Correspondence

Equivalent Models for Queueing Analysis of Deterministic Service Time Tree Networks

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***Abstract*—In this correspondence, we analyze feedforward tree networks of queues serving fixed-length packets. Using sample path conservation properties and stochastic coupling techniques, we analyze these systems without making any assumptions about the nature of the underlying input processes. In the case when the server rate is the same for all queues, the exact packet occupancy distribution in any queue of a multistage network is obtained in terms of a reduced two-stage equivalent model. Simple and exact expressions for occupancy mean and variance are derived from this result, and the network is shown to exhibit a natural traffic smoothing property, where preliminary stages act to smooth or improve traffic for downstream nodes. In the case of heterogeneous server rates, a similar type of smoothing is demonstrated, and upper bounds on the backlog distribution are derived. These bounds hold for general input streams and are tighter than currently known bounds for leaky bucket and stochastically bounded bursty traffic.**

***Index Terms*—Network calculus, stochastic coupling.**

I. Introduction

Many modern data networks transmit information using fixed-length packets. Often this takes place at lower network protocol layers, where variable-length packets from a source are segmented into fixed-length cells for transmission over a subnetwork. Such data segmentation facilitates network design and control and allows for many practical advantages in terms of pipelining gains, congestion control, and fairness issues. Fixed-length packets are also advantageous from a queueing theory perspective, as they minimize queue backlog among all packet length distributions with the same mean [3]. It is thus important to develop methods for understanding and analyzing networks of deterministic service time queues.

In this correspondence, we consider feedforward tree networks with arbitrary traffic streams exogenously entering each node (Fig. 1). Packets from these streams flow through the multiple stages of the tree toward a single-output port at the head node. All packets have length L bits, and hence have deterministic service times $T_i = L/\mu_i$, where μ_i is the service rate of packets in node i (in units of bits per second). In the first half of this correspondence, we assume processing rates μ_i are identical for all queues. Under this assumption, we use stochastic equivalence and stochastic inequality relationships to show that the steady-state occupancy distribution at the head node of a multistage tree network is exactly preserved in a reduced two-stage equivalent model. Exact expressions for the mean and variance of packet occupancy are derived from this result, and the number of packets in the head node of the tree is shown to be stochastically less than the number there would be if all preliminary nodes were removed. We then consider networks with heterogeneous server rates μ_i, and develop a simple upper bound on the packet occupancy distribution at each of the nodes. This analysis contributes to a stochastic network calculus for analyzing general tree systems in terms of simpler one-queue or two-queue equivalent models.

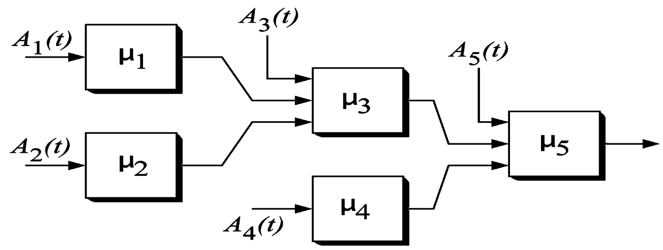

Fig. 1. A multistage tree network with multiple exogenous sources.

As queueing networks are nonlinear systems driven by stochastic events, exact analytical results are largely limited to systems with the special structure of *time reversibility* [23], [24]. Exact analysis of nonreversible queueing systems is usually confined to small networks (see [4]–[6] for analysis of a single discrete time queue with general inputs, and [7] for a moment generating function analysis of a two-queue tandem with independent and identically distributed (i.i.d.) arrivals every timeslot). An approximation method is developed in [8] for modeling discrete-time tandems with arrivals and departures at each stage in the special case when inputs have a specified Markovian structure. Bounding techniques for general networks are developed in [9]–[11] using a deterministic calculus of network service curves, and bounds using stochastic calculus are developed in [13], [14].

An exact analysis of fluid tandems with a single input having exponentially distributed ON and OFF periods is presented in [15] using stochastic Itô calculus. An exact waiting time analysis is developed in [16], [17] for a tandem of deterministic service time queues with Poisson inputs. This analysis uses a simple input–output invariance property for tandems with nonincreasing service rates. A similar property is proven in [19] for discrete-time trees with slotted service at each queue and with independent inputs. This result is used in [18] to compute the average delay in each node of a discrete-time tree when inputs are Poisson. Our work uses a similar input–output invariance property, but extends the techniques to demonstrate equivalence of packet occupancy distributions (rather than just averages) and to treat networks with general stochastic inputs. Exact analysis is provided for networks with homogeneous server rates, and bounding analysis is provided for systems with heterogeneous server rates (without requiring the nonincreasing service rate assumption).

This correspondence is structured as follows: In Section II, we develop an input–output relationship for tree networks with a single bottleneck. In Section III, we consider networks with homogeneous server rates and show that the steady-state packet occupancy distribution of any node in a multistage tree is identical to the corresponding distribution in a simpler two-stage equivalent model. This analysis combines sample path queueing properties with stochastic equivalence relations.

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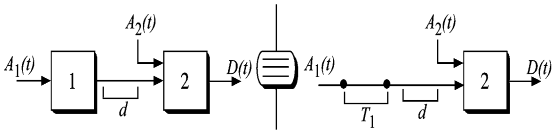

Fig. 2. A two-queue system with deterministic service times T_1, T_2 (with $T_1 \le T_2$), and its equivalent model with queue 1 replaced by a time delay.

In Sections IV and V, exact expressions for the mean and variance of packet occupancy are derived, and a traffic smoothing result is developed to establish upper bounds on the backlog distribution at any node. In Section VI, this bounding analysis is extended to systems with heterogeneous servers. The resulting bounds exploit the special structure of deterministic service time queues as well as the probabilistic nature of the underlying input processes, and are shown to be tighter than the network calculus bounds derived for special classes of traffic with variable length packets in [9]–[14].

II. Equivalent Models

Here we demonstrate an important input–output sample path property for deterministic service time queues. We begin by presenting two simple lemmas. Consider a single queue with a general input stream consisting of fixed-length packets arriving at times $\tau_1, \tau_2, \ldots$, with arbitrarily distributed and correlated interarrival times. Packets are served according to any nonpreemptive service discipline, such as the first-in-first-out (FIFO) policy. Let $A(t)$ represent the arrival process, formally written as a sum of shifted impulse functions: $A(t) = \sum_{n=1}^{\infty} \delta(t-\tau_n)$. We assume that $\tau_1 > 0$, so that all packets arrive after time 0. Let T represent the service time of each packet in the server of the queue. Assume the system is initially empty, and let $D(t)$ represent the corresponding process of queue departures during the interval $[0, t]$. Suppose the departures next pass through a *delay line* of duration d, for some value $d \ge 0$. The delay line accepts incoming packets, stores them, and releases them after exactly d seconds, resulting in an output process $D(t-d)$.

Lemma 1 (Delay Permutation): The resulting output process $D(t-d)$ is unchanged if the delay line is situated *before* the queue, rather than after it.

This elementary lemma follows because queues are time-invariant systems, so that an input of $A(t-d)$ leads to an output of $D(t-d)$. For the next lemma, consider the two-queue tandem of Fig. 2. Packets arrive to the first queue according to an input stream $A_1(t)$, and these packets are then delivered to the second queue after passing through an intermediate delay line of duration $d \ge 0$. Additional packets arrive directly to the second queue according to an input stream $A_2(t)$. Let $D(t)$ represent the resulting output process of the second queue. Let T_1 and T_2 represent the service times of packets in the first queue and second queue, respectively.

Lemma 2 (Queue Replacement): If both queues are initially empty and if $T_1 \le T_2$, then $D(t)$ is unchanged if the first queue is replaced by a pure delay line of duration T_1.

The modified system is simpler and forms an *equivalent model* of the original system. Lemma 2 was independently proven in [16] and [1] for the case when there is no intermediate delay between the two queues. The case of an intermediate delay line of duration $d > 0$ follows directly from the result for no delay line by using Lemma 1 to switch the order of the delay and the first queue. Intuitively, the result of Lemma 2 follows because the final node serves packets no faster than the first node can deliver them. Hence, the busy period of the final node of the original system cannot finish before the busy period of the final node

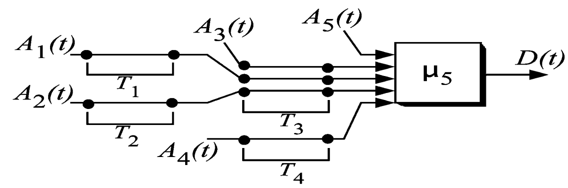

Fig. 3. Replacing all preliminary nodes of the network in Fig. 1 with delay lines. The output process $D(t)$ is unchanged if $\mu_i \ge \mu_5$ for all $i \in \{1, \ldots, 5\}$.

in the equivalent model. Note that although the $D(t)$ function is unchanged, it is possible for the packet departure order to be different in the equivalent model.

A. Tree Reduction Principle

Consider a multistage tree network with M queues and heterogeneous server rates μ_i, so that packets have service times $T_i = L/\mu_i$ in each queue $i \in \{1, \ldots, M\}$. The inputs to each queue i consist of an exogenous arrival process $A_i(t)$ together with the endogenous departure process of upstream queues (Fig. 1). Let T_M represent the service time of the final node (corresponding to service rate μ_M), and let $D(t)$ represent the departure process of this node.

Theorem 1 (Output Invariance): If $T_M \ge T_i$ for all preliminary nodes i, then $D(t)$ is unchanged if all other nodes $i \ne M$ are replaced with pure delay lines of duration T_i, as in Fig. 3.

Proof: The proof follows by iteratively applying Lemma 2. Specifically, consider the two-queue system composed of the final node together with any other node i situated immediately behind it (possibly with a delay line in between). Let the preliminary node represent "queue 1" of Lemma 2, and let all of its arrivals collectively represent "$A_1(t)$" from the lemma. Likewise, let all other streams into the final node collectively represent "$A_2(t)$." Then the departure process of the final node is unchanged if the preliminary node is replaced by a delay line of duration equal to T_i. Recursively repeating this procedure replaces all preliminary nodes with pure delay lines without affecting the overall departure process of the system. □

Theorem 1 allows the departure function of the final node in a multistage tree network to be modeled by the departures of a single queue with a set of delayed input streams, provided that the service rate of the final node is less than or equal to all other service rates. We note that a similar reduction principle for trees was developed in [19] for the special case of discrete-time systems with slotted service and independent inputs. In Section III, we use this theorem to analyze tree networks with homogeneous servers.

As an aside, we note that for heterogeneous networks, Theorem 1 implies that routing according to a shortest path tree is optimal when sending fixed-length packets from multiple sources to a single bottleneck node M, where $\mu_M \le \mu_i$ for all nodes i. Specifically, consider a network of queueing nodes defined by an arbitrary graph (with no tree structure yet determined), and define the *path length* of a route between any two nodes as the sum of service times $T_i = L/\mu_i$ over all nodes of the route. It is clear that under any routing policy, every exogenously arriving packet is delayed from entering the final node by at least the sum of service times over its shortest path. Each packet is delayed by exactly this amount in the shortest path tree model where all preliminary nodes are replaced by pure time delays (Fig. 3). Hence, at every instant of time the number of departures in this reduced model is greater than or equal to the number of departures under any routing scheme in the actual network. However, Theorem 1 implies that the departures of this reduced model are identical to the actual departures in the network under shortest path tree routing, and hence shortest path tree routing minimizes the total number of packets in the system at every instant of time.

III. Trees With Homogeneous Server Rates

Here we consider tree networks with homogeneous server rates, so that packet service times satisfy $T_i = T$ for all i.

Theorem 2: If $T_i = T$ for all i, then at every instant of time the number of packets in the final node of the tree is the same as the number of packets in the final node of a reduced two-stage tree, where all nodes more than one stage beyond the final node are replaced with delay lines of duration T.

Proof: The endogenous inputs to the final node are the departure processes from the previous stages. By Theorem 1, these departure processes are unchanged when their preliminary queues are replaced with time delays. Because the endogenous and exogenous inputs to the final node remain the same in the reduced system, the number of packets in the final node is unchanged. □

Theorem 2 allows the occupancy dynamics in the final node of a tree to be modeled by a reduced two-stage system with preliminary time delays. Intuitively, it is clear that if the inputs $A_i(t)$ are stationary and independent of each other, then the steady-state occupancy distribution in the final node of the reduced system is preserved when the delay lines are removed. Here we use stochastic coupling to formally prove this result, and show that steady-state behavior exists in the equivalent model if and only if it exists in the original system. We first present the basic concepts of stochastic coupling theory.

A. Stochastic Dominance and Equivalence

Definition 1: A random variable N_1 is *stochastically greater* than another random variable N_2 if there exists a third "coupling variable" $\tilde{N}_1$ such that $N_1 \geq \tilde{N}_1$, and $\tilde{N}_1$ has the same distribution as N_2. In this case, we write $N_1 \underset{\text{st.}}{\geq} N_2$.

An equivalent definition can be stated by coupling with respect to N_2, so that there is an external variable $\tilde{N}_2$ with the same distribution as N_1 and such that $\tilde{N}_2 \geq N_2$. It is well known that $N_1 \underset{\text{st.}}{\geq} N_2$ if and only if $Pr[N_1 \geq \alpha] \geq Pr[N_2 \geq \alpha]$ for all real numbers α (see [22]). This fact immediately implies that stochastic inequality relations are *transitive*: If $N_1 \underset{\text{st.}}{\geq} N_2$ and $N_2 \underset{\text{st.}}{\geq} N_3$, then $N_1 \underset{\text{st.}}{\geq} N_3$.

If N_1 and N_2 have the same distribution, we write $N_1 \underset{\text{st.}}{=} N_2$, and say that the random variables are *stochastically equivalent*. It is also essential to have a notion of stochastic equivalence for random processes:

Definition 2: A random process $A_1(t)$ is *stochastically equivalent* to another process $A_2(t)$ if $\mathbb{E}\{h(A_1)\} = \mathbb{E}\{h(A_2)\}$ for all measurable operators $h(\cdot)$ that map a process $A(t)$ to a single real number.

It is clear that if stochastically equivalent input processes $A_1(t)$ and $A_2(t)$ are applied to identical queues at time 0, they produce stochastically equivalent packet occupancy processes $N_1(t)$ and $N_2(t)$, and that at any particular time instant τ, the random variables $N_1(\tau)$ and $N_2(\tau)$ satisfy $N_1(\tau) \underset{\text{st.}}{=} N_2(\tau)$. Indeed, this can be seen from the definition by defining the operator $h(\cdot)$ that maps an arrival process to the number of packets $N(\tau)$ at time τ via the queueing law.

B. Removing Delays Via Stochastic Coupling

Consider now the two-node tandem with nodes 1 and 2 and input streams $A(t)$, $B(t)$, and $C(t)$ delivering packets destined for node 2, as shown in Fig. 4. This system represents any two sequential nodes of a multistage tree, where the general inputs consist of the combined exogenous streams and endogenous streams from other nodes. As before, we assume all arrivals occur after time 0, so that $A(t) = B(t) = C(t) = 0$ for all $t \leq 0$.

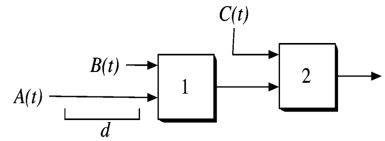

Fig. 4. A canonical two-queue system with input $X(t)$ delayed by time d.

Note that the $A(t)$ process is explicitly shown with a time delay of duration d. We show that if $A(t)$, $B(t)$, and $C(t)$ are independent and stationary, the delay can be removed without affecting the steady-state distribution of packets in node 1 or node 2. We first define the notions of steady state and stationarity.

Definition 3: Let the stochastic process $N(t)$ represent the number of packets in a queue as a function of time. The steady-state distribution $F[n]$ for the queue is defined as follows:

$$F[n] \triangleq \lim_{t\to\infty} \frac{1}{t} \int_{\tau=0}^{t} Pr[N(\tau) \leq n] d\tau, \qquad n \in \{0, 1, 2, \ldots\} \quad (1)$$

whenever the limit exists.

We now define the notion of stationarity for processes that only have arrivals after time 0. For any arrival process $A(t)$ and any positive delay d, we define the *partially deleted process* $\tilde{A}_d(t)$ as follows:

$$\tilde{A}_d(t) \triangleq \begin{cases} 0, & \text{if } t \leq d \\ A(t), & \text{if } t > d. \end{cases}$$

Thus, $\tilde{A}_d(t)$ can be viewed as a version of the $A(t)$ data stream in which packets during the first d seconds are thrown away.

Definition 4: An arrival process $A(t)$ is *stationary* if for any positive delay d, the delayed process $A(t-d)$ is stochastically equivalent to $\tilde{A}_d(t)$.

Note that for two stationary arrival processes $A(t)$ and $B(t)$ that are also independent, the superposition $A(t-d) + B(t)$ is stochastically equivalent to the superposition $\tilde{A}_d(t) + B(t)$, and hence, either superposition applied to a queue yields the same steady-state packet occupancy distribution, provided that the distribution exists. We further note that the total number of packets in a deterministic service time queue cannot increase if some packets from the input stream are deleted [2].

Theorem 3: For any general inputs $A(t)$, $B(t)$, $C(t)$ that are independent and stationary, the steady-state occupancy distribution in node 1 of Fig. 4 exists if and only if the steady-state occupancy distribution exists when the time delay on the $A(t)$ input stream is removed. If the distributions exist, they are identical.

Similarly, the steady-state distribution in node 2 is the same with or without the time delay on the $A(t)$ stream, provided the distribution exists.

Proof: It is useful to define $N_{[A(t)]}(\tau)$ as the number of packets at time τ in a queue that is initially empty with a general arrival process $A(t)$ applied at time 0, where $A(t)$ could represent a superposition of processes. Note that $N_{[A(t)]}(\tau)$ is always greater than or equal to the number of packets in a queue at time τ with the same input process but with some of the arriving packets deleted. Hence, the following inequalities hold deterministically for all time instants τ:

$$N_{[\tilde{A}_d(t)+\tilde{B}_d(t)]}(\tau) \leq N_{[\tilde{A}_d(t)+B(t)]}(\tau) \leq N_{[A(t)+B(t)]}(\tau). \quad (2)$$

The random variable $N_{[A(t)+B(t)]}(\tau)$ on the right of the above inequality represents the number of packets in node 1 of Fig. 4 at time τ

in the case when the $A(t)$ and $B(t)$ streams are applied directly with no time delay, while the middle term of the above inequality represents the corresponding number of packets when arrivals during the first d seconds are deleted from the $A(t)$ stream. Likewise, the leftmost term considers the case when packets from both the $A(t)$ and $B(t)$ streams are deleted during the first d seconds.

However, by stationarity, the arrival process $\tilde{A}_d(t) + \tilde{B}_d(t)$ is stochastically equivalent to the process $A(t-d) + B(t-d)$, which represents a delayed version of $A(t) + B(t)$. Likewise, by independence, the arrival process $\tilde{A}_d(t) + B(t)$ is stochastically equivalent to the process $A(t-d) + B(t)$. We thus have the following stochastic equalities for all time instants τ:

$$N_{[\tilde{A}_d(t)+\tilde{B}_d(t)]}(\tau) \underset{\text{st.}}{=} N_{[A(t)+B(t)]}(\tau - d)$$
$$N_{[\tilde{A}_d(t)+B(t)]}(\tau) \underset{\text{st.}}{=} N_{[A(t-d)+B(t)]}(\tau).$$

Using these stochastic inequalities in (2) yields

$$N_{[A(t)+B(t)]}(\tau - d) \underset{\text{st.}}{\leq} N_{[A(t-d)+B(t)]}(\tau) \underset{\text{st.}}{\leq} N_{[A(t)+B(t)]}(\tau).$$

The upper and lower bounds in the above inequality are time delayed versions of the same process, namely, the process of packets in node 1 of Fig. 4 when $A(t)$ and $B(t)$ are applied directly. It follows that their time average distributions (defined in (1)) are equal, and converge if and only if the middle term converges. The middle term represents the process of packets in node 1 when the $A(t)$ stream first passes through the d-second delay. Thus, the steady-state distribution in node 1 is unchanged if the time delay is removed.

The proof of the corresponding property for node 2 is similar, and follows from the fact that, because both nodes have the same service time T, the number of packets in node 2 is greater than or equal to the resulting number of packets if some arrivals from $A(t)$, $B(t)$, or $C(t)$ are deleted before entering the system (see the Appendix). It follows that for all instants τ

$$N_1(\tau) \leq N_2(\tau) \leq N_3(\tau)$$

where $N_3(\tau)$ represents the packet occupancy in the second node of Fig. 4 when the streams $A(t)$, $B(t)$, and $C(t)$ are applied with no time delays; $N_2(t)$ represents the corresponding occupancy in the second node in the case when all packets arriving from $A(t)$ during the first d seconds are deleted; and $N_1(t)$ represents the occupancy in the case when arrivals from all streams are deleted for the first d seconds. The result follows by noting that

$$N_1(\tau) \underset{\text{st.}}{=} N_3(\tau - d)$$

and that $N_2(\tau)$ is stochastically equal to the number of packets in the final node of Fig. 4 (with $A(t)$ delayed by d seconds). □

C. The Two-Stage Reduction Theorem

We can now present the main reduction theorem for homogeneous tree networks with general stationary and independent arrival processes. Consider a multistage tree network with exogenous arrival processes $A_i(t)$ at each node i, and define its two-stage equivalent model as the system with the same inputs, but with all queues more than one stage behind the final node removed.

Theorem 4 (Two-Stage Equivalent Models): If exogenous inputs $A_i(t)$ are stationary and independent of each other, a steady-state occupancy distribution exists in the final node of the original network if and only if a steady-state occupancy distribution exists in the two-stage equivalent model. Furthermore, if the distributions exist, they are exactly the same.

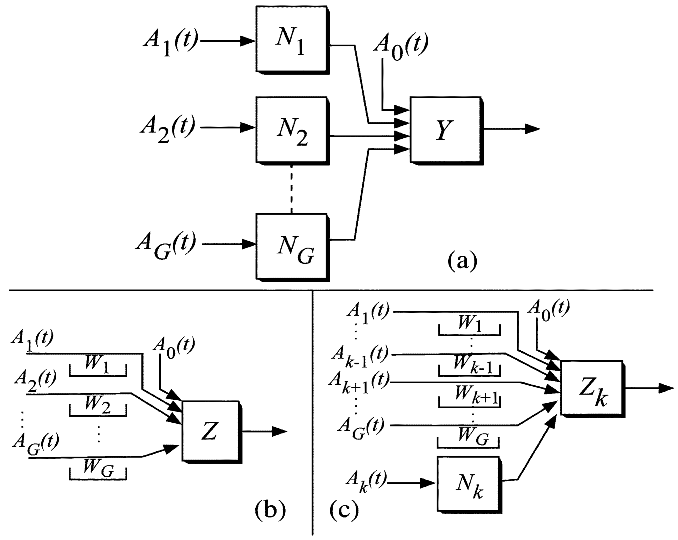

Fig. 5. (a) The canonical two-stage equivalent model of a homogeneous tree network, and (b), (c) reduced systems with the same total packets.

Proof: Reduce the multistage tree network to a two-stage network in tandem with delay lines, as described in Theorem 2. Note that this does not change the packet occupancy process $N(t)$ in the final node. By the delay removal theorem (Theorem 3), we can iteratively remove each of the delay lines without changing the steady-state distribution in the final node. The resulting system has no time delays and is exactly the two-stage equivalent model. □

Note that any node i of a tree can be viewed as the final node of the smaller network consisting only of nodes with arrival streams that pass through node i. Hence, the steady-state occupancy distribution of any node of a tree network can be exactly analyzed according to a two-stage equivalent model.

IV. Mean and Variance Analysis

Here we use the two-stage reduction theorem to develop simple expressions for the mean and variance of packet occupancy in terms of the corresponding moments in systems with only one queue and two queues, respectively. Consider a multistage tree with homogeneous service times T in each node, and with stationary and independent exogenous arrival processes $A_i(t)$. By Theorem 4, the steady-state occupancy in any such network can be analyzed by a two-stage equivalent model. The canonical two-stage model is shown in Fig. 5(a), and has G first stage queues. In this model, the inputs $A_1(t), \ldots A_G(t)$ represent superpositions of the exogenous inputs of the original multi-stage network, and $A_0(t)$ is the exogenous input to the final node. Assume this system exhibits steady state behavior, and let $N_1, \ldots, N_G$ represent the steady state number of packets in the first stage queues. Let Y represent the steady state occupancy in the final node. That is, the collection $\{N_1, \ldots, N_G, Y\}$ can be viewed as random variables with joint distribution given by the steady-state system occupancy distribution. Altenatively, these random variables can be viewed as samples of queue occupancy at a time when the system is in steady state.

A. Mean Occupancy

Here we compute $\mathbb{E}\{Y\}$, the mean occupancy in the final node of the canonical two-stage tree of Fig. 5(a). Assume inputs $A_0(t)$, $A_1(t), \ldots, A_G(t)$ are rate ergodic with arrival rates $\lambda_0, \lambda_1, \ldots, \lambda_G$, and define the sum rate $\lambda = \lambda_0 + \cdots + \lambda_G$. Define $Q_T(A_i + \cdots + A_j)$ to be the expected occupancy in a single queue with deterministic service time T and with a superposition of the arrival streams $A_i(t) + \cdots + A_j(t)$ entering as inputs. Thus, the mean occupancy in the first stage queues of Fig. 5(a) can be written as $\mathbb{E}\{N_i\} = Q_T(A_i)$ for $i \in \{1, \ldots, G\}$. Now consider the system of Fig. 5(b), where the first stage queues are replaced by delay lines of duration T. Let $W = W_1 + \cdots + W_G$ represent the sum number of packets in the delay lines (where W_i represents the number of packets in the delay line from stream $A_i(t)$). Let Z represent the number in the final node. We have

$$\begin{aligned}\mathbb{E}\{N_1+\cdots+N_G+Y\} &= Q_T(A_1)+\cdots+Q_T(A_G)+\mathbb{E}\{Y\}\\ \mathbb{E}\{W+Z\} &= \mathbb{E}\{W\}+Q_T(A_0+A_1+\cdots+A_G)\\ &= (\lambda-\lambda_0)T+Q_T(A_0+A_1+\cdots+A_G).\end{aligned}$$

By Theorem 1, we know that the departures of both the system of Fig. 5(a) and (b) are the same for all times, and hence, the total number of packets in the two systems is always the same. The equalities above can thus be equated, yielding the following exact expression for $\mathbb{E}\{Y\}$:

$$\mathbb{E}\{Y\} = (\lambda-\lambda_0)T+Q_T(A_0+A_1+\cdots+A_G)-\sum_{i=1}^{G} Q_T(A_i).$$

The preceding equation expresses the expected occupancy in any node of a homogeneous multistage tree network in terms of the expected occupancy in single-queue systems with a superposition of the original exogenous inputs, and can be evaluated whenever the average occupancy in such single-queue systems can be computed.

We note that expected occupancy for tree networks with Poisson inputs was derived previously in [18]. We can obtain the result of [18] from the above formula by using the $Q_T(A)$ function for the special case of Poisson inputs. In this case, $Q_T(A)$ can be written as a pure function of the arrival rate λ, and is given by the standard Pollaczek–Khinchin formula for expected occupancy in an M/D/1 queue: $Q_T(\lambda) = \lambda T + \frac{(\lambda T)^2}{2(1-\lambda T)}$.

B. Occupancy Variance

To compute the variance $\mathrm{Var}(Y)$ for packet occupancy in the final node, consider the following alternative modification of the canonical two-stage system in Fig. 5(a): Replace all preliminary nodes $i \in \{1, \ldots, G\} - \{k\}$ with time delays of duration T, but keep the preliminary node k unchanged. By Theorem 1, this modification also contains the same aggregate number of packets as the original system. Let Z_k represent the corresponding number of packets in the final node of this modified system.

Lemma 3: The variables Y, Z, $\{Z_k\}$, $\{W_k\}$, and $\{N_k\}$ satisfy

$$Y^2 = \sum_{k=1}^{G} Z_k^2 - (G-1)Z^2 + \sum_{i\neq j}(W_i - N_i)(W_j - N_j). \quad (3)$$

Proof: Using the fact that all three systems in Fig. 5 have the same total number of packets within them, we have

$$\begin{aligned}N_1 + \cdots + N_G + Y &= W_1 + \cdots + W_G + Z\\ &= N_k + Z_k + \sum_{i\in\{1,\ldots,G\}-\{k\}} W_i.\end{aligned}$$

The above equalities hold for all $k \in \{1, \ldots, G\}$, and hence,

$$W_k - N_k = Z_k - Z, \qquad \text{for all } k \in \{1, \ldots, G\} \quad (4)$$

$$Y = Z + \sum_{k=1}^{G}[W_k - N_k] = Z + \sum_{k=1}^{G}[Z_k - Z]. \quad (5)$$

Squaring both sides of (5) (working only with the Z_k and Z variables) and then using (4) establishes the result. □

Theorem 5 (Variance): If $A_0(t), A_1(t), \ldots, A_G(t)$ are stationary and independent of each other, then

$$\mathrm{Var}(Y) = \sum_{k=1}^{G}\mathrm{Var}(Z_k) - (G-1)\mathrm{Var}(Z). \quad (6)$$

Proof: Note that (3) of the preceding lemma is simply an algebraic statement about any variables Y, Z, $\{Z_k\}$, $\{W_k\}$, and $\{N_k\}$ that satisfy (4) and (5). Any random variables satisfying these two linear equations will also satisfy these equations in their expected values. Hence, we can replace Y, Z, $\{Z_k\}$, $\{W_k\}$, and $\{N_k\}$ in (4) and (5) with their expectations $\mathbb{E}\{Y\}$, $\mathbb{E}\{Z\}$, $\{\mathbb{E}\{Z_k\}\}$, $\{\mathbb{E}\{W_k\}\}$, and $\{\mathbb{E}\{N_k\}\}$ to find that the lemma also implies

$$\mathbb{E}\{Y\}^2 = \sum_{k=1}^{G}\mathbb{E}\{Z_k\}^2 - (G-1)\mathbb{E}\{Z\}^2 + \sum_{i\neq j}\mathbb{E}\{W_i - N_i\}\,\mathbb{E}\{W_j - N_j\}. \quad (7)$$

Taking expectations over (3) and subtracting (7) yields

$$\begin{aligned}\mathrm{Var}(Y) = {} & \textstyle\sum_{k=1}^{G}\mathrm{Var}(Z_k) - (G-1)\mathrm{Var}(Z)\\ & + \textstyle\sum_{i\neq j}\mathbb{E}\{(W_i - N_i)(W_j - N_j)\}\\ & - \textstyle\sum_{i\neq j}\mathbb{E}\{W_i - N_i\}\,\mathbb{E}\{W_j - N_j\}.\end{aligned}$$

Because the input processes are stationary and independent, $(W_i - N_i)$ is independent of $(W_j - N_j)$ whenever $i \neq j$. The last two terms of the above equation thus cancel, proving the theorem. □

Expressions for the variance $\mathrm{Var}(Z_k)$ for a tandem of 2 queues with Poisson inputs and for a discrete-time tandem with i.i.d. arrivals and general arrival distributions are presented in [2] using a moment generating technique developed in [7]. These expressions can be used with Theorem 5 to yield exact variance expressions for any node of a multistage tree with such inputs.

V. Traffic Smoothing

Here we show that tree networks of homogeneous, deterministic service time queues naturally act to "smooth" traffic, making the patterns better for downstream nodes to receive. All inputs are again assumed to be independent and stationary.

Theorem 6 (Smoothing): The steady-state packet occupancy in the final node of a homogeneous tree is stochastically less than its resulting occupancy when all preliminary nodes are removed and the exogenous inputs are applied directly to the final stage.

Proof: By Theorem 4, we can first reduce the multistage tree to its canonical two-stage equivalent model (as in Fig. 5(a)), where the inputs to the equivalent model represent superpositions of the inputs to the tree. The total number of packets in this two-stage system is the same as the total number of packets in a modified system where all nodes at the first stage are replaced by time delays of duration T, as in Fig. 5(b). However, it is clear that the number of packets in the set of first stage queues is always greater than or equal to the number in the corresponding delay lines. It follows that the number of packets Y in the final node of the two-stage system is less than or equal to the number of packets Z in the final node of the modified system. Thus, $Y \le Z$. However, from Theorem 3, we know that $Z \underset{\text{st.}}{=} Z'$, where Z' represents the steady-state occupancy of the modified system when the preliminary delay lines are removed. Hence, $Y \underset{\mathrm{st.}}{\leq} Z'$, proving the theorem. □

This result provides a simple bound on the occupancy distribution of the final node of a homogeneous tree in terms of the occupancy in a single queue with the same exogenous inputs.

VI. Trees With Heterogeneous Service Rates

Note that the results of the previous section are derived from the output invariance properties of tree networks with homogeneous service rates, as characterized by Theorem 1. The theorem holds whenever the service rate of a given node of the tree is less than or equal to the rates of all of its preliminary nodes. Hence, all results for homogeneous trees derived in the previous section apply equally to trees with nonincreasing service rates on every path to the destination. However, it is common for service rates to *increase* at one or more stages, so that downstream nodes have the ability to support the sum traffic load from earlier queues. In this section, we consider the general case of trees with arbitrary service rates μ_i at each node i. Specifically, we consider a tree with M nodes with labels $i \in \{1, \ldots, M\}$, where the final node is labeled as node M (see Fig. 6(a) for the case $M = 5$). Let $A_i(t)$ represent a general arrival process of packets exogenously entering the network at node i, and let λ_i represent the arrival rate. All packets have fixed lengths L with service times L/μ_i in each node i. Let N_M represent the steady-state packet occupancy in the final node. For each $i \in \{1, \ldots, M\}$ let $N_i^{(\gamma_i)}$ represent the steady-state occupancy in a virtual queue with service rate γ_i and input process $A_i(t)$ entering it alone (we assume throughout that a steady state exists).

Theorem 7 (Stochastic Occupancy Bound): For any virtual service rates $\{\gamma_i\}$ such that $\gamma_1 + \cdots + \gamma_M = \mu_M$, we have the following.

a) If exogenous inputs $\{A_i(t)\}$ are stationary and independent, then variables $\left\{N_i^{(\gamma_i)}\right\}$ are independent, and

$$N_M \underset{\mathrm{st.}}{\leq} N_1^{(\gamma_1)} + N_2^{(\gamma_2)} + \cdots + N_M^{(\gamma_M)} \tag{8}$$

b) If exogenous inputs are not stationary and independent, then there exist variables $\{\hat{N}_i^{(\gamma_i)}\}$ such that

$$N_M \leq \hat{N}_1^{(\gamma_1)} + \hat{N}_2^{(\gamma_2)} + \cdots + \hat{N}_M^{(\gamma_M)} \tag{9}$$

$$\hat{N}_i^{(\gamma_i)} \underset{\mathrm{st.}}{=} N_i^{(\gamma_i)} \text{ for all } i \in \{1, \ldots, M\}. \tag{10}$$

The stochastic bounds described above can be easily visualized in terms of the parallel queue system of Fig. 6(d), where each of the M exogenous inputs of the tree is given a separate virtual queue. Notice that the rates γ_i of the virtual queues must sum to the rate of the final node, but are otherwise left unspecified. Hence, the rates can be chosen for convenience, or can be optimized to achieve the tightest upper bound. Before proving the theorem, we demonstrate its implications.

Example 1 (Averages): Let $Q_{L/\gamma_i}(A_i)$ represent the average occupancy in a single queue with service rate γ_i, packet length L, and input stream $A_i(t)$. From Theorem 7, we have the following upper bound for the average occupancy in the final node of a multistage tree:

$$\mathbb{E}\{N_M\} \leq \min_{\sum_{i=1}^{M} \gamma_i = \mu_M} \sum_{i=1}^{M} Q_{L/\gamma_i}(A_i).$$

The $Q_{L/\gamma_i}(A_i)$ function can be shown to be a convex function of γ_i [20], and hence, the minimization above represents a convex optimization. A simpler bound is obtained by the assignment $\gamma_i = \mu_M \frac{\lambda_i}{\lambda}$

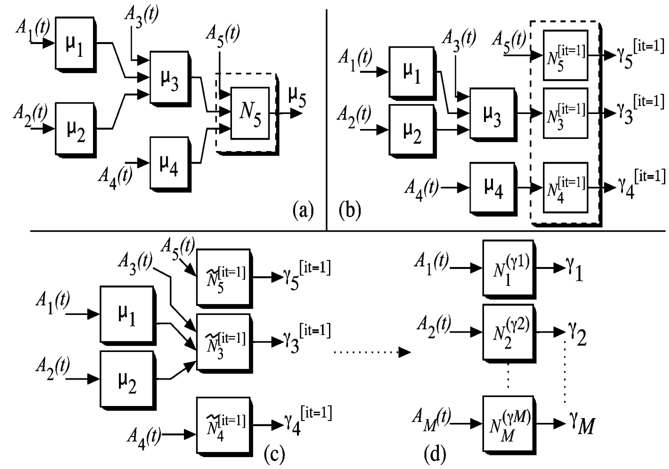

Fig. 6. An illustration of the iterative reduction technique to stochastically bound the occupancy in the final node. Note that $\sum_i \gamma_i = \mu_0$.

(where $\lambda = \lambda_1 + \cdots + \lambda_M$ is the sum input rate). This choice of the γ_i values ensures all virtual queues are stable whenever the original network is stable.

Example 2 (Leaky Bucket Inputs): Suppose the exogenous inputs $\{A_i(t)\}$ are leaky bucket constrained with rate and burst parameters (λ_i, σ_i) [9], so that with probability 1 a queue with input $A_i(t)$ and server rate γ_i will never have more than σ_i packets, provided that $\gamma_i \geq \lambda_i L$. If the sum arrival rate λ from all inputs to the tree satisfies $\lambda L \leq \mu_M$, then using the proportional rate allocation $\gamma_i = \mu_M \lambda_i / \lambda$ guarantees each virtual queue i receives a service rate $\gamma_i \geq \lambda_i L$. Theorem 7 thus verifies the well-known result that the number of packets in the head node is always less than or equal to $\sum_{i=1}^{M} \sigma_i$ [10], [12]. Furthermore, the probability of achieving this worst case backlog can be bounded if more detailed statistical information about the arrival processes is available.

Example 3 (Moment Generating Functions): If inputs $\{A_i(t)\}$ are stationary and independent, then we have from part a) of Theorem 7 that for any rates $\{\gamma_i\}$ that sum to μ_M

$$\mathbb{E}\left\{e^{rN_M}\right\} \leq \prod_{i=1}^{M} \mathbb{E}\left\{e^{rN_i^{(\gamma_i)}}\right\}$$

for any value $r \geq 0$. Hence, the moment generating function for the number of packets in the final node of a tree is less than or equal to the product of moment generating functions for queues with single inputs $A_i(t)$ and server rates γ_i.

If inputs are not necessarily stationary or independent, then part b) of Theorem 7 implies there exist variables $\{\hat{N}_i^{(\gamma_i)}\}$ such that the occupancy N_M in the final node of the tree satisfies

$$N_M \leq \hat{N}_1^{(\gamma_1)} + \cdots + \hat{N}_M^{(\gamma_M)}$$

where $\hat{N}_i^{(\gamma_i)} \underset{\mathrm{st.}}{=} N_i^{(\gamma_i)}$ for each $i \in \{1, \ldots, M\}$. Let $\{p_1, p_2, \ldots, p_M\}$ be any collection of nonnegative numbers that sum to 1. Using the fact that

$$\sum_{i=1}^{M} x_i \leq \max_{i \in \{1, \ldots, M\}} [x_i/p_i]$$

for any values $\{x_i\}$, we have

$$\mathbb{E}\left\{e^{rN_M}\right\} \leq \mathbb{E}\left\{e^{r \max_i \left[\hat{N}_i^{(\gamma_i)}/p_i\right]}\right\} \leq \sum_{i=1}^{M} \mathbb{E}\left\{e^{r\hat{N}_i^{(\gamma_i)}/p_i}\right\}$$

for any $r \geq 0$. Because $\hat{N}_i^{(\gamma_i)} \underset{\text{st.}}{=} N_i^{(\gamma_i)}$, the moment generating function satisfies

$$\mathbb{E}\left\{e^{rN_M}\right\} \leq \sum_{i=1}^{M} \mathbb{E}\left\{e^{rN_i^{(\gamma_i)}/p_i}\right\}.$$

Example 4 (Complementary Occupancy Distribution): A bound on the complementary occupancy distribution can similarly be derived for the general case when inputs are not necessarily stationary or independent. Again let $\{p_1, p_2, \ldots, p_M\}$ be a collection of nonnegative numbers that sum to 1. Using the same technique as [14], we observe the following inclusion of events:

$$\left\{\hat{N}_1^{(\gamma_1)} + \cdots + \hat{N}_M^{(\gamma_M)} > k\right\} \subset \left\{\hat{N}_1^{(\gamma_1)} > p_1 k\right\} \cup \cdots \cup \left\{\hat{N}_M^{(\gamma_M)} > p_M k\right\}.$$

Hence, because $Pr\left[\hat{N}_i^{(\gamma_i)} > x\right] = Pr\left[N_i^{(\gamma_i)} > x\right]$ for any x, we have from the union bound

$$Pr[N_M > k] \leq \min_{\sum \gamma_i = \mu_M, \sum p_i = 1} \left[Pr\left[N_1^{(\gamma_1)} > p_1 k\right] + \cdots + Pr\left[N_M^{(\gamma_M)} > p_M k\right]\right]. \quad (11)$$

The above bound is simpler and tighter than the bounds derived for variable-length packet systems in [13], [14] with inputs that are characterized by a class of exponentially bounded bursty arrivals (EBB) or stochastically bounded bursty arrivals (SBB). Indeed, in the special case when inputs are EBB or SBB, then neglecting the optimization over γ_i and simply choosing the proportional rate assignment $\gamma_i = \mu_M \lambda_i / \lambda$, the bound of (11) becomes identical to the bound offered in [13], [14] for the occupancy distribution of a *single queue* with a superposition of EBB or SBB sources. When this queue is situated within a multistage network, the bounding techniques of [13], [14] can be used recursively to compute updated bounds by considering the effects of each stage. However, these updated bounds require more computation and have the disadvantage of getting progressively larger and larger at each stage—suggesting that congestion may increase with the size of the network in systems with variable-length packets. The bound in (11) holds equally well for any node at any stage of the network. Hence, it is always tighter and is immune to any pejorative effects when the size of the network is scaled. This independence to scaling is due to the intrinsic traffic smoothing properties of deterministic service time queues. Furthermore, the bound of (11) holds for any general input processes, and incorporates the particular statistical properties of each process by relating performance to the resulting backlog in a system of single queues with each of these inputs applied individually.

A. Derivation of Theorem 7

To prove Theorem 7, we use two preliminary lemmas that hold for one-queue and two-queue systems with fixed-length packets. Consider a queue with server rate μ and with a superposition of M input processes $A_1(t) + \cdots + A_M(t)$. Let $N(t)$ represent the number of packets in this queue as a function of time (assuming the system is initially empty). For a given rate $\gamma_i \leq \mu$, let $N_i(t)$ represent the corresponding number of packets in a queue with server rate γ_i and with input stream $A_i(t)$ alone.

Lemma 4 (Multiplexing Inequality): For arbitrary input streams $\{A_i(t)\}$ and for any rates $\{\gamma_i\}$ such that $\gamma_1 + \cdots + \gamma_M = \mu$, we have at every instant of time

$$N(t) \leq N_1(t) + \cdots + N_M(t).$$

The lemma is an immediate consequence of the multiplexing results proven in [21]. Intuitively, Lemma 4 follows by observing that the single-queue system is either empty or is processing data at a rate that is greater than or equal to the sum processing rate of the combined system of M queues.

Next consider two tandem queues with a single input. Packets of length L bits arrive to the first queue according to an arrival process $A(t)$ with rate λ, and these packets enter the second queue after service at the first. Service rates of the first and second queues are μ_1 and μ_2, respectively. Assume the system exhibits a steady state, and let random variable N_2 represent the steady-state number of packets in queue 2. Let $\tilde{N}_2$ represent the corresponding steady-state occupancy in the case when the first queue is removed and all packets directly enter queue 2.

Lemma 5 (Smoothing in Single-Input Tandems): For arbitrary service rates μ_1 and μ_2, we have $N_2 \underset{\text{st.}}{\leq} \tilde{N}_2$. That is, removing the first queue creates a stochastically greater packet occupancy at the second queue.

Proof: If $\mu_1 \geq \mu_2$, then the proof is the same as the proof of Theorem 6, as the total number of packets in the tandem is unchanged if the first queue is replaced by a pure delay line. In the case $\mu_1 < \mu_2$, there is never more than one packet in the second queue. Thus,

$$Pr[N_2 > k] = 0 \leq Pr[\tilde{N}_2 > k]$$

for all integers $k \geq 1$. Furthermore, by Little's theorem, we have that

$$Pr[N_2 > 0] = Pr[\tilde{N}_2 > 0] = \lambda L / \mu_2.$$

Thus, N_2 is stochastically less than $\tilde{N}_2$. □

These two lemmas can be used iteratively to bound packet occupancy in any node of a heterogeneous tree. Consider the final node in the network of Fig. 6(a), which has rate μ_5. Let N_5 represent the random number of packets in this final node when the system is in steady state. We see from the figure that there are three separate streams flowing into this node. Thus, in the *first iteration* of our reduction technique we split this node into three *virtual subnodes* with rates $\gamma_5^{[\text{it}=1]}$, $\gamma_3^{[\text{it}=1]}$, and $\gamma_4^{[\text{it}=1]}$ that individually service the three streams (where $\gamma_5^{[\text{it}=1]} + \gamma_3^{[\text{it}=1]} + \gamma_4^{[\text{it}=1]} = \mu_5$, see Fig. 6(b)). From Lemma 4, the resulting number of packets $N_5^{[\text{it}=1]}$, $N_3^{[\text{it}=1]}$, $N_4^{[\text{it}=1]}$ in the subnodes satisfy

$$N_5 \leq N_5^{[\text{it}=1]} + N_3^{[\text{it}=1]} + N_4^{[\text{it}=1]}.$$

(Here, the "$\text{it} = 1$" superscript designates values obtained on the first iteration of the reduction, and the subscript index represents the highest numbered exogenous input corresponding to the particular subnode).

Now notice that several two-queue tandem situations have been created (Fig. 6(b)). Consider, for instance, the queue with rate μ_3 in tandem with the $N_3^{[\text{it}=1]}$ queue. From Lemma 5, removing this front queue creates a stochastically greater occupancy $\tilde{N}_3^{[\text{it}=1]}$. Likewise, the queue with rate μ_4 can be removed to generate a new variable $\tilde{N}_4^{[\text{it}=1]}$ that is stochastically greater than $N_4^{[\text{it}=1]}$, creating the simplified system in Fig. 6(c). The number of stages in this simplified system is one less than the original.

For a second iteration, the same procedure can be applied to split queue $\tilde{N}_3^{[\text{it}=1]}$ into queues with rates $\gamma_1^{[\text{it}=2]}$, $\gamma_2^{[\text{it}=2]}$, $\gamma_3^{[\text{it}=2]}$ such that

$$\gamma_1^{[\text{it}=2]} + \gamma_2^{[\text{it}=2]} + \gamma_3^{[\text{it}=2]} = \gamma_3^{[\text{it}=1]}.$$

Proceeding this way, we remove nodes and split nodes until we are left with a parallel collection of 5 queues of rates $\gamma_1, \ldots, \gamma_5$ such that $\gamma_1 + \cdots + \gamma_5 = \mu_5$, and each new node i has its own exogenous input stream $A_i(t)$. At each step of the iteration, the component variables N_i are stochastically increasing. Thus, we are left with variables $N_1^{(\gamma_1)}, \ldots, N_5^{(\gamma_5)}$, where each $N_i^{(\gamma_i)}$ is distributed as a packet occupancy in a single queue with input stream $A_i(t)$ and processing rate γ_i (Fig. 6(d)).

In this way, we can bound packet occupancy in the head node of a multistage heterogeneous tree with arbitrary exogenous inputs $A_1(t), \ldots, A_M(t)$ by simply using the parallel queue picture of Fig. 6(d). However, there is a subtlety in the above iteration procedure: While it is true that

$$N_5 \leq N_5^{[\mathrm{it}=1]} + N_3^{[\mathrm{it}=1]} + N_4^{[\mathrm{it}=1]}$$

when the network is changed from Fig. 6(a) and (b), and it is true that

$$N_5^{[\mathrm{it}=1]} \underset{\mathrm{st.}}{\leq} \tilde{N}_5^{[\mathrm{it}=1]}$$
$$N_3^{[\mathrm{it}=1]} \underset{\mathrm{st.}}{\leq} \tilde{N}_3^{[\mathrm{it}=1]}$$
$$N_4^{[\mathrm{it}=1]} \underset{\mathrm{st.}}{\leq} \tilde{N}_4^{[\mathrm{it}=1]}$$

when the network is changed from Fig. 6(b) and (c), it is not necessarily the case that

$$N_5 \underset{st.}{\leq} \tilde{N}_5^{[\mathrm{it}=1]} + \tilde{N}_3^{[\mathrm{it}=1]} + \tilde{N}_4^{[\mathrm{it}=1]}.$$

This issue is formally treated by using stochastic coupling to introduce the auxiliary variables $\hat{N}_i^{(\gamma_i)}$ of Theorem 7. We first require two lemmas.

Lemma 6: If N, A, and B are random variables such that $N \underset{\mathrm{st.}}{\leq} A+B$, then there exist variables $\hat{A}$ and $\hat{B}$ such that $N \leq \hat{A} + \hat{B}$, and $\hat{A} \underset{\mathrm{st.}}{=} A$, $\hat{B} \underset{\mathrm{st.}}{=} B$. Furthermore, if A and B are independent, then $\hat{A}$ and $\hat{B}$ can be chosen to be independent.

Proof: By definition of stochastic inequality, there exists a random variable Z such that $N \leq Z$ and $Z \underset{\mathrm{st.}}{=} A + B$. Given this Z variable, form the pair of random variables $(\hat{A}, \hat{B})$ by choosing them according to the joint conditional distribution $p_{A,B|Z}(a, b \mid A + B = Z)$. Thus, the vector $(\hat{A}, \hat{B})$ is coupled to the Z variable through the joint distribution function for the pair (A, B). It follows that $\hat{A} + \hat{B} = Z \geq N$, and that $\hat{A} \underset{\mathrm{st.}}{=} A$, $\hat{B} \underset{\mathrm{st.}}{=} B$. If A and B are independent, then $\hat{A}$ and $\hat{B}$ will also be independent. □

Lemma 7: If N, A, and B are random variables such that $N \underset{\mathrm{st.}}{\leq} A+B$, and if there exist variables $\tilde{A}$, $\tilde{B}$ such that $A \underset{\mathrm{st.}}{\leq} \tilde{A}$ and $B \underset{\mathrm{st.}}{\leq} \tilde{B}$ then there exist variables $\hat{A}$, $\hat{B}$ such that $N \leq \hat{A} + \hat{B}$, and $\hat{A} \underset{\mathrm{st.}}{=} \tilde{A}$, $\hat{B} \underset{\mathrm{st.}}{=} \tilde{B}$.

Proof: By definition of stochastic inequality, because $A \underset{\mathrm{st.}}{\leq} \tilde{A}$ and $B \underset{\mathrm{st.}}{\leq} \tilde{B}$, there must exist variables A' and B' such that $A \leq A'$, $B \leq B'$, and $A' \underset{\mathrm{st.}}{=} \tilde{A}$, $B' \underset{\mathrm{st.}}{=} \tilde{B}$. It follows that $N \underset{\mathrm{st.}}{\leq} A' + B'$. By the previous lemma, there must be variables $\hat{A}$ and $\hat{B}$ such that $N \leq \hat{A} + \hat{B}$ and $\hat{A} \underset{\mathrm{st.}}{=} A'$, $\hat{B} \underset{\mathrm{st.}}{=} B'$. By transitivity, we have $\hat{A} \underset{\mathrm{st.}}{=} \tilde{A}$ and $\hat{B} \underset{\mathrm{st.}}{=} \tilde{B}$, proving the lemma. □

We can now prove Theorem 7.

Proof of Theorem 7: We proceed by induction on the iterative procedure outlined above. At the beginning of iteration k, assume we have a set of subnodes I_k with steady-state occupancies $\tilde{N}_i^{[\mathrm{it}=k]}$ ($i \in I_k$) and corresponding variables $\hat{N}_i^{[\mathrm{it}=k]}$ such that

$$N_M \leq \sum_{i \in I_k} \hat{N}_i^{[\mathrm{it}=k]}$$

and $\hat{N}_i^{[\mathrm{it}=k]} \underset{\mathrm{st.}}{=} \tilde{N}_i^{[\mathrm{it}=k]}$ for all $i \in I_k$. After splitting each node with occupancy $\tilde{N}_i^{[\mathrm{it}=k]}$ into a set of $S(i)$ parallel subnodes with new occupancies $N_j^{[\mathrm{it}=k+1]}$ ($j \in S(i)$) such that

$$\tilde{N}_i^{[\mathrm{it}=k]} \leq \sum_{j \in S(i)} N_j^{[\mathrm{it}=k+1]}$$

it follows that

$$\hat{N}_i^{[\mathrm{it}=k]} \underset{\mathrm{st.}}{\leq} \sum_{j \in S(i)} N_j^{[\mathrm{it}=k+1]}, \qquad \forall\, i \in I_k. \tag{12}$$

Next, the inputs to the parallel subnodes are "un-smoothed" by removing any preliminary queues, and new variables $\tilde{N}_j^{[\mathrm{it}=k+1]}$ are formed that are stochastically greater than $N_j^{[\mathrm{it}=k+1]}$, that is, for each $i \in I_k$ and each $j \in S(i)$

$$N_j^{[\mathrm{it}=k+1]} \underset{\mathrm{st.}}{\leq} \tilde{N}_j^{[\mathrm{it}=k+1]}. \tag{13}$$

Applying Lemma 7 to (12) and (13), for all i we can find auxiliary variables $\hat{N}_j^{[\mathrm{it}=k+1]}$ for each $j \in S(i)$ such that

$$\hat{N}_i^{[\mathrm{it}=k]} \leq \sum_{j \in S(i)} \hat{N}_j^{[\mathrm{it}=k+1]}$$

and $\hat{N}_j^{[\mathrm{it}=k+1]} \underset{\mathrm{st.}}{=} \tilde{N}_j^{[\mathrm{it}=k+1]}$ for all $j \in S(i)$. Defining the set $I_{k+1} \triangleq \cup_{i \in I_k} S(i)$ establishes the induction hypothesis for the next iteration, proving the theorem. □

VII. Conclusion

We have used sample path observations and stochastic coupling techniques to analyze deterministic service time tree networks with arbitrary input streams. The analysis yields quantitative probabilistic expressions for network congestion in terms of simpler systems. For homogeneous tree networks with multiple stages, a reduced two-stage equivalent model was developed and shown to exactly preserve the steady-state occupancy distribution. Exact expressions for occupancy mean and variance in any node were obtained using this analysis, and a smoothing result was proven, showing that packet occupancy at any node is stochastically increased if its preliminary nodes are removed. For tree networks with heterogeneous server rates $\{\mu_i\}$, a simple upper bound on the packet occupancy of any node was developed in terms of the occupancy in a set of virtual queues that individually serve the exogenous input streams $A_i(t)$. The bound is independent of the location of the node within the tree, and is tighter than previous bounds derived for variable length packet systems with traffic envelope constraints. This work contributes to a theory of stochastic network calculus, and provides powerful techniques for analyzing complex queueing systems.

Appendix

Here we show that the number of packets in the second node of the two-queue tandem in Fig. 4 is greater than or equal to the number there would be if some arrivals from any of the inputs are deleted before entering the system. To see this, first note by Theorem 1 that the total number of packets in the tandem is unchanged if the first node is replaced by a pure delay line of duration T. The total number of packets in this equivalent model (delay line plus queue) cannot increase if some arrivals are deleted, and hence the total number in the two-queue tandem cannot increase with deleted arrivals. Thus, if the first node of the two-queue tandem is empty, then the total number of packets in node 2 is greater than or equal to the corresponding amount if some arrivals were deleted. It is not difficult to show that this property is preserved during times when the first stage node is busy. Indeed, during such busy periods, the first node delivers at the maximum rate of one packet per T seconds, and the number of packets in the second node cannot decrease below the corresponding number in the deleted system.

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On Distances in Uniformly Random Networks

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***Abstract*—The distribution of Euclidean distances in Poisson point processes is determined. The main result is the density function of the distance to the n-nearest neighbor of a homogeneous process in $\mathbb{R}^m$, which is shown to be governed by a generalized Gamma distribution. The result has many implications for large wireless networks of randomly distributed nodes.**

***Index Terms*—Poisson point process, random graphs, stochastic geometry, wireless networks.**

I. INTRODUCTION

For the capacity and performance analysis and comparison of protocols and algorithms for wireless networks with unknown location of the terminals, in particular for *ad hoc* and sensor networks, it is important that the distribution of the distances between the terminals be known. Only few results are available in the literature: In [1], distance distributions of uniformly and Gaussian distributed nodes in a rectangular area are presented. In [2], the mean L_1 distance in a square random network of unit size is determined to be $2/3$. Mean distances for Manhattan networks, hypercubes, and shufflenets are presented in [3]. In this correspondence, we provide closed-form expressions for the distributions in m-dimensional homogeneous Poisson point processes (or, equivalently, infinite networks with uniformly random distributions).

II. EUCLIDEAN DISTANCES IN INFINITE NETWORKS

In a homogeneous m-dimensional Poisson point process of intensity λ, the probability of finding k nodes in a bounded Borel $A \subset \mathbb{R}^m$ is given by

$$\mathbb{P}[k \text{ nodes in } A] = e^{-\lambda\mu(A)} \frac{(\lambda\mu(A))^k}{k!} \tag{1}$$

where $\mu(A)$ is the standard Lebesgue measure of A. This permits the calculation of the distance to an nth neighbor in a straightforward manner.

Theorem 1 (Euclidean Distance to nth Neighbor): In a Poisson point process in $\mathbb{R}^m$ with intensity λ, the distance R_n between a point and its nth neighbor is distributed according to the generalized Gamma distribution

$$f_{R_n}(r) = e^{-\lambda c_m r^m} \frac{m(\lambda c_m r^m)^n}{r\Gamma(n)} \tag{2}$$

where $c_m r^m$ is the volume of the m-dimensional ball of radius r.

Proof: Let $B_m(r) := c_m r^m$ be the volume of the m-dimensional ball of radius r. The coefficient c_m is given by

$$c_m = \begin{cases} \frac{\pi^{\frac{m}{2}}}{(\frac{m}{2})!}, & \text{for even } m \\ \frac{\pi^{\frac{m-1}{2}} 2^m (\frac{m-1}{2})!}{m!}, & \text{for odd } m. \end{cases} \tag{3}$$

Let S_k be the kth coefficient in the Poisson distribution: $S_k := (\lambda B_m(r))^k/k!$. The complementary cumulative distribution function

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