Nonlinear Control Synthesis by Convex Optimization

Stephen Prajna, Pablo A. Parrilo, and Anders Rantzer

***Abstract*— A stability criterion for nonlinear systems, recently derived by the third author, can be viewed as a dual to Lyapunov's second theorem. The criterion is stated in terms of a function which can be interpreted as the stationary density of a substance that is generated all over the state space and flows along the system trajectories towards the equilibrium.**

The new criterion has a remarkable convexity property, which in this paper is used for controller synthesis via convex optimization. Recent numerical methods for verification of positivity of multivariate polynomials based on sum of squares decompositions are used.

I. Introduction

LYAPUNOV functions have long been recognized as one of the most fundamental analytical tools for analysis and synthesis of nonlinear control systems. See for example [2], [3], [4], [6], [7], [9].

There has also been a strong development of computational tools based on Lyapunov functions. Many such methods are based on convex optimization and solution of matrix inequalities, exploiting the fact that the set of Lyapunov functions for a given system is convex.

A serious obstacle in the problem of controller synthesis is however that the joint search for a controller $u(x)$ and a Lyapunov function $V(x)$ is not convex. Consider the synthesis problem for the system

$$\dot{x} = f(x) + g(x)u.$$

The set of u and V satisfying the condition

$$\frac{\partial V}{\partial x}[f(x) + g(x)u(x)] < 0$$

is not convex. In fact, for some systems the set of u and V satisfying the inequality is not even connected [14].

Given the difficulties with Lyapunov based controller synthesis, it is most striking to find that the new convergence criterion presented in [15] based on the so-called density function ρ (cf. Section II) has much better convexity properties. Indeed, the set of $(\rho, u\rho)$ satisfying

$$\nabla \cdot [\rho(f + gu)] > 0 \tag{1}$$

is convex. In this paper, we will exploit this fact in the computation of stabilizing controllers. For the case of systems with polynomial or rational vector fields, the search for a candidate pair $(\rho, u\rho)$ verifying the inequality (1) can be done using the methods introduced in [12]. In particular, a recently available software SOSTOOLS [13] can be used for this purpose.

II. The convergence criterion

The main result of [15] can be stated as follows:

Theorem 1: Given the system $\dot{x}(t) = f(x(t))$, where $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ and $f(0) = 0$, suppose there exists a non-negative $\rho \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R})$ such that $\rho(x)f(x)/|x|$ is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$ and

$$[\nabla \cdot (\rho f)](x) > 0 \qquad \text{for almost all } x \tag{2}$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

The proof is based on the following lemma, which can be viewed as a version of Liouville's theorem [1], [10].

Lemma 1: Let $f \in \mathbf{C}^1(D, \mathbf{R}^n)$ where $D \subset \mathbf{R}^n$ is open and let $\rho \in \mathbf{C}^1(D, \mathbf{R})$ be integrable. For $x_0 \in \mathbf{R}^n$, let $\phi_t(x_0)$ be the solution $x(t)$ of $\dot{x} = f(x)$, $x(0) = x_0$. For a measurable subset Z, assume that $\phi_\tau(Z) = \{\phi_\tau(x) : x \in Z\}$ is a subset of D for all τ between 0 and t. Then

$$\int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(z)dz = \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (\rho f)](x)dxd\tau$$

Proof of Theorem 1, second statement. Here it is assumed that $x = 0$ is a stable equilibrium, while ρ may take negative values. The proof for the other case can be found in [15].

Rather than exploiting that $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$, we will actually prove the result under the weaker condition that $f \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R}^n)$ and $f(x)$ is locally Lipschitz continuous at $x = 0$. Given any $x_0 \in \mathbf{R}^n$, let $\phi_t(x_0)$ for $t \geq 0$ be the solution $x(t)$ of $\dot{x}(t) = f(x(t))$, $x(0) = x_0$. Assume first that ρ is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$ and $|f(x)|/|x|$ is bounded. Then ϕ_t is well defined for all t. Given $r > 0$, define

$$Z = \cap_{l=1}^{\infty} \{x_0 \; : \; |\phi_t(x_0)| > r \text{ for some } t > l\} \tag{3}$$

Notice that Z contains all trajectories with $\limsup_{t\to\infty} |x(t)| > r$. The set Z, being the intersection of a countable number of open sets, is measurable. Moreover, $\phi_t(Z) = \big\{\phi_t(x) \;\big|\; x \in Z\big\}$ is equal to Z for every t. By stability of the equilibrium $x = 0$, there is a positive lower

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S. Prajna is with Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125, USA. Email: prajna@cds.caltech.edu

P. A. Parrilo is with the Automatic Control Laboratory, ETH Zürich, CH-8092 Zürich, Switzerland. Email: parrilo@aut.ee.ethz.ch

A. Rantzer is with the Department of Automatic Control, Lund Institute of Technology, Box 118, S-221 00 Lund, Sweden. Email: rantzer@control.lth.se

bound ϵ on the norm of the elements in Z, so Lemma 1 with $D = \{x \ : \ |x| > \epsilon\}$ gives

$$\begin{aligned} 0 &= \int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(z)dz \\ &= \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (\rho f)](x)dxd\tau \end{aligned} \tag{4}$$

By the assumption (2), this implies that Z has measure zero. Consequently, $\limsup_{t\to\infty} |x(t)| \leq r$ for almost all trajectories. As r was chosen arbitrarily, this proves that $\lim_{t\to\infty} |x(t)| = 0$ for almost all trajectories.

When $|f(x)|/|x|$ is unbounded, there may not exist any nonzero t such that $\phi_t(z)$ is well defined for all z. We then introduce

$$\begin{aligned} \rho_0(x) &= \left[\frac{e^{-|x|}}{1+|\rho(x)|^2} + \frac{|f(x)|^2}{|x|^2}\right]^{1/2} \rho(x) \\ f_0(x) &= \frac{f(x)\rho(x)}{\rho_0(x)} \end{aligned}$$

Then $|f_0(x)|/|x|$ is bounded and ρ_0 is integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$, so the argument above can be applied to f_0 together with ρ_0 to prove that $\lim_{\tau\to\infty} |y(\tau)| = 0$ for almost all trajectories of the system $dy/d\tau = f_0(y(\tau))$. However, modulo a transformation of the time axis

$$t = \int_0^\tau \frac{\rho(y(s))}{\rho_0(y(s))} ds$$

the trajectories are identical: $x(t) = y(\tau)$. This, together with local Lipschitz continuity of $f(x)$ at $x = 0$, also shows that $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$ provided that $\lim_{\tau\to\infty} |y(\tau)| = 0$. Hence the proof of the second statement in Theorem 1 is complete.

III. A COMPUTATIONAL APPROACH

In order to understand the possibilities and limitations of computational approaches to nonlinear stability, an issue that has to be addressed is how to deal numerically with functional inequalities such as the standard Lyapunov one, or the divergence inequality (1).

Even in the restricted case of polynomial functions, it is well-known that the problem of checking global nonnegativity of a polynomial of quartic (or higher) degree is computationally hard. For this reason, we need tractable sufficient conditions that guarantee nonnegativity, and that are not overly conservative. A particularly interesting sufficient condition is given by the existence of a sum of squares decomposition: can the polynomial $P(x)$ be written as $P(x) = \sum_i p_i^2(x)$, for some polynomials $p_i(x)$? Obviously, if this is the case, then $P(x)$ takes only nonnegative values. Notice that in the case of quadratic forms, for instance, the two conditions (nonnegativity and sum of squares) are equivalent.

In this respect, it is interesting to notice that many methods used in control theory for constructing Lyapunov functions (for example, backstepping) use either implicitly of explicitly a sum of squares approach.

As shown in [12], the problem of checking if a given polynomial can be written as a sum of squares can be solved via convex optimization, in particular semidefinite programming. We refer the reader to that work for a discussion of the specific algorithms. For our purposes, however, it will be enough to know that while the standard semidefinite programming machinery can be interpreted as searching for a positive semidefinite element over an affine family of quadratic forms, the new tools provide a way of *finding a sum of squares, over an affine family of polynomials*. For instance, these tools can be used in the computation of Lyapunov functions for proving that a nonlinear system is stable [12], [11].

To apply these tools to the stabilization problem addressed in this paper, consider the following parameterized representation for ρ and $u\rho$:

$$\rho(x) = \frac{a(x)}{b(x)^\alpha}, \qquad u(x)\rho(x) = \frac{c(x)}{b(x)^\alpha},$$

where $a(x), b(x), c(x)$ are polynomials, $b(x)$ is positive, and α is chosen large enough so as to satisfy the integrability condition in Theorem 1. Note that by choosing this particular representation, we presuppose that we will be searching for ρ and u that are rationals. In this case, condition (1) can be written as:

$$\begin{aligned} \nabla \cdot [\rho(f+gu)] &= \nabla \cdot [\frac{1}{b^\alpha}(fa+gc)] \\ &= \frac{1}{b^{\alpha+1}}[b\nabla \cdot (fa+gc) - \alpha \nabla b \cdot (af+gc)]. \end{aligned}$$

Since b is positive, we only need to satisfy the inequality:

$$b\nabla \cdot (fa+gc) - \alpha \nabla b \cdot (af+gc) > 0. \tag{5}$$

For fixed b, α, the inequality is linear in a, c. If instead of checking positivity, we check that the left-hand side is a *sum of squares*, the problem can be solved using semidefinite programming.

Some numerical examples will be presented in the next sections to illustrate how the controller synthesis can be performed. The sum of squares conditions corresponding to these problems are solved using SOSTOOLS [13].

IV. SOME EXAMPLES

A. Example 1

A simple numerical example is the following:

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^3 + x_1^2 \\ \dot{x}_2 &= u \end{aligned}$$

The function $b(x)$ is chosen based on the linearization of the system. We picked $b(x) := 3x_1^2 + 2x_1x_2 + 2x_2^2$, which is a control Lyapunov function (CLF) for the linearized system, and therefore, $b(x)^{-\alpha}$ (for some α) will be a good choice for a ρ function near the origin. Since we will be using a cubic polynomial for $c(x)$, and $a(x)$ is taken to be a constant, we choose $\alpha = 4$ to satisfy the integrability condition.

In this case, after solving the sum of squares inequality (5), we obtain an explicit expression for the controller, as a third order polynomial in x_1 and x_2. The optimization criterion chosen

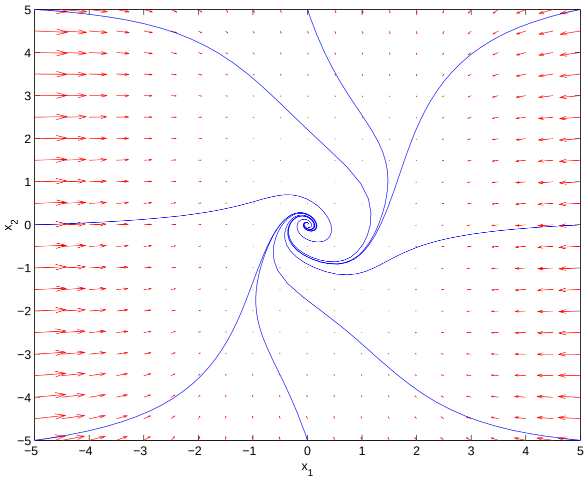

Fig. 1. Phase plot of the closed-loop system in Example 1. Solid curves are trajectories of the system.

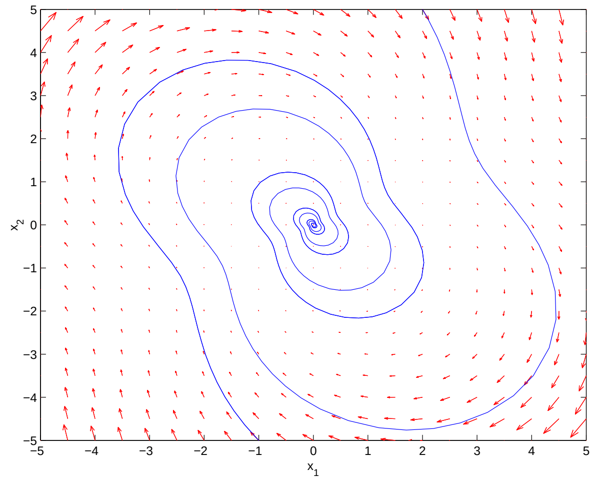

Fig. 2. Phase portrait of the closed-loop system in Example 2. Solid curves are trajectories of the system.

is the ℓ_1 norm of the coefficients. This way, we approximately try to minimize the number of nonzero terms [5]. The expression for the final controller is:

$$u(x_1, x_2) = -1.22x_1 - 0.57x_2 - 0.129x_2^3.$$

A phase plot of the closed-loop system is presented in Figure 1.

This example has been chosen for its relative simplicity: in this particular case, it is possible to solve it directly using other methodologies. For instance, it can be noted that in this particular case $b(x)$ is actually a CLF for the full nonlinear system, and from that we can obtain a controller, e.g. using Sontag's formula. There is no requirement in the present framework that requires $b(x)$ to be a CLF, as we will see in the following subsections. The main difference would be in terms of the computational difficulty of approximating the controller when the choice of the denominator $b(x)$ is not optimal.

B. *Example 2*

Consider the following homogeneous system, whose linearization is not stabilizable:

$$\begin{aligned}\dot{x}_1 &= 2x_1^3 + x_1^2x_2 - 6x_1x_2^2 + 5x_2^3\\ \dot{x}_2 &= u.\end{aligned}$$

Since a CLF cannot be found for the linearized system, we will simply use a "generic" function $b(x)^\alpha = (x_1^2 + x_2^2)^\alpha$ as the denominator of our density function. Notice in particular that this function (as well as other generic denominators such as $x_1^n + x_2^n$ and $(x_1^n + x_2^n)^\alpha$, where n is an even positive integer) is *not* a CLF for the system.

For a controller that is a polynomial of degree 3 (the same degree as the drift vector field) and $\rho(x) = \frac{a(x)}{b(x)^\alpha}$, the integrability condition is fulfilled if the degree of $a(x)$ satisfies $\deg(a(x)) \leq 2\alpha - 5$. We choose $\alpha = 2.5$ and use a constant $a(x)$. For the chosen α and $b(x)$, the positivity of $\nabla \cdot (f + gu)$ is established by $a(x) = 1$ and

$$c(x) = -3.6345x_1^3 + 4.4439x_1^2x_2 - 7.5113x_1x_2^2 - 3.5452x_2^3.$$

The phase portrait of the closed-loop system with controller $u(x) = c(x)$ is shown in Figure 2. The origin is globally asymptotically stable, as can also be proven using a polynomial Lyapunov function of degree 6.

C. *Example 3*

For the third example, we consider the system

$$\begin{aligned}\dot{x}_1 &= -6x_1x_2^2 - x_1^2x_2 + 2x_2^3,\\ \dot{x}_2 &= x_2u.\end{aligned}$$

It is straightforward to see that the equilibrium at the origin cannot be made asymptotically stable, because every $x = (x_1, x_2)$ with $x_2 = 0$ will necessarily be an equilibrium of the closed-loop system. Nevertheless, it is possible to design a controller which makes almost all trajectories converge to the origin.

Lyapunov design using non-strict Lyapunov function (i.e., a Lyapunov function whose time derivative is only negative semidefinite) combined with LaSalle's invariance principle will only prove that the trajectories of the closed-loop system converge to $\mathcal{D} = \{x \mid x_2 = 0\}$. Therefore we will instead resort to the method described in this paper to design a controller that makes the origin almost globally attractive (i.e., almost all trajectories converge to the origin). Choosing $b(x) = x_1^2 + x_2^2$ and $\alpha = 3$, we find that the positivity of $b\nabla\cdot(fa+gc) - \alpha\nabla b\cdot(fa+gc)$ is fulfilled for $a(x) = 1$ and $c(x) = 2.229x_1^2 - 4.8553x_2^2$. Since the integrability condition is also satisfied, we conclude that the controller $u(x) = \frac{c(x)}{a(x)} = 2.229x_1^2 - 4.8553x_2^2$ renders the origin almost globally attractive. The phase portrait of the closed loop system is shown in Figure 3.

V. Application: Attitude Control of a Rigid Body

We will now look at the attitude control of a rigid body using three inputs as a physically motivated example. The complete attitude dynamics of a rigid body can e.g. be described using the following state equations [16]:

$$\begin{aligned}\dot{\omega} &= J^{-1}S(\omega)J\omega + J^{-1}u,\\ \dot{\psi} &= H(\psi)\omega,\end{aligned}$$

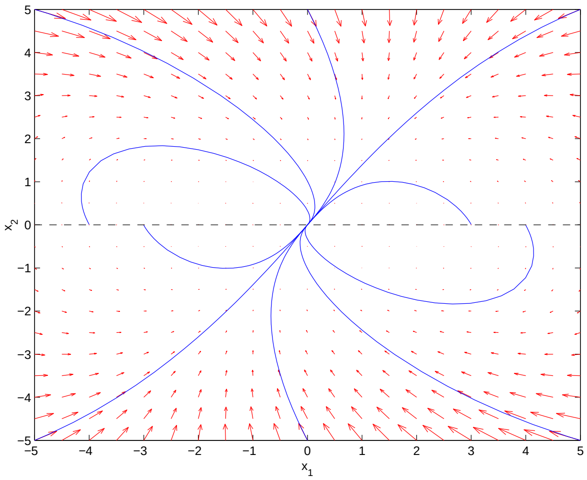

Fig. 3. Phase portrait of the closed-loop system in Example 3. Solid curves are trajectories; dashed line is the set of equilibria.

with $\omega \in \mathbf{R}^3$ the angular velocity vector in a body-fixed frame, $\psi \in \mathbf{R}^3$ the Rodrigues parameter vector, and $u \in \mathbf{R}^3$ the control torque. The matrix J is the positive definite inertia matrix, while $S(\omega)$ and $H(\psi)$ are given by

$$
\begin{aligned}
S(\omega) &= \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}, \\
H(\psi) &= \frac{1}{2}(I - S(\psi) + \psi\psi^T).
\end{aligned}
$$

We will apply the method described in the previous sections to numerically construct a stabilizing controller for this system. Synthesis of stabilizing controller for this system can also be performed e.g. using backstepping [8]. In our construction, the matrix $J = \text{diag}(4, 2, 1)$ will be chosen as the inertia matrix.

First, a density function of the following type is used:

$$\rho(\omega, \psi) = \frac{a(\omega, \psi)}{(\|\omega\|^2 + \|\psi\|^2)^\alpha}, \tag{6}$$

where $a(\omega, \psi)$ is obtained from convex optimization. Using this density function and $\alpha = 6$, it is possible to obtain a controller of the form:

$$u_i(\omega, \psi) = \frac{c_i(\omega, \psi)}{a(\omega, \psi)}, \qquad i = 1, 2, 3,$$

with $a(\omega, \psi)$ being positive definite. In fact, the function $a(\omega, \psi)$ is a homogeneous polynomial of degree 2, whereas the $c_i(\omega, \psi)$'s are polynomials of degree 5. Since the lowest degree of the monomials in $c_i(\omega, \psi)$ is equal to 3, we have $\lim_{(\omega,\psi)\to 0} u_i(\omega, \psi) = 0$, and thus we may set $u_i(0, 0) = 0$ to obtain a continuous controller as well as to make the origin an equilibrium of the closed loop system.

Controllers with simpler expressions can be obtained by choosing a CLF of the linearized system, such as

$$b(\omega, \psi) = \|\omega + \psi\|^2 + \|\psi\|^2, \tag{7}$$

or

$$b(\omega, \psi) = \|\omega + \psi\|^2 + \|\omega\|^2 \tag{8}$$

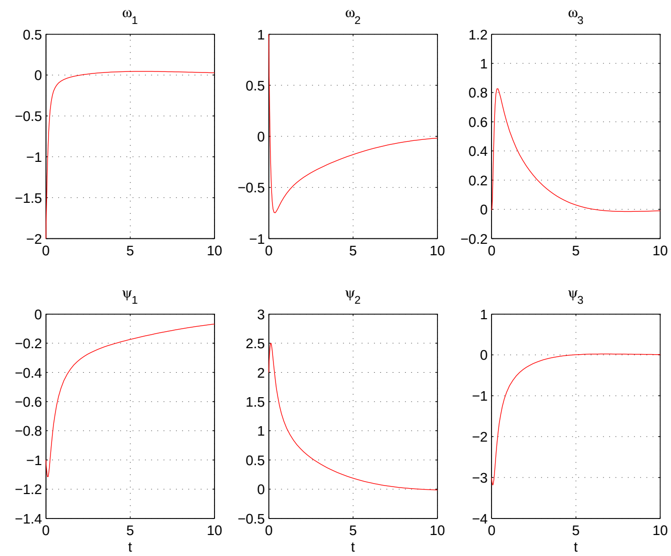

Fig. 4. Trajectory of the controlled rigid body.

for the denominator of the density function. Using (8) as the denominator and α again equal to 6, the controller obtained from convex optimization is given in (9).

A trajectory of the closed-loop system starting at $(\omega, \psi) = (-2, 1, 0, -1, 2, -3)$ is shown in Figure 4.

VI. Concluding Remarks

A new computational approach to nonlinear control synthesis has been introduced. The basis is a recent convergence criterion introduced by the third author. The criterion is closely related to earlier work on optimal control [18], [17] and makes it possible to state the synthesis problem in terms of convex optimization. Polynomials are used for parameterization and positivity is verified and certified using the ideas in [12] and the software [13].

In general, a controller designed using the proposed approach is only guaranteed (by Theorem 1) to make almost all trajectories of the closed-loop system tend to the origin. In many cases, however, such a controller will actually be globally asymptotically stabilizing. If necessary, global asymptotic stability of the closed-loop system can be verified by constructing a Lyapunov function, for which a similar computational approach can be utilized [12], [11].

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\begin{aligned}
u_1(\omega,\psi) &= -.49\psi_1^3 - .86\omega_1^3 - 1.2\omega_1\psi_1^2 - 1.5\omega_1\psi_2^2 - 1.1\omega_1\psi_3^2 + .37\omega_1^2\psi_1 - 2.6\omega_1 - .77\psi_1 + .035\omega_2\psi_1\psi_2 \\
u_2(\omega,\psi) &= -.28\psi_2^3 - .29\omega_2^3 - .27\omega_2\psi_1^2 + .17\omega_2^2\psi_2 - .37\psi_1^2\psi_2 - .69\omega_2\psi_2^2 - 1.1\omega_2\psi_3^2 - .45\psi_2\psi_3^2 - 1.1\omega_1^2\omega_2 \\
&\quad -.44\omega_1\psi_1\psi_2 - .46\psi_2 - 1.1\omega_2 + .24\omega_1\omega_2\psi_1, \\
u_3(\omega,\psi) &= -.14\psi_3^3 - .18\omega_3^3 - .44\omega_1^2\omega_3 - .34\omega_2^2\omega_3 - .55\omega_3\psi_2^2 + .11\omega_1^2\psi_3 + .052\omega_3^2\psi_3 - .18\psi_1^2\psi_3 - .039\psi_2^2\psi_3 \\
&\quad -.2\omega_2^2\psi_3 - .38\omega_3\psi_3^2 + .4\omega_2\omega_3\psi_2 + .37\omega_1\omega_3\psi_1 + .43\omega_2\psi_2\psi_3 - .69\omega_3 - .35\psi_3. \qquad (9)
\end{aligned}
$$

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