CONTROL OF AFFINE SYSTEMS WITH MEMORY

M.C. Delfour*

(Université de Montréal)

and

S.K. Mitter**

(Massachusetts Institute of Technology)

1. INTRODUCTION

In this paper we present a number of results related to control and estimation problems for affine systems with memory. The systems we consider are typically described by linear functional differential equations or Volterra integro-differential equations.

Our results may be divided into four categories:

(i) State-space description of systems with memory.

(ii) Feedback solution of the finite-time quadratic cost problem.

(iii) Feedback solution of the infinite-time quadratic cost problem.

(iv) Optimal linear filtering.

The main difficulty in the study of the systems considered in this paper

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is that the state spaces involved are infinite dimensional and that the equations describing the evolution of the state involve unbounded operators. Once an appropriate function space is chosen for the state space a fairly complete theory for the control and estimation problems for such systems can be given.

2. Affine Systems with Memory

In this paper we shall consider two typical systems: one with a fixed memory and one with a time varying memory.

Let X be the evolution space and U be the control space. We assume that X and U are finite-dimensional Euclidean spaces.

2.1. Constant Memory

Given an integer $N \geq 1$ and real numbers $-a = \Theta_N < \ldots < \Theta_1 < \Theta_0 = 0$ and $T > 0$, let the system with constant memory be described by:

$$(1) \quad \begin{cases} \dfrac{dx}{dt}(t) = A_{00}(t)x(t) + \displaystyle\sum_{i=1}^{N} A_i(t)x(t+\Theta_i) \\ \qquad + \displaystyle\int_{-a}^{0} A_{01}(t,\Theta)x(t+\Theta)d\Theta + f(t) \\ \qquad + B(t)v(t) \quad \text{in} \quad [0,T] \\ x(\Theta) = h(\Theta) \quad , \quad -a \leq \Theta \leq 0, \end{cases}$$

where A_{00}, A_i, A_{01} and B are strongly measurable and bounded, $f \in L^2(0,T; X)$ and $v \in L^2(0,T; U)$.

We first need to choose an appropriate space of initial data and an appropriate state space. It was shown in DELFOUR-MITTER [1], [2], that this can indeed be done provided that (1) is rewritten in the following form:

$$
(2)\quad \begin{cases}
\dfrac{dx}{dt} = A_{oo}(t)x(t) + \displaystyle\sum_{i=1}^{N} A_i(t) \begin{cases} x(t+\Theta_i) & , t+\Theta_i \geq 0 \\ h^1(t+\Theta_i) & , \text{otherwise} \end{cases} \\
\qquad + \displaystyle\int_{-a}^{o} A_{o1}(t,\Theta) \begin{Bmatrix} x(t+\Theta) & , & t+\Theta \geq 0 \\ h^1(t+\Theta) & , & \text{otherwise} \end{Bmatrix} d\Theta \\
\qquad + f(t) + B(t)v(t) \quad , \text{ in } [0,T], \\
x(0) = h^o.
\end{cases}
$$

We can pick initial data $h=(h^o,h^1)$ in the product space $X \times L^2(-a,0;X)$, where the solution of (2) is $x : [0,T] \to X$.

We can now define the <u>state at time t</u> as an element $\tilde{x}(t)$ of $X \times L^2(-a,0;X)$ as follows:

$$
(3)\quad \begin{cases}
\tilde{x}(t)^o = x(t) \\
\tilde{x}(t)^1(\Theta) = \begin{cases} x(t+\Theta) & , & t+\Theta \geq 0 \\ h^1(t+\Theta) & , & \text{otherwise.} \end{cases}
\end{cases}
$$

For additional details see DELFOUR-MITTER [1], [2]. System (1) has a memory of fixed duration $[-a,0]$.

2.2. Time Varying Memory

Consider the system

$$
(4)\quad \begin{cases} \dfrac{dx}{dt}(t) = A_o(t)x(t) + \displaystyle\int_0^t A_1(t,r)x(r)dr \\ \qquad\qquad + f(t) + B(t)v(t) \ , \text{ in } [0,T] \\ x(0) = h^o \text{ in } X, \end{cases}
$$

Where A_o, A_1 and B are strongly measurable and bounded. If we change the variable r to $\Theta = r-t$ and define

$$
(5)\quad \begin{cases} A_{oo}(t) = A_o(t) \\ A_{o1}(t,\Theta) = \begin{cases} A_1(t,t+\Theta) & , \ -t < \Theta < 0, \\ 0 & , \ -\infty < \Theta < t, \end{cases} \end{cases}
$$

equation (4) can be rewritten in the form

$$
(6)\quad \begin{cases} \dfrac{dx}{dt}(t) = A_{oo}(t)x(t) + \displaystyle\int_{-\infty}^{0} A_{o1}(t,\Theta) \begin{Bmatrix} x(t+\Theta) & , \ t+\Theta\geq 0 \\ h^1(t+\Theta) & , \ \text{otherwise} \end{Bmatrix} d\Theta \\ \qquad\qquad + f(t) + B(t)v(t) \quad \text{in} \quad [0,T] \\ x(0) = h^o \quad \text{in } X, \ h^1 \text{ in } L^2(-\infty,0;X), \end{cases}
$$

with $h^1 = 0$. In this form equation (6) is similar to equation (2). However here we consider the system to have a memory of infinite duration in order to accomodate the growing memory duration $[-t,0]$. The <u>state space</u> will be chosen to be the product $X \times L^2(-\infty,0;X)$. The <u>state</u> at time t is an element $\tilde{x}(t)$ of $X \times L^2(-\infty,0;X)$ which is defined as

$$(7)\quad \begin{cases} \tilde{x}(t)^{o} = x(t) \\ \tilde{x}(t)^{1}(\Theta) = \begin{cases} x(t+\Theta) & , -t < \Theta < 0 \\ h^{1}(t+\Theta) & , -\infty < \Theta < -t \end{cases} \end{cases}$$

3. State Equation

It will be more convenient to work with an evolution equation for the state of the system rather than equations (1) or (4). In order to obtain the state evolution equation corresponding to equation (1) let

$$(8)\quad \begin{cases} H = X \times L^{2}(-a,0;X) \\ V = \{(h(0),h) \mid h \in H^{1}(-a,0;X)\}. \end{cases}$$

The injection of V into H is continuous and V is dense in H. We identify H with its dual. Then if V' denotes the dual space of V, we have

$$V \subset H \subset V'.$$

This is the framework utilized by Lions (cf. J.L. LIONS) to study evolution equations. Define the unbounded operator $\tilde{A}(t)$: $V \to H$ by,

$$(9)\quad \begin{cases} (\tilde{A}(t)h)^{o} = A_{oo}(t)h(0) + \sum_{i=1}^{N} A_{i}(t)h(\Theta_{i}) + \int_{-a}^{0} A_{o1}(t,\Theta)h(\Theta)d\Theta \\ (\tilde{A}(t)h)^{1}(\Theta) = \frac{dh}{d\Theta}(\Theta), \end{cases}$$

and the bounded operator

$\tilde{B}(t)$: $U \to H$ by

$$(10)\quad \begin{cases} (\tilde{B}(t)u)^{o} = B(t)u \\ (\tilde{B}(t)u)^{1}(\Theta) = 0 \end{cases}$$

and $\tilde{f}(t) \in H$ by

$$(11)\quad \tilde{f}(t)^{o} = f(t), \quad \tilde{f}(t)^{1} = 0.$$

Then for all h in V, it can be shown that x is the unique solution in

$$(12)\quad W(0,T) = \{z \in L^2(0,T;V) \mid D\, z \in L^2(0,T;H)\}$$ [D denotes the distributional derivative]

of

$$(13)\quad \begin{cases} \dfrac{d\tilde{x}}{dt}(t) = \tilde{A}(t)x(t) + \tilde{B}(t)u(t) + \tilde{f}(t) \text{ in } [0,T] \\ \tilde{x}(0) = h. \end{cases}$$

Similarly in the case of equation (4) we let

$$(14)\quad \begin{cases} H = X \times L^2(-\infty,0;X) \\ V = \{(h(0),h) \mid h \in H^1(-\infty,0;X)\}. \end{cases}$$

We again have

$$V \subset H \subset V'.$$

We now define $\tilde{A}(t)$: $V \rightarrow H$ as follows:

$$(15)\quad \begin{cases} (\tilde{A}(t)h)^{o} = A_{oo}(t)h(0) + \displaystyle\int_{-\infty}^{o} A_{o1}(t,\Theta)h(\Theta)d\Theta \\ (\tilde{A}(t)h)^{1}(\Theta) = \dfrac{dh}{d\Theta}(\Theta), \end{cases}$$

$\tilde{B}(t)$ and $\tilde{f}(t)$ be as defined in (6) and (7). For all h in V, $\tilde{x}$ is the unique solution in

$$(16) \quad W(0,T) = \{z \in L^2(0,T;V) \mid D\,z \in L^2(0,T;H)\}$$

of

$$(17) \quad \begin{cases} \frac{d\tilde{x}}{dt}(t) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)v(t) + \tilde{f}(t), \text{ in } [0,T], \\ \tilde{x}(0) = h. \end{cases}$$

4. Optimal Control Problem in [0,T]

We now consider a quadratic cost function,

$$(18) \quad \begin{cases} J(v,h) = (\tilde{L}\,\tilde{x}(T),\, \tilde{x}(T))_H - 2(\tilde{\ell},\tilde{x}(T))_H \\ \quad + \int_0^T [(\tilde{Q}(t)\tilde{x}(t),\, \tilde{x}(t))_H - 2(\tilde{q}(t),\, \tilde{x}(t))_H]dt \\ \quad + \int_0^T (N(t)v(t),\, v(t))_U dt, \end{cases}$$

where $\tilde{L} \in \mathscr{L}(H)$, $\tilde{\ell} \in H$, $q \in L^2(0,T;H)$ and $\tilde{Q}$: $[0,T] \to \mathscr{L}(H)$ and N: $[0,T] \to \mathscr{L}(U)$ are strongly measurable and bounded. Moreover $\tilde{L}$, $\tilde{Q}(t)$ and N(t) are self adjoint and positive and there exists c > 0 such that

$$(19) \quad \forall t,\ \forall u,\ (N(t)u,u)_U > 0.$$

For this problem we know that given h in V, there exists a unique optimal control function u in $L^2(0,T;U)$ which minimizes J(v,h) over all v in $L^2(0,T;U)$. Moreover this optimal control can be synthesized via the feedback law

$$(20) \quad u(t) = -N(t)^{-1}\,\tilde{B}(t)^*\,[\pi(t)\tilde{x}(t) + r(t)],$$

where π and r are characterized by the following equations:

$$(21)\quad \begin{cases} \frac{d\pi}{dt}(t) = \tilde{A}(t)^*\pi(t) + \pi(t)\tilde{A}(t) - \pi(t)\tilde{R}(t)\pi(t) + \tilde{Q}(t) = 0, \text{ in } [0,T] \\ \pi(T) = \tilde{L} \end{cases}$$

$$(22)\quad \begin{cases} R(t) = B(t)^*N(t)B(t) \\ \tilde{R}(t) = \begin{bmatrix} R(t) & 0 \\ 0 & 0 \end{bmatrix} \end{cases}$$

and

$$(23)\quad \begin{cases} \frac{dr}{dt}(t) + [\tilde{A}(t) - \tilde{R}(t)\pi(t)]^*r(t) + [\pi(t)\tilde{f}(t) + \tilde{q}(t)] = 0, \text{ in } [0,T] \\ r(T) = \tilde{\ell}. \end{cases}$$

Here a solution of (21) is a map π: $[0,T] \rightarrow \mathscr{L}(H)$ which is weakly continuous such that for all h and k in V the map $t \rightarrow (h,\pi(t)k)_H$ is in $H^1(0,T;R)$; a solution of (23) is a map r: $[0,T] \rightarrow H$ such that $r \in L^2(0,T;H)$ and $D\, r \in L^2(0,T;V')$. For details see DELFOUR-MITTER [3] and BENSOUSSAN-DELFOUR-MITTER [1].

5. Optimal Control Problem in $[0,\infty]$

We can also give a complete theory for cost functions of the form

$$(24)\quad J_\infty(v,h) = \int_0^\infty [(\tilde{Q}\tilde{x}(t), \tilde{x}(t))_H + (Nv(t),v(t))_U]dt$$

with the following hypothesis:

1) $\tilde{Q} \in \mathscr{L}(H)$, $N \in \mathscr{L}(U)$ are positive and self adjoint and there exists $c > 0$ such that

$$\forall u \ (Nu,u)_U \geq c\, |u|_U^2 \ ;$$

2) $\tilde{x}$ is the solution of

$$(25)\quad \begin{cases} \frac{d\tilde{x}}{dt}(t) = \tilde{A}x(t) + \tilde{B}v(t) \quad \text{in} \quad [0,\infty] \\ \tilde{x}(0) = h \quad \text{in} \quad V; \end{cases}$$

3) (Stabilizability hypothesis) there exists a feedback operator $G \in \mathscr{L}(V,U)$ of the form

$$(26)\quad Gh = G_{oo}h(0) + \sum_{j=1}^{M} G_i h(\Theta_i) + \int_{-a}^{0} G_{o1}(\Theta)h(\Theta)d\Theta$$

such that the closed loop system

$$(27)\quad \begin{cases} \dfrac{d\tilde{x}}{dt}(t) = [\tilde{A} + \tilde{B}\, G]\tilde{x}(t) & \text{in} \quad [0,\infty] \\ \tilde{x}(0) = h \in V \end{cases}$$

be L^2-stable, that is

$$(28)\quad \forall\, h \in H, \quad \int_0^{\infty} ||\tilde{x}(t)||_H^2 \, dt < \infty.$$

For a study of the stabilizability problem see VANDEVENNE [1] [2].

When system (25) is stabilizable, there exists a unique u in $L^2{}_{loc}(0,\infty;U)$ which minimizes J_∞ (v,h) over all v in $L^2{}_{loc}(0,\infty;U)$ for a given h. Moreover this optimal u can be synthesized via a constant feedback law.

$$(29)\quad u(t) = -N^{-1}\, \tilde{B}^*\, \pi\, \tilde{x}(t),$$

where π is a solution of the algebraic Riccati equation

$$(30)\quad \tilde{A}^*\pi + \pi\tilde{A} - \pi\tilde{R}\pi + \tilde{Q} = 0.$$

A solution of (30) is a positive self adjoint element of $\mathscr{L}(H)$ such that (30) is verified as an equation in $\mathscr{L}(V,V')$. The operator π in $\mathscr{L}(H)$ can be decomposed into a matrix of operators

$$\begin{pmatrix} \pi_{oo} & \pi_{o1} \\ \pi_{1o} & \pi_{11} \end{pmatrix}$$

(since H is either X x $L^2(-a,0;X)$ or X x $L^2(-\infty,0;X)$) where

$$\begin{cases} \pi_{oo} \in \mathscr{L}(X) \quad , \quad \pi_{o1} \in \mathscr{L}(L^2(-a,0;X),X) \\ \pi_{1o} \in \mathscr{L}(X,L^2(-a,0;X)) \ , \ \pi_{11} \in \mathscr{L}(L^2(-a,0;X)). \end{cases}$$

Moreover

$$\begin{cases} \pi_{oo} A_{oo} + A_{oo}^*\pi_{oo} + \pi_{1o}(0) + \pi_{1o}(0)^* + Q - \pi_{oo}R\pi_{oo} = 0 \\ \pi_{oo}^* = \pi_{oo} \geq 0 \end{cases}$$

$$(\pi_{1o}h^o)(\alpha) = \pi_{1o}(\alpha)h^o \quad , \quad \alpha \to \pi_{1o}(\alpha) : [-a,0] \to \mathscr{L}(X)$$

$$\begin{cases} \frac{d\pi_{1o}}{d\alpha}(\alpha) = \pi_{1o}(\alpha)\ [A_{oo} - R\pi_{oo}] + \sum_{i=1}^{N-1} A_i{}^*\pi_{oo}\delta(\alpha-\Theta) \\ \qquad + A_{o1}(\alpha)^*\pi_{oo} + \pi_{11}(\alpha,0)\ ,\ \text{a.e. in } [-a,0] \\ \pi_{1o}(-a) = A_N{}^*\pi_{oo} \end{cases}$$

$$(\pi_{o1}h^1(\alpha) = \int_{-a}^{o} \pi_{1o}(\alpha)^*h^1(\alpha)d\alpha$$

$$\begin{cases} (\pi_{11}h^1)(\alpha) = \int_{-a}^{o} \pi_{11}(\alpha,\beta)h^1(\beta)d\beta \\ (\alpha,\beta) \to \pi_{11}(\alpha,\beta) : [-a,0] \times [-a,0] \to \mathscr{L}(X) \\ \left[\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right] \pi_{11}(\alpha,\beta) = A_{o1}(\alpha)^*\pi_{1o}(\beta)^* + \pi_{1o}(\alpha)A_{o1}(\beta) \\ \qquad + \sum_{i=1}^{N-1} A_i{}^*\pi_{1o}(\beta)^*\delta(\alpha-\Theta_i) + \sum_{j=1}^{N-1} \pi_{1o}(\alpha)A_j\delta(\beta-\Theta_i) \\ \qquad - \pi_{1o}(\alpha)\ R\pi_{1o}(\beta)^* \\ \pi_{11}(-a,\beta) = A_N{}^*\pi_{1o}(\beta)^*\ ,\ \pi_{11}(\alpha,-a) = \pi_{1o}(\alpha)A_N \\ \pi_{11}(\alpha,\beta) = \pi_{11}(\beta,\alpha)^*. \end{cases}$$

Under additional hypothesis on $\tilde{A}$ and $\tilde{Q}$ we can also describe the asymptotic behaviour of the closed loop system

$$(31)\quad \begin{cases} \frac{dx}{dt}(t) = [\tilde{A} - \tilde{R}\pi]\ \tilde{x}(t) \quad \text{in } [0,\infty] \\ \tilde{x}(0) = h \ \text{ in } \ V. \end{cases}$$

Definition Given a Hilbert space of observations Y and an observer $M \in \mathscr{L}(H,Y)$, System (25) is said to be observable by M if each initial datum h at time 0 can be determined from a knowledge of v in $L^2_{loc}(0,\infty;U)$ and the observation

$$(32)\quad z(t) = M\ x(t) \quad \text{in} \quad [0,\infty].$$

When System (15) is observable by $\tilde{Q}^{1/2}$, for each initial datum h

(33) $\tilde{x}(t) \to 0 \quad \text{as} \quad t \to \infty,$

where $\tilde{x}$ is the solution of the closed loop system (31).

In the special case where

$$\tilde{Q} = \begin{bmatrix} Q_{oo} & 0 \\ 0 & 0 \end{bmatrix}$$

and Q_{oo} is positive definite, the closed loop system (21) is L^2-stable. For further details see DELFOUR-MCCALLA-MITTER.

6. Optimal Linear Filtering and Duality

Let E and F be two Hilbert spaces. We consider the system

$$(34) \begin{cases} \frac{dx}{dt}(t) = A_{oo}(t)x(t) + \sum_{i=1}^{N} A_i(t)x(t+\Theta_i) + \int_{-a}^{o} A_{o1}(t,\Theta)x(t+\Theta)d\Theta \\ \qquad + B(t)\xi(t) + f(t), \\ x(0) = h^o + \zeta^o \\ x(\Theta) = h^1(\Theta) + \zeta^1(\Theta), \quad -a \leq \Theta < 0, \end{cases}$$

where $\zeta = (\zeta^o, \zeta^1)$ is the noise in the initial datum, and ξ is the input noise with values in F. We assume an observation of the form (with values in E)

(35) $z(t) = C(t)x(t) + \eta(t),$

where η represents the error in measurement and $C(t)$ belongs to $\mathscr{L}(X,E)$. As in BENSOUSSAN [1] $\{\zeta^o, \zeta^1, \xi, \eta\}$ will be modelled as a Gaussian linear random functional on the Hilbert space.

(36) $\Phi = X \times L^2(-a,0;X) \times L^2(0,T;E) \times L^2(0,T;F)$

with zero mean and covariance operator

$$(37) \quad \Lambda = \begin{bmatrix} P_o & & & \\ & P_1(\Theta) & & \\ & & Q(t) & \\ & & & R(t) \end{bmatrix}.$$

For each T we want to determine the best estimator of the linear random functional $x(T)$ with respect to the linear random functional $z(s)$, $0 \leq s \leq T$. For the solution to this problem see BENSOUSSAN [2] and BENSOUSSAN-DELFOUR-MITTER [2].

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