Remarks on Pathwise Nonlinear Filtering

by

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1. Introduction

This paper is concerned with an example of a nonlinear filtering problem where it is not known whether the pathwise equations of non-linear filtering can be used to construct the unnormalized conditional measure. For details about pathwise nonlinear filtering see CLARK [1978].

2. The Example

Consider the nonlinear filtering problem

$$(1)\qquad \begin{cases} dx_1(t) = dw_1(t) \\ dx_2(t) = dw_2(t) \end{cases} \qquad \underline{\text{state equation}}$$

$$(2)\qquad dy(t) = \left(x_1^3(t) + x_2^3(t)\right)dt + d\eta(t) \qquad \underline{\text{observation equation}}$$

where it is assumed that w_1, w_2 and η are independent Browian motions.

If $\rho(t,x)$ denotes the unnormalized conditional density of $x(t) = \left(x_1(t), x_2(t)\right)$ given $\mathscr{F}_t^Y = \sigma\left\{y(s) \mid o \le s \le t\right\}$ then ρ satisfies the Zakai equation

$$(3)\qquad d\rho(t,x) = \frac{1}{2}\left(\Delta - (x_1^3 + x_2^3)^2\right)\rho(t,x)\,dt + (x_1^3 + x_2^3)\rho(t,x)\circ dy(t),\quad \rho(o,x) = \rho_o(x),$$

where $\circ$ denotes the Stratonovich differential, and Δ is the two-dimensional Laplacian.

Defining

$$(4)\qquad \rho(t,x) = \exp\left((x_1^3 + x_2^3)\,y(t)\right)q(t,x),$$

$q(t,x)$ satisfies the parabolic partial differential equation, the so-called pathwise filtering equation

$$(5)\qquad \begin{cases} q_t = \frac{1}{2}\Delta q + g^y(x,t)\cdot q_x + V^y(x,t)q \\ q(o,x) = \rho^o(x) \end{cases}$$

where $g^y(x,t) = \begin{pmatrix} y(t)3x_1^2 \\ y(t)3x_2^2 \end{pmatrix}$

and

$$V^y = -\frac{1}{2}(x_1^3 + x_2^3)^2 + \frac{1}{2}y^2(t)\,(9x_1^4 + 9x_2^4)$$

The difficulty with studying existence and uniqueness of solutions to (5) is that V^y is not bounded above along the direction $x_1 = -x_2$. In the corresponding scalar case, the conditional measure has been constructed using the pathwise equations by FLEMING-MITTER [1982] and SUSSMANN [1981].

3. Existence and Uniqueness of Weak Solutions

We consider the equation

$$(6)\qquad \begin{cases} \dfrac{du}{dt} + Au = o \\ u(o) = u_o \end{cases}$$

where $A = -\Delta + V(x_1,x_2)$, with

$$V = V_1 - V_2 = (x_1^3 + x_2^3)^2 - (x_1^4 + x_2^4).$$

The same techniques will work for the slightly more general equation (5). The notation and terminology to be used is that of LIONS-MAGENES [1968] (Vol. 1, Chapter 3).

We define the bilinear form

$$a(\phi,\psi) = \int_{\mathbb{R}^2}\left[\frac{1}{2}\nabla\phi\cdot\nabla\psi + V(x)\phi\cdot\psi\right]dx$$

and the spaces

$$H_V^1(\mathbb{R}^2) = \left\{\phi : \mathbb{R}^2 \to \mathbb{R} \;\middle|\; D_{x_1}\phi \in L^2(\mathbb{R}^2),\ D_{x_2}\phi \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2}\left(V_1(x) + V_2(x)\right)|\phi(x)|^2 dx < \infty\right\}$$

with norm

$$\|\phi\|^2_{H_V^1} = \|D_{x_1}\phi\|^2_{L^2} + \|D_{x_2}\phi\|^2_{L^2} + \int_{\mathbb{R}} \Big(V_1(x) + V_2(x)\Big)|\phi(x)|^2 dx$$

and the corresponding scalar product.

$$L_V^2(\mathbb{R}^2) = \left\{\phi : \mathbb{R}^2 \to \mathbb{R} \;\Big|\; \int_{\mathbb{R}^2} \Big(V_1(x) + V_2(x)\Big)|\phi(x)|^2 dx < \infty\right\}$$

with the corresponding natural norm and scalar product.

It can be checked that H_V^1 and L_V^2 are complete with respect to their respective norms.

Denote by

$$\mathcal{H} = L_V^2(\mathbb{R}^2) \quad \text{and} \quad \mathcal{V} = H_V^1(\mathbb{R}^2) .$$

It is easy to check that $a(\phi,\psi)$ is a continuous bilinear form on $\mathcal{V}$ and furthermore there exists a λ such that

$a(\phi,\phi) + \lambda \|\phi\|^2_{\mathcal{H}} \geq \alpha \|\phi\|^2_{\mathcal{V}}$, $\alpha > o$. Hence by the variational theory of Parabolic equations thate exists a unique solution to the equation

$$(7) \qquad \begin{cases} \mathcal{A}u + \dfrac{du}{dt} = o \\ u(o) = u_o \in \mathcal{H}, \quad \mathcal{A} \in \mathcal{L}(\mathcal{V},\mathcal{V}') \end{cases}$$

in the space

$$W(o,T) = \left\{\phi \;\Big|\; \phi \in L^2(o,T;\mathcal{V}), \ \frac{d\phi}{dt} \in L^2(o,T;\mathcal{V}')\right\}$$

Furthermore $-\Delta + V$ generates ananalytic semigroup on $\mathcal{H}$ and also using a standard regularity result the equation (7) has a C^∞-solution. It is however an open problem whether we have the probabilistic representation (Feyman-Kac formula)

$$u(t,x) = E_x\left[u_o\Big(x(t)\Big)\exp\left(\int_o^t - V\Big(x(s)\Big)ds\right)\right]$$

where E_x denotes expectation with respect to 2-dimensional Browian motion. This case is not covered by the best results known in the Feyman-Kac formula (cf. SIMON [1979], p. 262).

Without a probabilistic representation as above it is unclear whether the conditional measure of $x(t)$ given $\mathcal{F}_t^y$ can be constructed using the pathwise filtering equations.

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