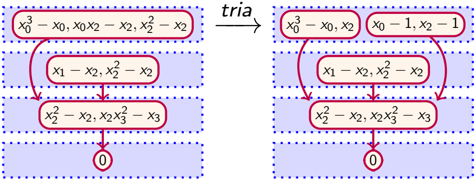
This represents the decomposition of I into the triangular sets

$$(x_3, x_2, x_1 - x_2, x_0^3 - x_0),$$
$$(x_3, x_2 - 1, x_1 - x_2, x_0 - 1),$$
$$(x_3 - 1, x_2 - 1, x_1 - x_2, x_0 - 1).$$

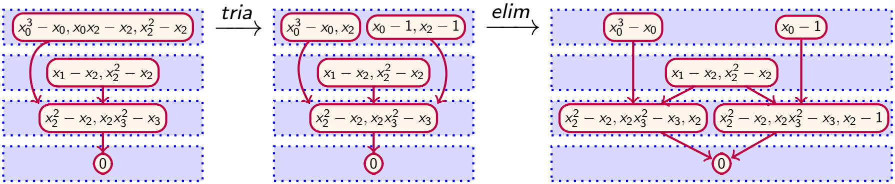
Computing chordal networks (Example)



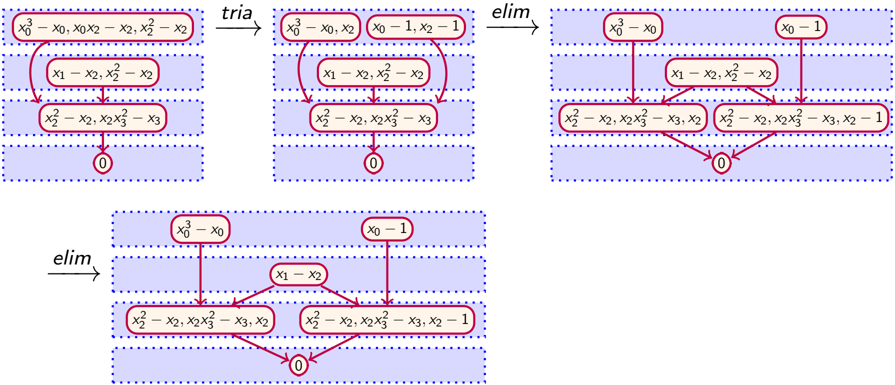
Computing chordal networks (Example)



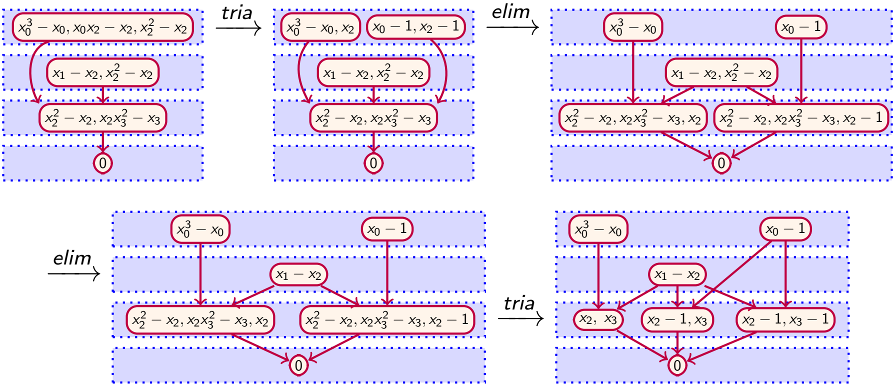
Computing chordal networks (Example)



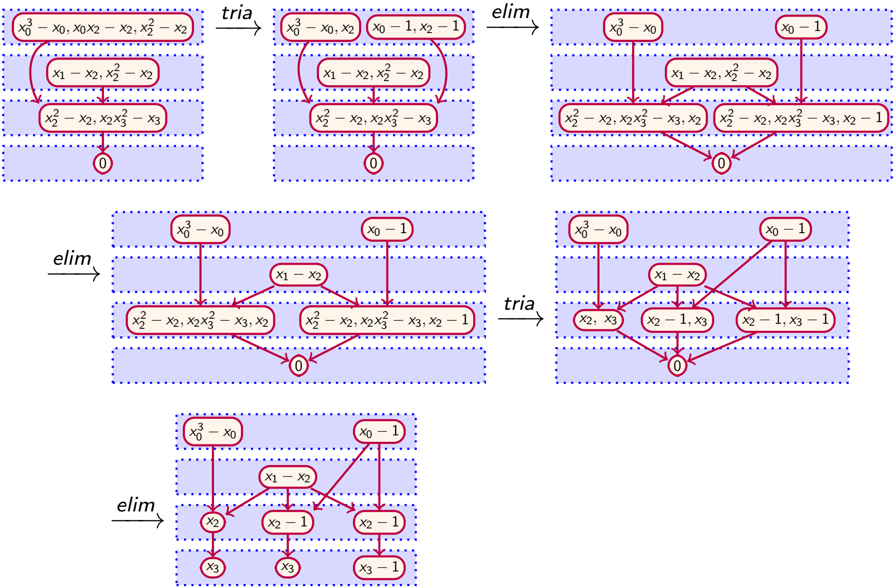
Computing chordal networks (Example)



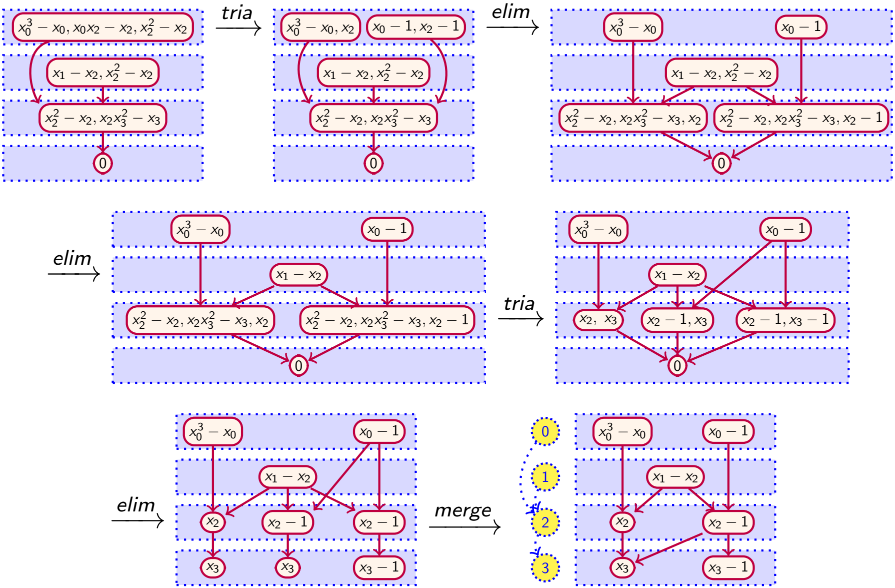
Computing chordal networks (Example)



Computing chordal networks (Example)



Computing chordal networks (Example)



Chordal networks in computational algebra

Given a triangular chordal network $\mathcal{N}$ of a polynomial system, the following problems can be solved in linear time:

- Compute the cardinality of $\mathcal{V}(I)$.
- Compute the dimension of $\mathcal{V}(I)$
- Describe the top dimensional component of $\mathcal{V}(I)$.

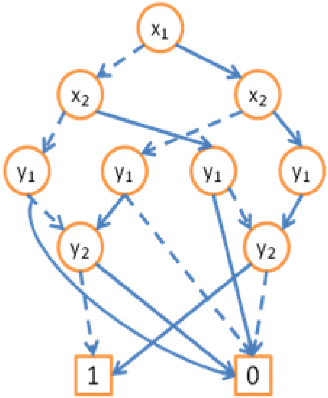
We also developed efficient algorithms to

- Solve the radical ideal membership problem ($h \in \sqrt{I}$?)
- Compute the equidimensional components of the variety.

Links to BDDs

Very interesting connections with *binary decision diagrams* (BDDs).

- A clever representation of Boolean functions/sets, usually much more compact than naive alternatives
- Enabler of very significant practical advances in (discrete) formal verification and model checking
- "One of the only really fundamental data structures that came out in the last twenty-five years" (D. Knuth)



For the special case of *monomial ideals*, chordal networks are equivalent to (reduced, ordered) BDDs. But in general, more powerful!

Implementation and examples

Implemented in Sage, using Singular and PolyBoRi (for $\mathbb{F}_2$).

- Graph colorings (counting q-colorings)
- Cryptography ("baby" AES, Cid *et al.*)
- Sensor Network localization
- Discretization of polynomial equations
- Reachability in vector addition systems
- Algebraic statistics

I - Vector addition systems

Given a set of vectors $\mathcal{B} \subset \mathbb{Z}^n$, construct a graph with vertex set $\mathbb{N}^n$ in which $u, v \in \mathbb{N}^n$ are adjacent if $u - v \in \pm\mathcal{B}$.

Ex: Determine whether $f_n \in I_n$, where

$$f_n := x_0 x_1^2 x_2^3 \cdots x_{n-1}^n - x_0^n x_1^{n-1} \cdots x_{n-1},$$
$$I_n := \{x_i x_{i+3} - x_{i+1} x_{i+2} : 0 \leq i < n\},$$

and where the indices are taken modulo n.

We compare our radical membership test with Singular (Gröbner bases) and Epsilon (triangular decompositions).

n	5	10	15	20	25	30	35	40	45	50	55
ChordalNet	0.7	3.0	8.5	14.3	21.8	29.8	37.7	48.2	62.3	70.6	84.8
Singular	0.0	0.0	0.2	17.9	1036.2	-	-	-	-	-	-
Epsilon	0.1	0.2	0.4	2.0	54.4	160.1	5141.9	17510.1	-	-	-

II - Algebraic statistics (Evans et al.)

Consider the binomial ideal I^{n,n_2} that models a 2D dimensional generalization of the birth-death Markov process. We fix $n_2 = 1$.

We can compute all irreducible components of the ideal faster than specialized packages (e.g., Macaulay2's "Binomials")

n		1	2	3	4	5	6	7
#components		3	11	40	139	466	1528	4953
time	ChordalNet	0:00:00	0:00:01	0:00:04	0:00:13	0:02:01	0:37:35	12:22:19
	Binomials	0:00:00	0:00:00	0:00:01	0:00:12	0:03:00	4:15:36	-

II - Algebraic statistics (Evans et al.)

Consider the binomial ideal I^{n,n_2} that models a 2D dimensional generalization of the birth-death Markov process. We fix $n_2 = 1$.

We can compute all irreducible components of the ideal faster than specialized packages (e.g., Macaulay2's "Binomials")

n		1	2	3	4	5	6	7
#components		3	11	40	139	466	1528	4953
time	ChordalNet	0:00:00	0:00:01	0:00:04	0:00:13	0:02:01	0:37:35	12:22:19
	Binomials	0:00:00	0:00:00	0:00:01	0:00:12	0:03:00	4:15:36	-

Our methods are particularly efficient for computing the highest dimensional components.

	Highest 5 dimensions					Highest 7 dimensions			
n	20	40	60	80	100	10	20	30	40
#comps	404	684	964	1244	1524	2442	5372	8702	12432
time	0:01:07	0:04:54	0:15:12	0:41:52	1:34:05	0:05:02	0:41:41	3:03:29	9:53:09

Summary

- (Hyper)graphical structure *may* simplify optimization/solving
- Under assumptions (treewidth + algebraic structure), tractable!
- New data structures: chordal networks
- Yields practical, competitive, implementable algorithms
- Ongoing and future work: other polynomial solving approaches (e.g., homotopies, full numerical algebraic geometry...)

Summary

- (Hyper)graphical structure *may* simplify optimization/solving
- Under assumptions (treewidth + algebraic structure), tractable!
- New data structures: chordal networks
- Yields practical, competitive, implementable algorithms
- Ongoing and future work: other polynomial solving approaches (e.g., homotopies, full numerical algebraic geometry...)

If you want to know more:

- D. Cifuentes, P.A. Parrilo, Exploiting chordal structure in polynomial ideals: a Groebner basis approach. *SIAM J. of Discrete Mathematics*, to appear. `arXiv:1411.1745`.
- D. Cifuentes, P.A. Parrilo, An efficient tree decomposition method for permanents and mixed discriminants, *Linear Algebra and Appl.*, 493:45–81, 2016. `arXiv:1507.03046`.
- D. Cifuentes, P.A. Parrilo, Chordal networks of polynomial ideals. `arXiv:1604.02618`.

Thanks for your attention!

Summary

- (Hyper)graphical structure *may* simplify optimization/solving
- Under assumptions (treewidth + algebraic structure), tractable!
- New data structures: chordal networks
- Yields practical, competitive, implementable algorithms
- Ongoing and future work: other polynomial solving approaches (e.g., homotopies, full numerical algebraic geometry...)

If you want to know more:

- D. Cifuentes, P.A. Parrilo, Exploiting chordal structure in polynomial ideals: a Groebner basis approach. *SIAM J. of Discrete Mathematics*, to appear. `arXiv:1411.1745`.
- D. Cifuentes, P.A. Parrilo, An efficient tree decomposition method for permanents and mixed discriminants, *Linear Algebra and Appl.*, 493:45–81, 2016. `arXiv:1507.03046`.
- D. Cifuentes, P.A. Parrilo, Chordal networks of polynomial ideals. `arXiv:1604.02618`.

Thanks for your attention! (and please come to MS111/MS126!)