Stabilization of Linear Systems With Limited Information

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***Abstract*—In this paper, we show that the coarsest, or least dense, quantizer that quadratically stabilizes a single input linear discrete time invariant system is logarithmic, and can be computed by solving a special linear quadratic regulator (LQR) problem. We provide a closed form for the optimal logarithmic base exclusively in terms of the unstable eigenvalues of the system. We show how to design quantized state-feedback controllers, and quantized state estimators. This leads to the design of hybrid output feedback controllers. The theory is then extended to sampling and quantization of continuous time linear systems sampled at constant time intervals. We generalize the definition of density of quantization to the density of sampling and quantization in a natural way, and search for the coarsest sampling and quantization scheme that ensures stability. We show that the resulting optimal sampling time is only function of the sum of the unstable eigenvalues of the continuous time system, and that the associated optimal quantizer is logarithmic with the logarithmic base being a universal constant independent of the system. The coarsest sampling and quantization scheme so obtained is related to the concept of minimal attention control recently introduced by Brockett. Finally, by relaxing the definition of quadratic stability, we show how to construct logarithmic quantizers with only finite number of quantization levels and still achieve *practical* stability of the closed-loop system. This final result provides a way to practically implement the theory developed in this paper.**

***Index Terms*—Hybrid systems, minimal information, optimal control, quantization.**

I. INTRODUCTION

IN THIS PAPER, our main goal is to develop a theory of stabilization of linear time-invariant (LTI) systems using only a finite number of fixed control values and finite number of measurement levels. The quantization of controls and measurements induces a quantization, or partition, in the system state-space.

We want to point out that our view fundamentally differs from the traditional view where the effects of quantization are seen as undesirable, either as noise, or state uncertainty, and must be reduced by often complex controllers [1]–[5].

In this paper instead, we seek to quantize the state of the system as coarsely as possible while maintaining the stability (and the performance) of the system. This problem is motivated by the lack of fundamental understanding of how to do systematic design of complex systems.

> For example, many hybrid phenomena (interaction between continuous dynamics and logic) are effects of information quantization [14], [16], [18], [22], [23]. In order to derive systematic design methods for hybrid systems, we need to understand how to systematically quantize information without losing stability and/or performance.
>
> In the hierarchical organization of systems, it is evident that higher levels in the hierarchy manipulate only quantized information about the dynamics at lower levels [21]. It is important to understand what is the minimum information needed in order to complete a given task.
>
> Complex systems are often spatially distributed aggregations of many subsystems. The coordination and control of such systems is achieved through communication channels. The number of the subsystems together with bandwidth limitations of the channels limit the information about the state of each subsystem available at the controller [9], [19].

In other words, we consider quantization useful, if not essential, instead of undesirable.

It is also worth mentioning that we are interested in the design of quantized closed-loop systems which are implicitly verified. This is in contrast with traditional stability analysis results obtained for a given quantizer already in place [6]–[8], [15], [17], and more along the line of [10].

The paper is organized as follows. We begin our study with discrete-time systems and quantizers with countable number of levels. As a first step we allow for countable quantizers, which makes the analysis and the notation simpler, captures the fundamental laws, and also provides important asymptotic results. We first solve the full-state feedback problem where the control values can take a countable number of (to be determined) fixed values. This is done in Section II which is the main section of the paper and contains the main ideas and results for the state-feedback case. In particular, assuming that the system is quadratically stabilizable, we show that the quantizer is logarithmic (the fixed levels follow a logarithmic law). Further, we characterize the coarsest, or least dense, (largest spacing between levels) logarithmic quantizer over all quadratic control Lyapunov functions in terms of the solution of a special linear quadratic regulator (LQR) problem. Then, in Section III, we show how the same approach and results apply to the design of state-observers using a countable number of quantized measurements. From these results we can design stabilizing output feedback controllers with quantized measurements and controls. In

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Section IV we consider sampled-quantized systems. We assume that the discrete-time system of the Sections I–III are obtained from constant sampling in time of a continuous-time system. We extend the definition of density of quantization to the density of sampling and quantization in a natural way, and search for the coarsest (least dense) sampling and quantization scheme. We show that the resulting optimal sampling time is only a function of the unstable eigenvalues of the continuous time system, and provide a formula for it. The associated optimal quantizer is logarithmic with the logarithmic base being a universal constant independent of the system. Finally, in Section V, we show that the system can be stabilized by finite logarithmic quantizers obtained by truncating the countable logarithmic quantizers. The reader should review the result of Section IV in the light of the result of Section V. In Section VI, we present an application of the theory to an example, and construct a finitely quantized output feedback discrete controller, which stabilizes a continuous time system. Finally, in Section VII, we present some conclusions and discuss future directions of research.

II. Quantized State Feedback

In this section we consider the problem of stabilizing an LTI discrete-time system with a possibly countable number of fixed control values to be determined.

The system is assumed unstable, single input, stabilizable, and governed by the following equation:

$$x^+ = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n \triangleq X$, x^+ denotes the system state at the next discrete-time, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times 1}$.

Since the system is stabilizable and linear, it is quadratically stabilizable, i.e., there is a control input u, function of x, that makes a quadratic function of the state a Lyapunov function for the closed-loop system. Such Lyapunov functions are called control Lyapunov functions (CLFs). For LTI systems, given a CLF, it is always possible to find a stabilizing control in the form of a linear static state-feedback control.

A. Approach and Problem Definition

Given a quadratic CLF $V(x) = x'Px$ with $P > 0$, P is always assumed to be symmetric in this paper, we propose to select a set of fixed control values such that $V(x)$ is still a Lyapunov function for the system, i.e., it decreases along the system's trajectories. In particular, we ask that for any $x \neq 0$

$$\Delta V(x) \triangleq V(x^+) - V(x) < 0.$$

More precisely, we want to solve the following problem.

Problem 2.1: For a given CLF $V(x) = x'Px$, $P > 0$, we want to find a set

$$\mathcal{U} = \{u_i \in \mathbb{R}: i \in Z\}$$

and a function

$$f: X \to \mathcal{U}$$

such that $f(x) = -f(-x)$, and such that for any $x \in X$, $x \neq 0$

$$\Delta V(x) = V(Ax + Bf(x)) - V(x) < 0.$$

With a slight abuse of terminology f is called the quantizer. Notice that the range of f induces a partition in the state-space of the system, where equivalence classes of states correspond to the same adopted control value, i.e., $\Omega_i = \{x \in X | f(x) = u_i\}$. Also, by requiring the quantizer to be a function we are implicitly assuming that to each x there corresponds only one element in $\mathcal{U}$. Notice finally that we consider only quantizers that are symmetric with respect to the origin and with an infinite countable number of levels. The first is not a restriction given the natural symmetry of the system and the Lyapunov function. The second is also not a restriction since, as will see in the development, such quantizers are required to solve Problem 2.1. By considering directly infinite countable quantizers we avoid needless, more elaborate, definitions that include quantizers with finite levels.

We assume that the values in the set $\mathcal{U}$ are ordered as follows $u_i < u_j$ for $i > j$, $i, j \in Z$.

u_{i+1} is called the immediate successor of u_i and u_i is called the immediate predecessor of u_{i+1}.

Definition 2.1: A quantizer taking the value $\beta_u \in \mathbb{R}$, $\beta_u > 0$ is such that $u_j = \beta_u \in \mathcal{U}$ for some j.

Lemma 2.1: Let $f: X \to \mathcal{U}$ be a quantizer that solves Problem 2.1, and let $\Omega_i = \{x \in X | f(x) = u_i\}$, $i \in Z$. Given any real number $\beta > 0$, define $\beta\mathcal{U} = \{\beta u_i : u_i \in \mathcal{U}\}$, and $\beta\Omega_i = \{y = \beta x : x \in \Omega_i\}$. Then $g: X \to \beta\mathcal{U}$ with $g(x) = \beta u_i$ for $x \in \beta\Omega_i$, and $i \in Z$, also solves Problem 2.1.

Proof: If f solves Problem 2.1, then, for any $i \in Z$, u_i is such that $V(Ax + Bu_i) - V(x) < 0$ for all $x \in \Omega_i$, $x \neq 0$. Form the linearity of (1), it follows that $V(Ax + B\beta u_i) - V(x) < 0$ for all $x \in \beta\Omega_i$. ■

The above lemma implies that there is no loss of generality in only considering quantizers with value $\beta_u = 1$, and, unless otherwise specified, value 1 will always be assumed.

We measure the coarseness of a quantizer by measuring its density defined next.

Definition 2.2: Given $V(x) = x'Px$, $P > 0$, a CLF for (1), let $\mathcal{Q}(V)$ denote the set of all quantizers that solve Problem 2.1. For $g \in \mathcal{Q}(V)$ and $0 < \epsilon \leq 1$, let $\#g[\epsilon]$ denote the number of levels that g has in the interval $[\epsilon, 1/\epsilon]$. Define

$$\eta_g = \limsup_{\epsilon \to 0} \frac{\#g[\epsilon]}{-\ln \epsilon}.$$

η_g is called the quantization density (of g). A quantizer f is said to be *coarsest* for $V(x)$ if it has the smallest density of quantization, i.e.,

$$f = \arg \inf_{g \in \mathcal{Q}(V)} \eta_g.$$

A quantizer which is *coarsest* for $V(x)$ need not be unique since, different sets $\mathcal{U}$ may satisfy the asymptotic property, and for the same $\mathcal{U}$, there may be different ways to define the function f mapping X into $\mathcal{U}$. Moreover, a quantizer which is *coarsest* for $V(x)$ may not be an element of $\mathcal{Q}(V)$. At any rate, since the quantizer induces a partition on X, the density of quantization induces a measure of coarseness on the partitions in the state-space X.

It is worth pointing out that the definition of density we are adopting in this paper allows us to measure quantizers for which the number of quantization values, although infinite, grows logarithmically, rather than linearly, with the length of the interval that includes them. Under this measure, the density of any uniform quantizer is infinity, and the density of any finite quantizer is zero.

The main result of this section is the characterization of a quantizer which is *coarsest* for $V(x)$. As we will describe presently, the main idea in the derivation of $\mathcal{U}$ and f is to consider the CLF as a robust Lyapunov function where, for a given fixed control value, we are interested in finding the set of all states for which $\Delta V(x)$ is negative enough.

We want to emphasize that the idea of using robust Lyapunov functions to design nonlinear control strategies, in particular quantizers, for linear systems is new and has great advantages with respect to traditional approaches based on optimal control which even in the case of a given fixed quantizer would often lead to intractable integer programming problems.

B. Problem Solution: Logarithmic Quantizer

In this section, we derive one of the main results of this paper. We show that, for a given CLF, there is a natural quantization of the control values and a partition of state-space which follows a logarithmic law. This result captures in a precise way the intuitive notion that, the farther from the origin the state is, the less precise the control action and knowledge about the location of the state in the state-space need to be.

Before we present the results, we need to introduce some notation. Given a CLF, $V(x) = x'Px$ $(P > 0)$, for (1), $\Delta V(x)$ is given by the following expression:

$$\Delta V(x) = x'(A'PA - P)x + 2x'A'PBu + u'B'PBu.$$

$'$ denotes transpose. For a given x, the control that makes $\Delta V(x)$ the most negative is given by

$$u = -\frac{B'PA}{B'PB}x \triangleq K_{GD}x.$$

Notice that the control is given in terms of a linear static feedback K_{GD} where GD stands for gradient descent, since it represents the controller that makes $V(x)$ decrease the most along trajectories.

Let $A_c = A + BK_{GD}$ be the resulting closed-loop state-transition matrix. The resulting $\Delta V(x)$ is given by

$$\Delta V(x) = x'\left(A_c'PA_c - P\right)x$$

For future reference, let

$$Q = P - A_c'PA_c = P - A'PA + \frac{A'PBB'PA}{B'PB}. \quad (2)$$

Note that $Q > 0$ since $x'Px$ is a CLF.

We are now ready to state the first main result of this section.

Theorem 2.1: Let $V(x) = x'Px, P > 0$ be a CLF for system (1). A quantizer $f : X \to \mathcal{U}$, $f(x) = u$, which is coarsest for $V(x)$, has fixed control values that follow a logarithmic law, and it is characterized as follows:

$$\mathcal{U} = \{\pm u_i, : u_{i+1} = \rho u_i,\ u_0 = \beta_u,\ i \in Z\} \cup \{0\}$$

the constant $0 \le \rho < 1$ is given by the following expression:

$$\rho = \frac{\sqrt{\frac{B'PAQ^{-1}A'PB}{B'PB}} - 1}{\sqrt{\frac{B'PAQ^{-1}A'PB}{B'PB}} + 1} \quad (3)$$

with

$$\eta_f = \frac{2}{\ln\left(\frac{1}{\rho}\right)}$$

and

$$\begin{aligned} f(x) &= -f(-x) \\ f(x) &= u_i \quad \forall\, x \in \Omega_i^+,\ i \in Z \\ f(x) &= 0 \quad \forall\, x \in \Omega_{\text{zero}} \end{aligned}$$

where

$$\begin{aligned} \Omega_i^+ &= \{x \in X | \alpha_{i+1} < K_{GD}x \le \alpha_i,\ \alpha_{i+1} = \rho\alpha_i \\ &\qquad \alpha_0 = \beta_\alpha,\ i \in Z\}, \\ \Omega_{\text{zero}} &= \{x \in X | K_{GD}x = 0\} \end{aligned}$$

and

$$\beta_\alpha = \frac{1+\rho}{2\rho}\beta_u.$$

Moreover, given any $0 < \epsilon < 1-\rho$, let $\rho_\epsilon = \rho + \epsilon$, and construct f_ϵ as f, but with ρ_ϵ instead of ρ. Then $f_\epsilon \in \mathcal{Q}(V)$, and therefore solves Problem 2.1.

In proving this Theorem, we will actually prove a stronger result. First, we need the result stated in the following Lemma.

Lemma 2.2: Let $V(x) = x'Px, P > 0$, be a CLF for system (1). For any $x \in X$ $x \ne 0$, denote by $U(x)$ the following set:

$$U(x) = \{u \in \mathbb{R} | \Delta V(x) < 0\}.$$

Then, $U(x)$ is equivalently characterized by the following open interval:

$$U(x) = \left\{u \in \mathbb{R} | u^{(1)} < u < u^{(2)}\right\}$$

where $u^{(1)}$ and $u^{(2)}$ are the roots of the second order equation in u

$$x'(A'PA - P)x + 2x'A'PBu + u'B'PBu = 0$$

and is given by the following expression:

$$u^{(1),(2)} = K_{GD}x \pm \sqrt{\frac{x'Qx}{B'PB}}.$$

Proof: See Appendix. ∎

$U(x)$ is nothing but the set of control values u that can be selected (for the given x) to ensure that the Lyapunov function is still decreasing along trajectories.

Proof of Theorem 2.1: $U(x)$ has the following important properties whose verification is immediate:

P1) *scaling* $U(\alpha x) = \alpha U(x)$ for $\alpha > 0$;
P2) *symmetry* $u^{(1)} = -u^{(2)}$ for all x such that $K_{GD}x = -B'PAx/B'PB = 0$.

From Property $(P2)$, it follows that $u = 0$ can be used for all $x \perp K'_{GD}$ to ensure that the Lyapunov function decreases along trajectories. This also implies that, in searching for a quantizer which is coarsest, we can restrict our attention to the partition induced by the quantizer in the direction K'_{GD} as explained by the following argument.

Let $Y_{GD} \subset X$ denote the subspace of X generated by K'_{GD}

$$Y_{GD} = \left\{ x \in X : x = y\frac{K'_{GD}}{K_{GD}K'_{GD}},\, y \in \mathbb{R} \right\}.$$

Given any quantizer $g : X \to \mathcal{U}$, $g \in \mathcal{Q}(V)$, consider the restriction of g on Y_{GD}

$$\begin{aligned} h : Y_{GD} \to & \mathcal{U}_{Y_{GD}} \\ h(x) = & \{g(x) | x \in Y_{GD}\} \end{aligned}$$

and the extension g^{GD} of h given by

$$\begin{aligned} g^{GD} : X \to & \mathcal{U}_{Y_{GD}} \\ g^{GD}(x) = & h(K_{GD}x). \end{aligned}$$

From Property $(P2)$ it follows that $g^{GD} \in \mathcal{Q}(V)$. Note that $\mathcal{U}_{Y_{GD}} \subseteq \mathcal{U}$, therefore

$$\# g^{GD}[\epsilon] \le \# g[\epsilon] \text{ for all } 0 < \epsilon \le 1.$$

Thus, we only need to look for the coarsest quantizer in the direction K'_{GD}, or equivalently

$$\inf_{g \in \mathcal{Q}(V)} \eta_g = \inf_{g^{GD} \in \mathcal{Q}(V)} \eta_{g^{GD}}.$$

To simplify the derivations ahead, it is more convenient to do the following change of coordinates. Let

$$z = Q^{1/2}x.$$

Note that $Q^{1/2}$ is well defined since $Q > 0$. In this new coordinate system, the boundary points $u^{(1),(2)}$ of $U(z)$ are given by the following expression:

$$u^{(1),(2)} = K_{GD}Q^{-1/2}z \pm \sqrt{\frac{z'z}{B'PB}}.$$

Note that $K_{GD}Q^{-1/2} \triangleq \overline{K}_{GD}$ is the gradient descent controller in the new coordinate system. This suggests the following natural decomposition of the state-space in the z-coordinates.

$$z = Q^{-1/2}A'PB\alpha + w\beta \quad w \perp Q^{-1/2}A'PB.$$

The boundary points $u^{(1),(2)}$ of $U(z)$ in terms of α and β are given by the following expression:

$$\begin{aligned} u^{(1),(2)} = & \alpha\frac{B'PAQ^{-1}A'PB}{B'PB} \\ & \pm\sqrt{\alpha^2\frac{B'PAQ^{-1}A'PB}{B'PB} + \beta^2\frac{1}{B'PB}}. \end{aligned}$$

We see that the smallest, or worst-case, interval $U(\alpha, \beta)$ is obtained by setting $\beta = 0$. In other words, the worst direction of quantization, the one with the most restricted choice of control values, is the one parallel to $\overline{K'}_{GD}$, and, for any $\beta > 0, U(\alpha, \beta)$ grows symmetrically around $U(\alpha, 0)$.

Note that the change of variable implies that $K_{GD}x = \alpha(B'PAQ^{-1}A'PB/B'PB)$ for some $\alpha \in \mathbb{R}$, with the right-hand side being a more convenient representation to handle.

We can now use the scaling property $(P1)$ to show that the coarsest covering in the direction $\overline{K}'_{GD}$ follows a logarithmic law. Consider without loss of generality the set $U(1, 0)$. Define

$$\rho \triangleq \inf_{U(\alpha,0)\cap U(1,0) \neq \emptyset} \alpha.$$

In other words, ρ tells us what is the maximum range of states along the direction $\overline{K}'_{GD}$ for which there is a common control value that still decreases the Lyapunov function in the next step.

From Property $(P1)$ we have that

$$\rho = \inf_{\alpha U(1,0)\cap U(1,0) \neq \emptyset} \alpha.$$

Since the boundary points corresponding to $U(1, 0)$ are

$$u^{(1)} = \frac{B'PAQ^{-1}A'PB}{B'PB} - \sqrt{\frac{B'PAQ^{-1}A'PB}{B'PB}}$$

and

$$u^{(2)} = \frac{B'PAQ^{-1}A'PB}{B'PB} + \sqrt{\frac{B'PAQ^{-1}A'PB}{B'PB}}$$

it turns out that

$$\rho = \inf_{\alpha u^{(2)} > u^{(1)}} \alpha$$

or

$$\rho = \frac{u^{(1)}}{u^{(2)}} = \frac{\sqrt{\frac{B'PAQ^{-1}A'PB}{B'PB}} - 1}{\sqrt{\frac{B'PAQ^{-1}A'PB}{B'PB}} + 1}.$$

Note that $0 \le \rho < 1$.

The relationship between β_u and β_α can be derived as follows. $u_0 = \beta_u$ is the common value of u to be used for all the values of $K_{GD}x \in (\rho\alpha_0, \alpha_0)$. Let $\alpha_0 = \beta_\alpha = \alpha\left(B'PAQ^{-1}A'PB/B'PB\right)$ for some α. Then

$$\begin{aligned} u_0 = & \alpha\rho\frac{B'PAQ^{-1}A'PB}{B'PB} + \sqrt{\rho^2\alpha^2\frac{B'PAQ^{-1}A'PB}{B'PB}} \\ = & \alpha\frac{B'PAQ^{-1}A'PB}{B'PB} - \sqrt{\alpha^2\frac{B'PAQ^{-1}A'PB}{B'PB}}. \end{aligned}$$

Substituting in the last equality for α we obtain

$$u_0 = \beta_\alpha\left(1 - \frac{1}{\sqrt{\frac{B'PAQ^{-1}A'PB}{B'PB}}}\right)$$

and since

$$\sqrt{\frac{B'PAQ^{-1}A'PB}{B'PB}} = \frac{1+\rho}{1-\rho}$$

we have that

$$\beta_\alpha = \frac{1+\rho}{2\rho}\beta_u \text{ or } \alpha_0 = \frac{1+\rho}{2\rho}u_0.$$

Now, from the previous derivation we know that $u_0 = \beta_u$ guarantees the non increasing of $V(x)$ for all $x \in \overline{\Omega}_0^+ = \{x \in X : \rho\alpha_0 \le K_{GD}x \le \alpha_0\}$. Moreover, we know that $\overline{\Omega}_0^+ \cap Y_{GD} \supset \Omega_0 \cap Y_{GD}$, for any Ω_0 which can be associated to u_0 by any quantizer in $\mathcal{Q}(V)$. Thus, from Lemma 2.1, of which the scaling Property $(P1)$ is a consequence, we have that $u_1 = \rho u_0$ guarantees the non increasing of $V(x)$ for all $x \in \rho\overline{\Omega}_0^+ = \overline{\Omega}_1^+$. Furthermore, $u = u_1$ is the smallest value that can be an immediate predecessor of u_0. Fig. 1 gives a visual proof of this statement and helps seeing that if $u_1 < \rho u_0$ then there is gap in the covering of K_{GD}, that cannot be covered by any set Ω associated with any value of control u which is either $u > u_0$ or $u < u_1$. The same argument can be repeated for u_2 and by induction for any u_i. Ω_i^+ in the theorem statement are derived from the corresponding $\overline{\Omega}_i^+$ by replacing the nonstrict inequality $\le$ with a strict one, so that f is a well defined function, and Ω_{zero} is the natural closure of the partition. Thus, we have that the sequence for both positive and negative control values is given by

$$\pm u_{i+1} = \pm u_i \rho \quad i \in Z \quad \text{with } u_0 = \beta_u. \tag{4}$$

and the resulting quantization in the state-space is given by

$$\pm \alpha_{i+1} = \pm \alpha_i \rho \quad i \in Z \quad \text{with } \alpha_0 = \beta_\alpha. \tag{5}$$

From the structure of f it follows that:

$$\eta_f = \frac{2}{\ln \frac{1}{\rho}} < \infty$$

and from the construction we have that for any $g \in \mathcal{Q}(V)$ (with value u_0)

$$\#f[\delta] \le \#g[\delta] \text{ for all } 0 < \delta \le 1.$$

Finally, since for any $0 < \epsilon < 1 - \rho$, and $\rho_\epsilon = \rho + \epsilon$, f_ϵ constructed as f but with ρ_ϵ instead of ρ belongs to $\mathcal{Q}(V)$ (left to the reader), we have that f is *coarsest* for $V(x)$. ■

Fig. 2 shows the resulting logarithmic partition in the z coordinate system.

Remark 2.1: Several remarks are in order. First, this theorem is the first to capture in very precise terms the intuitive argument that when the system is far from the equilibrium we do not need precise knowledge of the state and, therefore, we can use imprecise controls to steer the system in the right direction.

Second, the scaling property at the basis of the logarithmic law of quantization is not only a requisite of quadratic control Lyapunov functions but also of any seminorm, and therefore extends to more general CLFs.

Third, requiring the Lyapunov function to be strictly decreasing along trajectories has the critical role of regularizing the problem and drastically reduces the complexity of finding the right partition. This would have been otherwise intractable even in the single input case considered here.

Finally, we want to remind the reader that quantizer f constructed in the theorem is in general non unique in the sense

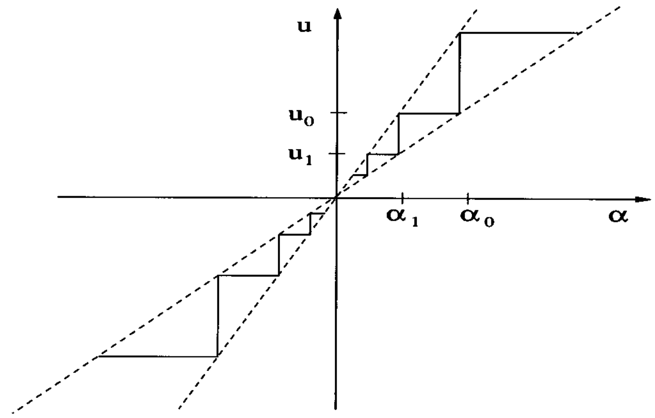

Fig. 1. Logarithmic partition in the α, u plane.

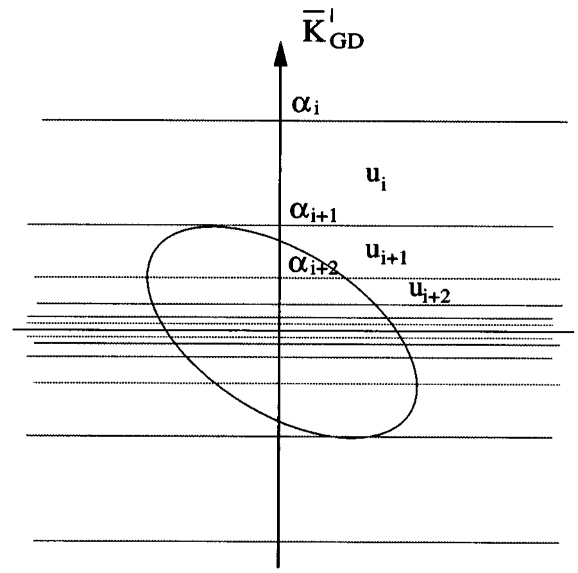

Fig. 2. Logarithmic partition in the (α, β)-coordinate.

that other partitions could be associated to the same set of control values. For example, assume that the system dimension is $n \ge 2$, and consider the new partition defined as follows. Let $\overline{\Omega}_{\text{zero}} = \{x \in \mathbb{R}^n | x'(A'PA - P)x \le 0\}$. This is a cone in the state space that includes Ω_{zero} as defined in Theorem 2.1. It is left to the reader to verify that the inclusion is usually strict. Let $\overline{\Omega}_i^+ = \Omega_i^+ \backslash \overline{\Omega}_{\text{zero}}$, where $\backslash$ means set exclusion. Then, the resulting quantizer will still be coarsest for $V(x)$ by construction. This new partition could be preferable to the one of Theorem 2.1 since the zero control value is used for a larger set of states. However, it is more complex to implement than the one of the theorem. Clearly the two quantizers are the same if and only if $\overline{\Omega}_{\text{zero}} = \Omega_{\text{zero}}$. This will be the case for the coarsest quantizer over all quadratic Lyapunov functions characterized in the next chapter.

Corollary 2.1: If system (1) is stable, then there exists a CLF for which $\rho = 0$.

Proof: Immediate and left to the reader. ■

The next corollary has important implications that will be exploited later on in this paper.

Corollary 2.2: ρ is invariant under linear coordinate transformations.

Proof: Left to the reader. ■

C. Optimal Quantization Over All Quadratic CLF

In this section, we characterize the coarsest quantizer (smallest ρ) by searching over all quadratic CLFs. The optimal quantizer is related to a special LQR problem.

Define

$$\rho^* = \inf_{\substack{V=x'Px \\ P>0 \\ V \text{CLF}}} \rho.$$

Theorem 2.2: Assume that system (1) is unstable, and let $\lambda_i^u \; i = 1,\ldots,k \leq n$ denote the eigenvalues of the matrix A with magnitude greater or equal than 1. Then

i) The optimal quadratic Lyapunov function corresponding to ρ^* is given by the positive–semidefinite solution of the following Riccati equation:

$$P^* = A'P^*A - \frac{A'P^*BB'P^*A}{B'P^*B+1} \tag{6}$$

which is also the solution to the special LQR problem

$$\min_{\substack{x_{k+1}=Ax_k+Bu_k \\ x(0)=x_0}} \text{stable} \sum_{k=0}^{\infty} u_k^2$$

corresponding to the minimum energy control that stabilizes the system.

ii) $K_{GD}^* = -(B'P^*A/B'P^*B)$ and it is parallel to $K_{\text{LQR}} = -B'P^*A/B'P^*B+1$ the LQR optimal controller.

iii) ρ^* is given by the following equation:

$$\rho^* = \frac{\prod_{1\leq i\leq k} |\lambda_i^u| - 1}{\prod_{1\leq i\leq k} |\lambda_i^u| + 1}. \tag{7}$$

Proof: Let

$$\gamma^2 = \frac{B'PAQ^{-1}A'PB}{B'PB}.$$

Then, (3) becomes

$$\rho = \frac{\gamma - 1}{\gamma + 1} \tag{8}$$

which is monotonically increasing for $\gamma \geq 0$. Thus, minimizing ρ is equivalent to minimizing γ (or γ^2) over $P > 0 \; V = x'Px$ CLF for system (1).

Thus, we focus on the equivalent problem

$$\gamma^* = \inf_{\substack{(B'PAQ^{-1}a'PB/B'PB)\leq\gamma^2 \\ V=x'Px \\ P>0 \\ V \text{CLF}}} \gamma.$$

However, the following implications follow immediately, given that $Q^{-1/2}A'PB \in \mathbb{R}^n$:

$$\begin{aligned}
&\frac{B'PAQ^{-1}A'PB}{B'PB} \leq \gamma^2 \\
\Leftrightarrow &\text{Trace}\left\{\frac{B'PAQ^{-1}A'PB}{B'PB}\right\} \\
&= \text{Trace}\left\{Q^{-1/2}A'PB(B'PB)^{-1}B'PAQ^{-1/2}\right\} \leq \gamma^2 \\
\Leftrightarrow &\lambda_{\max}\left(Q^{-1/2}A'PB(B'PB)^{-1}B'PAQ^{-1/2}\right) \leq \gamma^2 \\
\Leftrightarrow &Q^{-1/2}A'PB(B'PB)^{-1}B'PAQ^{-1/2} \leq \gamma^2 I \\
\Leftrightarrow &A'PB(B'PB)^{-1}B'PA \leq \gamma^2 Q \quad \text{(substituting for } Q\text{)} \\
\Leftrightarrow &A'PB(B'PB)^{-1}B'PA \\
&\quad \leq \gamma^2 \times \left(P - A'PA + A'PB(B'PB)^{-1}B'PA\right) \\
\Leftrightarrow &P - A'PA + A'PB\left(\frac{\gamma^2 B'PB}{\gamma^2 - 1}\right)^{-1} B'PA \geq 0.
\end{aligned}$$

By a simple rearrangement it follows that we have to find the smallest γ such that there is a quadratic CLF, $V = x'Px$ with $P > 0$ for which

$$P - A'PA + A'PB\left(B'PB + \frac{B'PB}{\gamma^2 - 1}\right)^{-1} B'PA \geq 0 \tag{9}$$

Notice that γ^2 must be strictly greater than 1, otherwise there is no way that this last expression can be positive–semidefinite, given that the systems is unstable, that $P > 0$, and $V = x'Px$ is a CLF for the system.

Now, let

$$\beta = \frac{B'PB}{\gamma^2 - 1}.$$

Inequality (9) becomes

$$P - A'PA + A'PB(B'PB + \beta)^{-1}B'PA \geq 0. \tag{10}$$

For a given fixed $\beta > 0$, the above expression is a Riccati inequality. Since the inequality is not affected by positive scaling of P, we can, without loss of generality assume that $\beta = 1$. It is well known [25] that any $P > 0$ satisfying (10) is such that $P \geq R$ where R is the solution to the corresponding Riccati equation

$$R - A'RA + A'RB(B'RB + 1)^{-1}B'RA = 0. \tag{11}$$

Since $B'PB/(\gamma^2 - 1) = 1$, this also implies that the smallest γ is obtained by R.

Equation (11) is the same Riccati equation associated with the solution of the following LQR problem

$$\min_{\substack{x_{k+1}=Ax_k+Bu_k \\ x(0)=x_0}} \text{stable} \sum_{k=0}^{\infty} u_k^2 = x_0'Rx_0$$

where R is Riccati solution to the problem of minimum energy control to stabilize the system. This proves *i*) with $P^* = R$. *ii*) follows from the expression of K_{GD} associated with R, i.e.,

$$K_{GK}^* = -\frac{B'RA}{B'RB} \text{ is parallel to } K_{\text{LQR}} = -\frac{B'RA}{B'RB+1}.$$

In order to prove iii), we use the invariance under transformation of ρ^* stated by Corollary 2.2. We already know that if the system is stable, $\rho^* = 0$. There is no loss of generality in assuming that all the eigenvalues of A are outside the unit disc. We can transform the system into the following block diagonal form through a coordinate transformation

$$\tilde{A} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} B_s \\ B_u \end{bmatrix}$$

where A_u describes the dynamics of the unstable modes of the system (all its eigenvalues have magnitude greater than 1, while A_s describes the stable dynamics. Now, the minimal energy controller, in these coordinates, has the form $K^*_{GD} = [K^*_{GDs} K^*_{GDu}]$ with $K^*_{GDs} = 0$, i.e., it will not put any effort stabilizing A_s. This property will become useful in the extension of this theory to quantized estimators developed in Section III. Thus, we can concentrate in designing the quantizer to stabilize A_u without affecting the stability of A_s. Having noticed this, we proceed by assuming for the rest of the proof that all the eigenvalues of A have magnitude greater than 1 (e.g., concentrating only on the pair (A_u, B_u)).

We can also assume, without loss of generality, that the system is in the controllable canonical form.

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \ddots \\ 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

From the property of the expensive control case of the LQR problem [24], we know that the optimal controller, K_{LQR} will place the closed-loop poles in the mirror image of the unstable open-loop poles. This implies that the closed-loop state transition matrix has the following form:

$$A_c = A + BK_{\mathrm{LQR}} = \begin{bmatrix} 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \ddots \\ 0 & \dots & 0 & 1 \\ -\frac{1}{a_0} & -\frac{a_{n-1}}{a_0} & \dots & -\frac{a_1}{a_0} \end{bmatrix}$$

thus, K_{LQR} must have the following expression

$$K_{\mathrm{LQR}} = \left[a_0 - \frac{1}{a_0}, a_1 - \frac{a_{n-1}}{a_0}, \dots, a_{n-1} - \frac{a_1}{a_0} \right]. \quad (12)$$

From the previous derivations it follows that the optimal γ^* has the following value

$$\gamma * = \sqrt{B'RB + 1} = \sqrt{R_{nn} + 1} \quad (13)$$

where R_{nn} denotes the $R(n, n)$ element of R. Thus, we need to compute R_{nn} in order to find an alternative expression for ρ^* based on (8).

Notice that from the expression of K_{LQR} we have that

$$K_{\mathrm{LQR}} = -\frac{B'RA}{B'RB + 1}.$$

By equating the first element of K_{LQR} above with the first element in (12) we obtain the following equation in R_{nn}:

$$\frac{a_0 R_{nn}}{R_{nn} + 1} = \frac{a_0^2 - 1}{a_0}$$

which when solved for R_{nn} gives

$$R_{nn} = a_0^2 - 1.$$

Substituting in the expression (13) for γ^* we obtain that

$$\gamma^* = |a_0|.$$

Note that a_0 is nothing but the product of the eigenvalues of A. Thus, we obtain the desired result by substituting for γ^* in the expression for ρ^*

$$\rho^* = \frac{\gamma^* - 1}{\gamma^* + 1}.$$

■

Theorem 2.2 together with Theorem 2.1 provides a complete characterization of the coarsest quantizer that guarantees quadratic stability of the closed-loop system in terms of the unstable eigenvalues of the open-loop system. Stated more formally, we have the following.

Definition 2.3: Given $V(x) = x'P^*x$ with P^* satisfying (6), the quantizer constructed according to Theorem 2.1 with $\rho = \rho^*$ given by (7) is denoted by f^*, and is called the *coarsest* quantizer (for quadratic stability).

Given the structure of the coarsest quantizer, it is useful to introduce the following.

Definition 2.4: Let f^* be the coarsest quantizer. Then $h^* : Y_{GD} \to \mathcal{U}$ denotes the function with the property that $f^*(x) = h^*(K_{GD}x)$.

Comment: Whenever we use nonlinear feedback, like a quantizer, new equilibrium points may be created in the closed-loop system, and we need to be aware of their effects on the overall trajectory dynamics. Note however that quadratic stability of the closed-loop system does not allow multiple equilibrium points to exist. This also gives an alternative characterization of ρ^* as the largest value for which new equilibria are generated by the quantizer in the closed-loop system dynamics.

Perturbed Nonsingular Problem: Once again, we want to stress that ρ^* may not be an achieved infimum over all quadratically stabilizing quantizers. Although we can construct f^*_ϵ according to Theorem 2.1, and f^*_ϵ can be shown to be stabilizing, we want to describe another way of constructing f^*_ϵ, which will naturally extend to the construction of finite quantizers in Section V.

In constructing a feasible (quadratically stabilizing) quantizer the main issue is to provide a quadratic Lyapunov function $V(x) = x'Px$, $P > 0$, and to guarantee that for any $x \neq 0$, $\Delta V(x) < 0$. A natural candidate would be P^*, the Riccati solution associated to the optimal logarithmic base, ρ^*. Unfortunately P^* is only positive–semidefinite if the system in not completely unstable, therefore is not a valid Lyapunov function. Moreover, even in the case $P^* > 0$, guaranteed if the system is completely unstable, the quadratic stability assumption is not guaranteed by the $V(x) = x'P^*x$ since $\Delta V^*(x) = x'Q^*x \leq 0$. This happens because the matrix Q^* given by (2) is of rank 1. In fact, Q^* is a scalar multiple of $K^*_{GD}{}' K^*_{GD}$. This is easy to verify, and left to the reader, and implies that $\Delta V^*(x) = 0$ for all x orthogonal to K'_{GD}.

Having noticed that, we only need to slightly perturb the problem in order to achieve a feasible Lyapunov function. The singularity is due to the fact that P^* solves the *expensive control*

LQR problem, which is the limit as $\epsilon \rightarrow 0$ of the standard LQR problem. Thus, we just back up a little bit, and solve the Riccati equation associated with the following problem:

$$\min_{\substack{x_{k+1}=Ax_k+Bu_k \text{ stable} \\ x(0)=x_0}} \sum_{k=0}^{\infty} \epsilon x_k' S x_k + u_k^2 = x_0' R x_0$$

where $S > 0$, and ϵ is a very small positive number.

The resulting Riccati equation is the following:

$$P = A'PA - \frac{A'PBB'PA}{B'PB} + \epsilon S.$$

For ϵ small enough, the solution of this equation, the associated K_{GD}, and the associated $\rho \triangleq \rho_\epsilon^*$ get arbitrarily close to those associated with the expensive control case with the resulting $Q_\epsilon^* > 0$.

D. Stability With Guaranteed Decay Rate

The result of the Section II-C-I can be generalized to include rate constraints on the convergence of the trajectories to the origin. This introduces some primitive performance measure into the framework.

It is well known that if the Lyapunov function, $V(x)$ is such that

$$V(x^+) < \alpha^2 V(x) \tag{14}$$

for $0 < \alpha \leq 1$ then

$$\|x_k\|_2 \leq \alpha^k \|x_0\|_2 .$$

We can now extend the results of the Sections II-A–C by applying the same arguments to Inequality (14). Here, we briefly summarize the development.

Equation (14) implies that

$$x'(A'PA - \alpha^2 P)x + 2x'A'PBu + u'B'PBu < 0$$

must hold.

The analogoue of (2) is given by the following equation:

$$Q = \alpha^2 P - A_c' P A_c = \alpha^2 P - A'PA + \frac{A'PBB'PA}{B'PB} \tag{15}$$

and the formula for ρ given by (3) is unchanged although the value of ρ will depend on the different value of Q and possibly on a different P.

Finally, the derivation of Theorem 2.2 also follows in a straightforward way. The reader, may verify that Inequality (9) is changed into the following one:

$$\alpha^2 P - A'PA + A'PB \left(B'PB + \frac{B'PB}{\gamma^2 - 1} \right)^{-1} B'PA \geq 0. \tag{16}$$

Now dividing by α^2, we can rewrite the above inequality as follows:

$$P - \frac{A'}{\alpha} P \frac{A}{\alpha} + \frac{A'}{\alpha} PB \left(B'PB + \frac{B'PB}{\gamma^2 - 1} \right)^{-1} B'P \frac{A}{\alpha} \geq 0.$$

Following the development, we see that the optimal solution P which provides the smallest ρ and guarantees that the trajectories have a decay rate α given by the optimal P which provides the smallest ρ for the system

$$x^+ = \frac{A}{\alpha} x + Bu. \tag{17}$$

It also follows that, the optimal ρ with a decay rate α for system (1) is given by the optimal ρ for System (17).

III. Quantized State Estimation

In Section II, we saw that the optimal quantizer is closely related to the optimal *expensive control* LQR controller. Such a controller has the property that, in closed loop, it places the unstable open-loop poles at their mirror images, and leaves the stable ones in their original location in the closed-loop control. In this section, we show how the properties and ideas described in Section II apply to the problem of designing e quantized state estimator. We claim that, in this framework, we need to quantize the estimator error rather than the measurements. This is also what is done in [9] for the problem of control with communication constraints.

Consider a traditional linear state estimator for a discrete-time system.

$$\hat{x}^+ = A\hat{x} + Bu + L(C\hat{x} - y).$$

We assume that the system is single output, so $y = Cx$ is a scalar, and it is observable.

The estimator error $e = x - \hat{x}$ follows the following dynamics:

$$e^+ = Ae + LCe.$$

The estimation error goes to zero as the discrete time progresses only if $A + LC$ has all the eigenvalues strictly inside the unit disc.

This clearly resembles the situation of the state-feedback problem with e in place of x, L in place of B, and C in place of K. However, before we were given B and we had to find the coarsest quantizer, here denoted by $f_S^*(x)$, for the system

$$x^+ = Ax + Bu$$

which turned out to have the structure $f_S^*(x) = h_S^*(K_{GD}^* x)$, for some K_{GD}^*. Now, we need to find, over all feasible estimator gains L, the coarsest quantizer, denoted by $f_E^*(e)$, for the system

$$e^+ = Ae + L\xi \tag{18}$$

with $f_E(e) = h_E^*(Ce)$, and with C given.

The following theorem describes how to solve this problem.

Theorem 3.1: Given $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{1 \times n}$ with (A, C) detectable. Let f^* be the *coarsest* quantizer for the system

$$w^+ = A'w + C'v \tag{19}$$

i.e., $v = f^*(w) = h^*(L^{*\prime} w)$, $L^{*\prime} = -(CS^*A'/CS^*C', \in \mathbb{R}^{n \times 1})$ is the optimal direction of quantization, and S^* is the symmetric positive–semidefinite solution of

$$S^* - AS^*A' + AS^*C'(CS^*C' + 1)^{-1} CS^*A' = 0.$$

Then, the coarsest quantizer for the system

$$e^+ = Ae + L\xi$$

denoted by $f_E^*(e)$ is obtained by setting $L = L^*$, it has the form $f_E^*(x) = h_E^*(Ce)$ with C' in place the optimal direction of quantization, it has the optimal ρ^* given by (7), and it is constructed according to Definition 2.3.

Proof: We first argue that there is no loss of generality in considering the error system completely unstable.

We can assume, without loss of generality, that System (18), through a coordinate transformation, has the following block diagonal form:

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \quad L = \begin{bmatrix} L_s \\ L_u \end{bmatrix} \text{ and } C = [C_s, C_u]$$

where A_u describes the dynamics of the unstable modes of the system (all its eigenvalues have magnitude greater than 1), while A_s describes the stable dynamics.

For any L, such that the system is stabilizable from ξ, the coarsest quantizer f^* for quadratic stability is given by Theorem 2.2, and has the form

$$f^*(e) = h^*(C_{GD}^*(L)e).$$

We are using C_{GD}^* instead of K_{GD}^* for obvious reasons, and showing its dependence on L. $C_{GD}^*(L)$ is the minimal energy controller, (measurement vector) for the given L, and we have already argued that $C_{GD}^*(L)$ must have the following structure:

$$C_{GD}^*(L) = [C_{GDs}^*(L)\, C_{GDu}^*(L)]\,, \text{ with } C_{GDs}^*(L) = 0$$

i.e., no effort is exerted in stabilizing A_s.

At first sight, it seems that unless the given matrix C happens to make the stable modes unobservable, $C_s = 0$, C' can not be an optimal direction of quantization. In this problem however, we have to select L. If we select L^* as described in the theorem statement, then

$$L^* = [L_s^*, L_u^*] \tag{20}$$

will be such that $L_s^* = 0$, which makes all the stable modes of the system uncontrollable. Let

$$C_{GD}^* = [0,\, C_{GDu}^*]$$

denote the optimal direction of quantization for $L = L^*$, let $f_E(e) = h_E^*\,(C_{GD}^* e)$ be the resulting optimal quantizer, and let

$$C^* = [C_s, C_{GDu}^*]\,, \text{ for some } C_s.$$

The above argument implies that quantizing along the direction C_{GD}^*, or along the direction C^*, i.e., with

$$\xi = h_E^*(C^* e)$$

instead of

$$\xi = h_E^*(C_{GD}^*)$$

does not affect stability, for any C_s, since the x_s, corresponding to the coordinates of the stable uncontrolled modes, will go to zero.

Thus, we only need to consider the unstable part of the system, and design L_u so that $C_{GDu}^* = C_u$. We claim that such an L_u is L_u^* in (20).

To simplify the notation we set $A = A_u$, $L = L_u$ and $C = C_u$.

From Theorem 2.2 applied to system (19), we have that L^* is the optimal direction of quantization associated with ρ^*, and moreover that $L^{*\prime}$ is equivalently characterized as the static gain vector that places all the eigenvalues of the closed-loop system

$$w^+ = (A' + C'L^{*\prime})w$$

at the mirror images of the eigenvalues of A', i.e., $\lambda_i(A' + C'L^{*\prime}) = (1/\lambda_i(A'))$. L^* is unique from the detectability assumption, and the fact that all eigenvalues of A need to be placed in a new location.

However, the eigenvalues of $(A' + C'L^{*\prime})$ are the same as the eigenvalues of $(A + L^*C)$, which means that C is the unique static gain vector that places the closed-loop eigenvalues of $(A + L^*C)$ at the mirror images of the eigenvalues of A. Therefore, C must be the optimal LQR controller associated with the problem

$$\min_{\substack{e_{k+1} = Ae_k + L^*\xi_k \\ e(0) = e_0}} \text{ stable } \sum_{k=0}^{\infty} \xi_k^2$$

and thus, from Theorem 2.2, C' is the optimal direction of quantization with optimal logarithmic base equal to ρ^*. ■

The above theorem suggests the following estimator structure where the estimator dynamics are driven by the logarithmically quantized estimator error:

$$\hat{x}^+ = A\hat{x} + Bu + L^* h_E^*(C\hat{x} - y).$$

A. Quantized Output Feedback

We can now construct (under the assumptions of Theorem 3.1) a quantized output feedback controller, based on the separation of the estimator and the state-feedback. Let $f_S^*(x) = h_S^*(K_{GD}^*)$ be the *coarsest* quantizer for the state feedback problem as defined in Definition 2.3, and Theorem 2.2, and $f_E^*(e) = h_E^*(Ce)$ the *coarsest* quantizer for the estimator problem as defined in Definition 2.3 and Theorem 3.1. Then we can obtain an output feedback controller by quantizing the state estimate by f_S^* instead of the actual plant state. For the SISO plant given by

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx \end{aligned}$$

the dynamic equations of the closed-loop system are the following:

$$\begin{aligned} x^+ &= Ax + Bh_S^*\,(K_{GD}^*\hat{x}) \\ \hat{x}^+ &= A\hat{x} + Bh_S^*\,(K_{GD}^*\hat{x}) + Lh_E^*(C\hat{x} - y). \end{aligned}$$

We will present a practical implementation of this controller in Section VI.

IV. Sampling and Quantization

In this section, we consider both sampling and quantization of a finite-dimensional LTI system. In particular, we extend the previous quantization results by studying the case of uniform (or linear) sampling and derive a criterion for optimal sampling and quantization.

We consider uniform sampling, i.e., a constant sampling interval, since it is the only sampling strategy that retains the time-invariance of the discrete-time system. Furthermore, it is what is used in practice.

In this section, we go back to the state-feedback case. We assume that the state of the linear system

$$\dot{x} = Fx + Gu$$

is sampled with sampling time T, and that the control input is held in the intersampling interval with zero-order hold. This is the typical situation encountered in practice. Furthermore, let $\lambda_i^u(F)$, $i = 1, \dots, k \le n$ denote the eigenvalues of the matrix F with positive real part. Let the resulting discrete LTI system be

$$x_d^+ = A_T x_d + B_T u_d \tag{21}$$

where $A_T = e^{FT}$ and $B_T = \int_0^T e^{A(T-\tau)} B d\tau$ are functions of T.

In this section, it is implicitly assumed that the discretized system is stabilizable, this is true for all but at most a countable set of sampling times [26]. The critical sampling times where there is a loss of stabilizability are not considered in what follows.

From the results in Section II, we have that, for each T, there is a *coarsest* quantizer that is logarithmic with base $\rho^*(T)$. Since $\lambda_i(A) = e^{\lambda_i(F)T}$, $\rho^*(T)$ must have the following expression:

$$\rho^*(T) = \frac{\prod_{1\le i\le k} \exp\left(\mathrm{Re}\{\lambda_i^u(F)\}T\right) - 1}{\prod_{1\le i\le k} \exp\left(\mathrm{Re}\{\lambda_i^u(F)\}T\right) + 1}. \tag{22}$$

This formula is saying that we need to quantize more finely if we sample slower in order to maintain stability of a given system. Since sampling is nothing but quantization in time, Section III-A, we generalize the concept of density to measure the coarseness of quantizers in space $\times$ time, and derive a criterion for optimality of sampling and quantization.

A. Density of Sampling and Quantization

Under the condition that in time we sample uniformly with sampling time T, the natural generalization of Definition 2.2 is the following.

Definition 4.1: Given $V(x) = x'Px$, $P > 0$, a CLF for System (21), let $\mathcal{Q}_T(V)$ denote the set of all quantizers that solve Problem 2.1 for the given T. For $g \in \mathcal{Q}_T(V)$ and $0 < \epsilon \le 1$, let $\#g[\epsilon]$ denote the number of levels that g has in the interval $[\epsilon, 1/\epsilon]$. Define

$$\begin{aligned}\eta_{g,T} &= \limsup_{t\to\infty} \limsup_{\epsilon\to 0} \frac{(\#\text{ of samples } \in [0,t])\,(\#g[\epsilon])}{-t\ln\epsilon} \\ &= \frac{1}{T} \limsup_{\epsilon\to 0} \frac{\#g[\epsilon]}{-\ln\epsilon}\end{aligned}$$

$\eta_{g,T}$ is called the density of sampling and quantization of (T, g).

Note that the order in which the limits are taken is immaterial in the above definition, and that the (linear) density of sampling is equal to $1/T$, which is the sampling frequency, i.e., the number of samples in the interval $[0, 1]$.

We can now define the couple (T^*, f^*) of sampling time and quantizer to be coarsest for quadratic stability if they minimize the density of sampling and quantization over all quadratic CLF. Formally, we have

Definition 4.2: The couple consisting of sampling time and quantizer (T^*, f^*) is coarsest for quadratic stability if

$$(T^*, f^*) = \arg \inf_{V(x)\mathrm{CLF}} \inf_{g\in\mathcal{Q}_T(V)} \eta_{g,T}.$$

For a given T, we know that the coarsest quantizer for quadratic stability, denoted by f_T^*, is logarithmic with base $\rho^*(T)$, where $\rho^*(T)$ is given by (22). Therefore, f_T^* gives the smallest density of sampling and quantization for the given T, or in other words

$$\eta_{f_T^*,T} \le \inf_{V(x)\mathrm{CLF}} \inf_{g\in\mathcal{Q}_T(V)} \eta_{g,T}.$$

with

$$\eta_{f_T^*,T} = \frac{2}{T\ln\frac{1}{\rho^*(T)}} \triangleq D(T).$$

Thus, in order to minimize the density of sampling and quantization we need to find the minimum of $D(T)$, with the optimal sampling time T^* given by $T^* = \arg\inf D(T)$, and the optimal quantizer $f^* = f_{T^*}^*$ being logarithmic with base $\rho^*(T^*)$.

For a given T, $D(T)$ is nothing but the density of the grid of discrete points in the continuum *state-space* $\times$ *time* that guarantees quadratic stability (of the discretized system). It provides a measure of the complexity of the control action.

A visual interpretation is obtained by looking at $D_\epsilon(T) = D(T)\ln\epsilon$ which is the number of boxes in the reference box $[\epsilon, 1/\epsilon] \times [0, 1]$ which guarantees quadratic stability. Therefore, by minimizing $D(T)$ over T we are minimizing the number of boxes in $[\epsilon, 1/\epsilon] \times [0, 1]$. This will be an important criterion in the derivation of finite logarithmic quantizers from infinite ones. A problem that will be addressed in Section V.

As a final note, we would like to suggest that the least dense grid can be interpreted as the grid of minimum attention in the sense of [20].

B. Optimal Density of Sampling and Quantization

The question is now the following: what is the optimal "least dense" grid needed?

Theorem 4.1: $D(T) : \mathbb{R}^+ \to \mathbb{R}^+$ has a unique minimum at T^*. The optimal sampling time T^* satisfies the following equation:

$$T^* \sum_{i=1}^{k} \lambda_i^u(F) = \ln(1 + \sqrt{2}) \tag{23}$$

and the corresponding optimal quantizer is logarithmic with base

$$\rho^*(T^*) = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$$

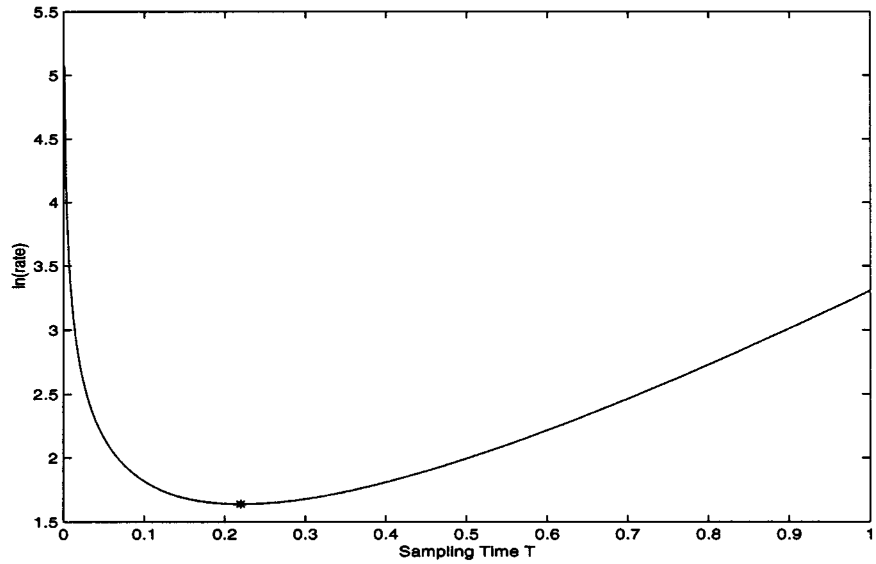

Fig. 3. Density versus sampling time for the case $F = 4$; $G = 1$.

which is independent of the system.

Proof: Let $w = T\sum_{i=1}^{k}\lambda_i^u(F)$, and consider the function

$$\bar{D}(T) = \frac{D(T)}{\sum_{i=1}^{k}\lambda_i^u(F)}.$$

Clearly $\bar{D}(T)$ and $D(T)$ are minimized at the same point. Rewriting $\bar{D}(T)$ in terms of w we have

$$\bar{D}(T) = \frac{2}{w\ln\left(\frac{e^w+1}{e^w-1}\right)}$$

which is minimized when $g(w) = w\ln(e^w+1)/(e^w-1)$ is maximized.

The derivative of $g(w)$ with respect to w is given by

$$\frac{dg(w)}{dw} = \ln\left(\frac{e^w+1}{e^w-1}\right) - \frac{2we^w}{e^{2w}-1} \triangleq h(w).$$

We argue that this function has only one zero as follows. It is easy to see that $h(w)$ is positive for w positive and close to 0, while $h(w)$ converges to 0 from the negative values as $w \to \infty$. The second derivative of $g(w)$, given by

$$\frac{dh(w)}{dw} = 2e^w\frac{e^{2w}(w-2)+w+2}{(e^{2w}-1)^2}$$

changes sign only once for $w \geq 0$. This is not difficult to show, and left to the reader. Therefore, $h(w)$ has only one minimum. It cannot cross the w axis more than once, because if it does then it should also have a maximum since as $w \to \infty$, $h(w) \to 0$ from the negative values.

It is easy to verify by substitution that, the derivative of $g(w)$ is equal to zero for

$$w = \ln(1+\sqrt{2})$$

which implies that

$$T^*\sum_{i=1}^{k}\lambda_i^u(F) = \ln(1+\sqrt{2})$$

and that the optimal density is equal to

$$D^* = D(T^*) = \frac{2\sum_{i=1}^{k}\lambda_i^u(F)}{\ln^2(1+\sqrt{2})}$$

$\rho^*(T^*)$ is obtained by substituting T^* in (22). ■

Note that the above theorem says that the product of the optimal sampling time, T^* and the sum of the unstable poles of the system is a constant independent of the system, and in this sense, *universal*. Also note that at T^*, $\rho^*(T^*) = \sqrt{2}-1$ is independent of the system. This indicates that $\rho^*(T^*)$ is the base of the *universal* logarithmic quantizer for a single input continuous time linear system for which the density of sampling and quantization is minimized. T^* is then appropriately selected from (23).

Fig. 3 shows the logarithm of $(1/2)D(T)$ versus the sampling time T for the first-order system $\dot{x} = 4x + u$.

We would like to point out that the optimization of the density seems also related to the optimal length of a block-code used to code the control input levels needed for stabilizing the system. This relationship is currently under investigation.

V. Finite Quantizers

In this section, we discuss how logarithmic quantizers with countable levels of the Sections I–IV can be replaced by logarithmic quantizers with finite number of levels and still maintain *practical* stability. The results are for discrete time systems but can be extended to sampled data systems with minor modifications.

Next, we define a finite symmetric quantizer, which with a slight abuse of terminology we will simply call finite quantizer.

Definition 5.1: A Finite Symmetric Quantizer (of order N) is a function $f_N : X \to \mathcal{U}_N$ which takes values in the finite set

$$\mathcal{U}_N = \{u_i \in \mathbb{R} :, i = -(N-1), \ldots, 0, \ldots, (N-1), \text{ and } u_i = -u_{-i}\}$$

A special class of finite quantizers is given by the following definition.

Definition 5.2: A Finite ρ-Logarithmic Quantizer (of order N) is a finite-symmetric quantizer where

$$\mathcal{U}_N = \left\{-u_0, -\rho u_0, \ldots, -\rho^{N-1}u_0, 0, \rho^{N-1}u_0, \ldots, \rho u_0, u_0, : u_0 \in \mathbb{R}\right\}$$

where $0 < \rho < 1$.

A. Relaxed Quadratic Stabilizability

We next define the notion of stabilizability we will use in this section. This is a relaxed version of quadratic stabilizability which we will call *practical* stabilizability. This concept is similar to that used in nonlinear system literature [11]–[13], and corresponds to what is called semiglobal practical quadratic stability.

The reason we need to abandon the notion of a trajectory asymptotically converging to the origin, is presented in [1], where it is shown that the set of initial conditions that give rise to trajectories converging to the origin asymptotically have measure zero. Instead, most of the trajectories will wander either in limit cycles or chaotically in a neighborhood of the origin.

Definition 5.3: System (1) is practically stabilizable, if there exists a Lyapunov function $V(x) = x'Px$, $P > 0$, such that, for any compact set $\mathcal{C}$ containing the origin, and any $\Omega_s \subset \mathcal{C}$ with $\Omega_s = \{x \in X : V(x) \leq \beta_s\}$, there is a state-feedback controller $f(x)$, function of $\mathcal{C}$ and β_s, such that $V(x) > V(x^+)$ for all $x \in \mathcal{C}\backslash\Omega_s$, and such that $x^+ \in \Omega_s$ whenever $x \in \Omega_s$.

By this definition, Ω_s is an attractor of $\mathcal{C}$. Trajectories starting in $\mathcal{C}$ and outside Ω_s will be attracted toward Ω_s, and will eventually enter it after finite time, and, those starting in Ω_s, never leave it.

B. Stability With Finite Logarithmic Quantizers

We are now ready to state the main theorem of this section.

Theorem 5.1: If system (1) is stabilizable, then it is *practically* stabilizable by a Finite ρ-Logarithmic Quantizer with $\rho < \rho^*$ arbitrarily close to ρ^*, and order N large enough.

In particular, given $\mathcal{C}$, β_s, with β_s such that $\Omega_s \subset \mathcal{C}$, and β_o such that

$$\Omega_o = \{x \in X : V(x) \leq \beta_o\} \supseteq \mathcal{C}$$

N, the order of the quantizer, can be taken to be

$$N \geq \frac{1}{2}\log_\rho\left(\frac{\beta_s}{\gamma\alpha_0^2}\right)$$

where

$$\alpha_0 = \max_{x\in\Omega_o} K_{GD}x$$

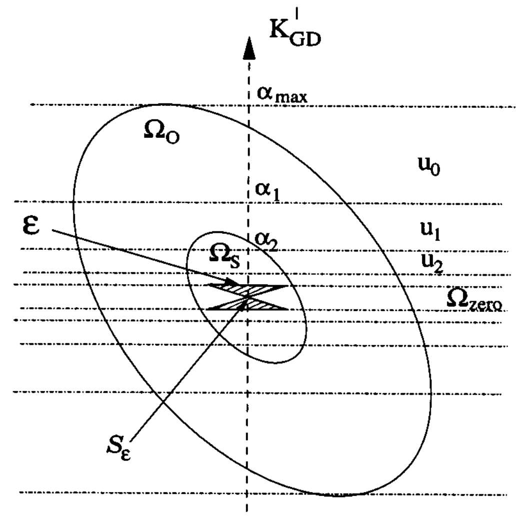

Fig. 4. Descriptive relation among Ω_o, Ω_s, $\mathcal{E}$, and $\mathcal{S}_\epsilon$. $\mathcal{E}$ represented in the figure is the ellipsoid (in this case the segment) obtained for $\eta = \epsilon$.

and γ is the solution of a convex optimization problem (given by (24)) which is a function of the problem data.

We devote the rest of this section to proving this theorem. Note that, in order to prove Theorem 5.1, we need to provide a quadratic Lyapunov function $V(x) = x'Px, P > 0$. We use the Lyapunov function obtained as in Section II-C-I by solving a slightly perturbed *Minimum Energy Control* problem. We have now a countable ρ-logarithmic quantizer and we need to describe how to obtain a Finite ρ-logarithmic quantizer out of it. The construction of such quantizer is a function of the given sets $\mathcal{C}$ and Ω_s.

1) Construction of the Finite Quantizer: Given $\mathcal{C}$ we can always select β_o arbitrarily among those for which

$$\Omega_o \supseteq \mathcal{C}.$$

We plan to obtain a finite quantizer from the countable one as follows: we start covering Ω_o with the equivalence classes of states induced by the countable quantizer. As shown in Fig. 4 these classes are stripes orthogonal to K'_{GD} that get smaller and smaller symmetrically with respect to the origin. Then, we stop after some N, essentially truncating the countable levels into a finite number of them, while the control value zero is used in the part of Ω_o (or $\mathcal{C}$) left uncovered.

In more detail, the structure of the finite quantizer is characterized by

$$\mathcal{U}_N = \left\{-u_0, -\rho u_0, \ldots, -\rho^{N-1}u_0, 0, \rho^{N-1}u_0, \ldots, \rho u_0, u_0, : u_0 \in \mathbb{R}\right\}$$

and the corresponding equivalent classes of states for which the same control values is used.

We know from Theorem 2.1 that the class $\Omega_j^{\pm}$ associated with $\pm u_0\rho^j$ is given by

$$\Omega^{\pm}{}_j = \{x : \pm\alpha_0\rho^{j+1}K_{GD}x \leq \pm\alpha_0\rho^j\}; \text{ for } j = 0, \ldots, N-1$$

while the zero control value is associated to the stripe

$$\Omega_{\text{zero}} = \left\{x : -\alpha_0\rho^N \leq K_{GD}x \leq \alpha_0\rho^N\right\}$$

which cover the part of Ω_o (or C) left uncovered by the other stripes.

The two design parameters are α_0 (or the associated u_0) and N the order of the quantizer. α_0 is function of $\mathcal{C}$ (or Ω_o), while N is function both Ω_s and $\mathcal{C}$. In order for the finite quantizer to cover Ω_o, α_0 must be

$$\alpha_0 \geq \max_{x\in\Omega_o} K_{GD}x.$$

From the construction, it is obvious that for any $x \in \Omega_o \cap \Omega_j^{\pm}$, $j = 0, \ldots, N-1$ we have that $V(x) - V(x^+) < 0$. Therefore, the only set we need to be concerned about is Ω_{zero} (or its intersection with Ω_o).

Now, let $\epsilon = \alpha_0\rho^N > 0$. By virtue of the truncation, the control value *zero* is used for any x in the central stripe

$$\Omega_{\text{zero}} = \{x : -\epsilon \leq K_{GD}x \leq \epsilon\}.$$

It is clear however that $u = 0$ will not be able to make $V(x)$ decreasing for all $x \in \Omega_{\text{zero}}$. Let $\mathcal{S}_\epsilon$ denote the set of states in Ω_{zero} for which the Lyapunov function is not strictly decreasing

$$\mathcal{S}_\epsilon = \{x \in \Omega_{\text{zero}} : V(x^+) \geq V(x)\}.$$

Looking ahead, we will next show that $\mathcal{S}_\epsilon$ is bounded, and its dependence on ϵ implies that it can be made arbitrarily small. Thus, its image under A, which is the set of reachable states in one step from $\mathcal{S}_\epsilon$ with zero control, is also bounded and its size also scales with ϵ. Therefore, both $\mathcal{S}_\epsilon$ and $A\mathcal{S}_\epsilon$ can be included in Ω_s for any $\beta_s > 0$ selecting ϵ small enough. This argument implies that Ω_s can be chosen arbitrarily small and within $\mathcal{C}$, and therefore, it will be control invariant, since for any $x \in \Omega_s$ either $V(x^+) < V(x)$ or $x^+ = Ax \in \Omega_s$.

2) Control Invariance of Ω_s: To give an explicit characterization of $\mathcal{S}_\epsilon$ it is convenient to make a coordinate transformation so that K'_{GD} is one of the basis elements and the others are orthogonal to it. Let

$$T = [W, K'_{GD}]$$

where the columns of W form an orthonormal basis for the null space of K_{GD}, and

$$x = T\begin{bmatrix}\xi\\ \eta\end{bmatrix}$$

where $\xi \in \mathbb{R}^{n-1}$ and $\eta \in \mathbb{R}$. Let $\bar{A} = T^{-1}AT$ and $\Pi = T'PT$ be the corresponding representation of A and P in the new basis. Let also $\Sigma = \bar{A}'\Pi\bar{A} - \Pi$. Finally, partition Π and Σ accordingly to $\begin{bmatrix}\xi\\ \eta\end{bmatrix}$

$$\Pi = \begin{bmatrix}\Pi_1 & \Pi_2\\ \Pi_2' & \Pi_3\end{bmatrix}.$$

and

$$\Sigma = \begin{bmatrix}\Sigma_1 & \Sigma_2\\ \Sigma_2' & \Sigma_3\end{bmatrix}$$

Then we can describe $\mathcal{S}_\epsilon$ as follows.

Lemma 5.1:

$$\mathcal{S}_\epsilon = \left\{\xi, \eta : -\epsilon \leq \eta \leq \epsilon, \text{ and } [\xi' \quad \eta]\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0\right\}.$$

For any $\eta \in [-\epsilon, \epsilon]$ the set

$$\mathcal{E} = \left\{\xi \in \mathbb{R}^{n-1} : \xi'\Sigma_1\xi + 2\xi'\Sigma_2\eta + \Sigma_3\eta^2 \geq 0\right\}$$

is an ellipsoid ($\Sigma_1 < 0$) which degenerates to a point for $\eta = 0$.

Proof: See Appendix. ■

Since $\mathcal{E}$ is a bounded ellipsoid for any $\eta\,[-\epsilon, \epsilon]$, $\mathcal{S}_\epsilon$ is bounded, and, moreover, $\mathcal{S}_{\epsilon_2} \supset \mathcal{S}_{\epsilon_1}$ for $\epsilon_2 > \epsilon_1$.

We are now ready to describe how to find Ω_s, i.e., $\beta_s(\epsilon)$ so that $\Omega_s \supset A\mathcal{S}_\epsilon$ and $\Omega_s \supset \mathcal{S}_\epsilon$.

Fig. 4 describes the relation among the various sets involved. For clarity, $A\mathcal{S}_\epsilon$ is not shown in the figure, but it also must be contained in Ω_s. The result of next Lemma shows that it is enough for Ω_s to include $A\mathcal{S}_\epsilon$ to guarantee that $\Omega_s \supset \mathcal{S}\epsilon$.

Define the $(n \times n)$ matrix Γ to be

$$\Gamma = \begin{bmatrix}0 & \cdots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & 0\\ 0 & \cdots & 0 & 1\end{bmatrix}.$$

Lemma 5.2: Given any $\epsilon > 0$, the smallest β_s^* such that $\Omega_{s^*} \supseteq A\mathcal{S}_\epsilon$ and $\Omega_{s^*} \supset \mathcal{S}_\epsilon$ is given by

$$\beta_s^* = \gamma\epsilon^2$$

where γ is the solution of the following LMI problem in γ and τ:

$$\min_{\substack{\gamma,\tau\\ \gamma\Gamma - \bar{A}'\Pi\bar{A} - \tau\Sigma \geq 0\\ \gamma>0, \tau>0}} \gamma. \tag{24}$$

Proof: We first describe how to compute the smallest β_s^o such that $\Omega_{s^o} \supseteq A\mathcal{S}_\epsilon$, we then show that in fact $\Omega_{s^o} \supset \mathcal{S}$ and therefore $\beta_s^o = \beta_s^*$.

We want to compute

$$\begin{aligned}\beta_s^o = \min \quad & \beta_s\\ \text{subject to: } & \left\{[\xi' \quad \eta]\bar{A}'\Pi\bar{A}\begin{bmatrix}\xi\\ \eta\end{bmatrix} \leq \beta_s;\right.\\ & \left.\forall\, \xi, \eta : [\xi' \quad \eta]\,\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0; -\epsilon \leq \eta \leq \epsilon\right\}\end{aligned}$$

but β_s^o is also given by

$$\begin{aligned}\beta_s^o = \epsilon^2 \min \quad & \gamma\\ \text{subject to: } & \left\{[\xi' \quad \eta]\bar{A}'\Pi\bar{A}\begin{bmatrix}\xi\\ \eta\end{bmatrix} \leq \epsilon^2\gamma;\right.\\ & \left.\forall\, \xi, \eta; [\xi' \quad \eta]\,\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0; -\epsilon \leq \eta \leq \epsilon\right\}.\end{aligned} \tag{25}$$

It is easy to verify, and it is left to the reader that Problem (25) is equivalent, having the same feasible set, to the following one:

$$\begin{aligned}\beta_s^o = \epsilon^2 \min \quad & \gamma\\ \text{subject to: } & \left\{[\xi' \quad \eta]\bar{A}'\Pi\bar{A}\begin{bmatrix}\xi\\ \eta\end{bmatrix} \leq \eta^2\gamma;\right.\\ & \left.\forall\, \xi, \eta : [\xi' \quad \eta]\,\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0; -\epsilon \leq \eta \leq \epsilon\right\}.\end{aligned} \tag{26}$$

Finally, given that no vector of the form $\begin{bmatrix}\xi\\0\end{bmatrix}$, satisfies $[\xi' \quad \eta]\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0$ (recall $\Sigma_1 < 0$), and the fact that the constraint set is balanced ($-(\xi,\eta)$ belong to the set if (ξ,η) do), it is sufficient to search over the following constraint set:

$$\left\{[\xi' \quad \eta]\,\bar{A}'\Pi\bar{A}\begin{bmatrix}\xi\\ \eta\end{bmatrix} \leq \eta^2\gamma;\right.$$
$$\left.\forall\, \xi,\eta : [\xi' \quad \eta]\,\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0;\ \eta = \epsilon\right\}$$

but this set is equivalent to the set where the constraint $\eta = \epsilon$ has been removed

$$\left\{[\xi' \quad \eta]\,\bar{A}'\Pi\bar{A}\begin{bmatrix}\xi\\ \eta\end{bmatrix} \leq \eta^2\gamma;\ \forall\, \xi,\eta : [\xi' \quad \eta]\,\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0\right\}.$$

This is once again due to the fact that any vector with $\eta = 0$ is not feasible, and any vector with $\eta \neq 0$ can be re-scaled to one with $\eta = \epsilon$ and vice-versa.

Problem (25) assumes now the following form:

$$\beta_s^o = \epsilon^2 \min \quad \gamma$$
$$\text{subject to: } \left\{[\xi' \quad \eta]\bar{A}'\Pi\bar{A}\begin{bmatrix}\xi\\ \eta\end{bmatrix} \leq \eta^2\gamma;\right.$$
$$\left.\forall\, \xi,\eta : [\xi' \quad \eta]\,\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0\right\} \quad (27)$$

Consider the following two quadratic forms which appear in the constraint set:

$$T_0(\xi,\eta) = [\xi' \quad \eta]\,(\gamma\Gamma - \bar{A}'\Pi\bar{A})\begin{bmatrix}\xi\\ \eta\end{bmatrix}$$

and

$$T_1(\xi,\eta) = [\xi' \quad \eta]\,\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix}.$$

A well know result known as the S-procedure [25] states the following in the case of two quadratic forms:

$$T_0(\xi,\eta) \geq 0, \quad \forall\, \xi,\eta : T_1(\xi,\eta) \geq 0$$

if and only if there exists a positive τ such that

$$T_0(\xi,\eta) \geq \tau T_1(\xi,\eta) \quad \forall\, \xi,\eta.$$

Therefore, our Problem (27) becomes

$$\beta_s^o = \epsilon^2 \min_{\gamma,\tau} \gamma \quad \text{subject to: } \begin{array}{c}\gamma\Gamma - \bar{A}'\Pi\bar{A} - \tau\Sigma \geq 0\\ \gamma > 0, \tau > 0\end{array} \quad (28)$$

which is an LMI problem that can be efficiently solved.

Substituting for $\Sigma = \bar{A}'\Pi\bar{A} - \Pi$ into the constraint

$$\gamma\Gamma - \bar{A}'\Pi\bar{A} - \tau\Sigma \geq 0$$

for τ and γ solutions of Problem (28), we obtain

$$\gamma\Gamma + \Pi\tau - \bar{A}'\Pi\bar{A}(1+\tau) \geq 0.$$

Let $\bar{\tau} = \tau + 1$, note that $\bar{\tau} > 0$. Then we have that the above expression can be rewritten as follows:

$$\gamma\Gamma + \Pi(\bar{\tau} - 1) - \bar{A}'\Pi\bar{A}\bar{\tau} \geq 0$$

which, after rearrangement becomes

$$\gamma\Gamma - \Pi - \Sigma\bar{\tau} \geq 0. \quad (29)$$

Applying the S-procedure again, we have that (29) holds if and only if

$$\left\{[\xi' \quad \eta]\,\Pi\begin{bmatrix}\xi\\ \eta\end{bmatrix} \leq \eta^2\gamma;\ \forall\, \xi,\eta : [\xi' \quad \eta]\,\Sigma\begin{bmatrix}\xi\\ \eta\end{bmatrix} \geq 0\right\}$$

which implies that $\Omega_s^o \supset \mathcal{S}_\epsilon$. Thus $\beta_s^* = \beta_s^o$. ■

Summarizing the development in this section, we have shown that by selecting

$$\alpha_0 \geq \max_{x\in\Omega_o} K_{GD}x$$

with $\Omega_o \supseteq \mathcal{C}$,

$$V(x) - V(x^+) < 0 \quad \forall\, x \in \mathcal{C}\backslash\Omega_s.$$

While, any $\Omega_s \subset \mathcal{C}$ can be made control invariant by selecting N so that

$$\alpha_0\rho^N < \sqrt{\frac{\beta_s}{\gamma}}.$$

Therefore, the Finite ρ-Logarithmic Quantizer so constructed *practically* stabilizes system (1). This ends the proof of Theorem 5.1.

Comment: We would like to point out that Theorem 5.1 provides a practical way to implement a quantized output feedback controller design based upon the theory developed in this paper.

VI. Example

In this section, we report the results of an application of the theory to the following second-order continuous time system:

$$\dot{x} = Fx + Gu$$
$$y = Hx$$

with

$$F = \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}; \quad G = \begin{bmatrix}0\\1\end{bmatrix}; \quad H = [1 \quad 1].$$

We sample it at the optimal T^* corresponding to the minimal density grid and given by (23). $T^* \approx 0.1763$ in this case. We use $\epsilon = 1\cdot 10^{-5}$ to perturb the *expensive* control Riccati solution, and obtain $\rho \approx 0.4142$.

The Riccati solutions for the state-feedback and estimator problems are, respectively

$$P_S = \begin{bmatrix}609.6579 & 127.4906\\127.4906 & 61.4880\end{bmatrix}, \text{ and}$$
$$P_E = \begin{bmatrix}3.9064 & 1.7509\\1.7509 & 1.1049\end{bmatrix}$$

and the associate directions of quantization are

$$K = -[12.4932 \quad 8.3679];\ \text{and}\ L = \begin{bmatrix}0.2172\\-2.4232\end{bmatrix}.$$

We selected $\beta_o = 200$, and $\beta_s = 0.01$ for both state-feedback and estimator. With these values the resulting order of the finite quantizer is $N = 15$ for both state-feedback and estimator.

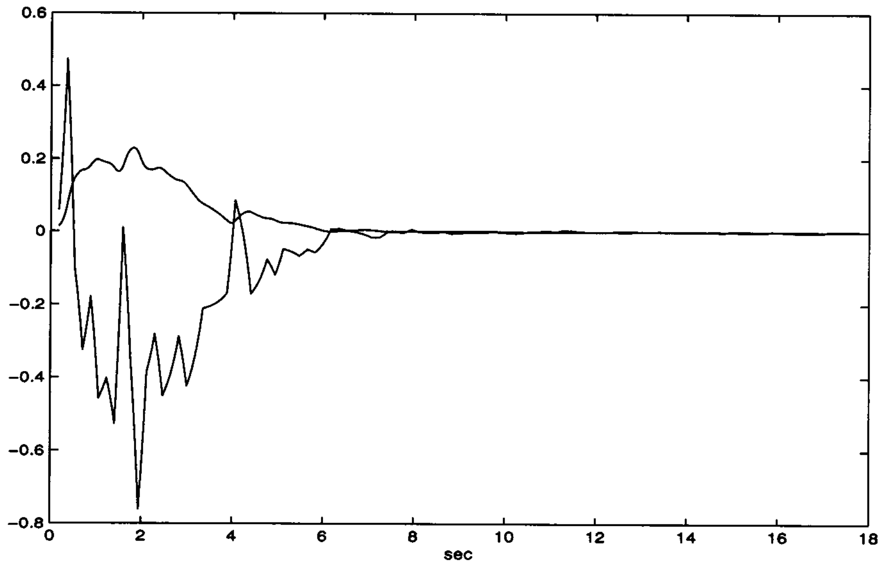

Fig. 5. Plant state evolution in closed loop.

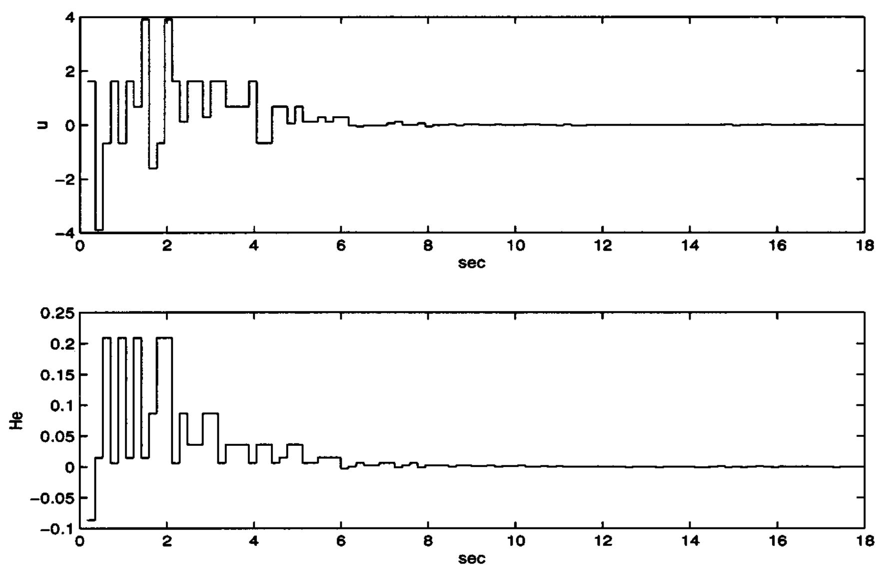

Fig. 6. Quantized control values and quantized estimation error along H'.

This means that we need $2N + 1$ levels to cover both positive, negative, and zero values, which correspond to 5 bit A/D logarithmic converter.

Fig. 5 shows the continuous-time evolution of the plant states starting from a random initial condition. We see that the state does not go to zero asymptotically but rather stays very close to it after the transient. This is the consequence of the *practical* stability. Fig. 6 show the quantized sequences of control input and estimation error used in this case.

VII. Conclusion and Future Work

In this paper, we have developed the basis for a theory of design of quantizers for SISO linear discrete-time systems. We have shown that quadratic Lyapunov functions for the systems induce a (countable) logarithmic quantization of controls or measurements and of the system state-space. We went further by looking for the best quadratic Lyapunov function that allows for the coarsest logarithmic quantizer, to show that it is the same as that arising in the solution of the *expensive control* LQR problem. Based on the properties of the *expensive control* LQR controller, we have derived a closed form expression for the smallest logarithmic base compatible with quadratic stability of the closed-loop system. The expression is exclusively in terms of the unstable eigenvalues of system. Both quantized state-feedback controller and estimator have been derived. The results of the analysis of discrete-time systems are a basis for the study of both sampling and quantization of continuous-time systems. We have shown that there is a sampling time with associated quantization that minimizes the density of both sampling and quantization and still ensures stability. The optimal sampling time depends exclusively on the unstable eigenvalues of the system. Perhaps even more interestingly, the base of the optimal logarithmic quantizer is independent of the system and therefore *universal*. Finally, we have shown how to construct logarithmic quantizers with only finite number of quantization levels, which still achieves *practical* closed-loop stability. This provided a way for a practical implementation of the theory developed in the paper.

There are many possible directions of research that arise from the results of this paper. Perhaps the most urgent one is the generalization to multi-variable systems. Another important issue is to generalize the method to include performance objectives other than just stability with decay rate. Given the tight connection with LQR theory, at this point it seems natural to look into quadratic type performance criteria. A perhaps not too remote possibility is to obtain a quantized version of LQG controllers. Also we have not considered the effects of noise as well as the effects of other model uncertainties. Since the approach proposed is essentially based on the idea of robust Lyapunov functions, it is conceivable that it can be extended in principle to nonlinear systems. Finally, we want to point out that the approach proposed can be applied to more general CLFs, as for example polytopic CLF. This should lead toward smaller values of ρ^* at the expenses of increased complexity of searching over these more general classes of CLFs.

APPENDIX

A. Proof of Lemma 2.2

Since $B'PB > 0$, $U(x)$ is characterized by the open interval between the roots of the second order equation in u

$$x'(A'PA - P)x + 2x'A'PBu + u'B'PBu = 0$$

which are given by

$$u^{(1),(2)} = -\frac{B'PAx}{B'PB} \pm \sqrt{\frac{x'A'PBB'PAx}{(B'PB)^2} - \frac{x'(A'PA-P)x}{B'PB}}.$$

From (2) is easy to see that the expression in the square root is equal to $(x'Qx/B'PB)$. Thus, we have that

$$u^{(1),(2)} = K_{GD}x \pm \sqrt{\frac{x'Qx}{B'PB}}$$

■

B. Proof of Lemma 5.1

$\mathcal{S}_\epsilon$ is given by the intersection of Ω_{zero} with the set of x such that $V(x^+) - V(x) = x'(A'PA - P)x \geq 0$. In the new coordinate

$$\mathcal{S}_\epsilon = \{\xi, \eta : -\epsilon \leq \eta \leq \epsilon\} \bigcap \left\{\xi, \eta : [\xi' \quad \eta]\, \Sigma \begin{bmatrix} \xi \\ \eta \end{bmatrix} \geq 0 \right\}.$$

In order to prove the second statement, it is sufficient to recall Property $(P2)$, and the fact that $Q > 0$, since we are solving the perturbed problem of Section II-C-I. $(P2)$ and $Q > 0$ imply that for any x orthogonal to K'_{GD}, $u = 0$ is sufficient to make $V(x^+) - V(x) < 0$. This, in the new coordinate system, implies that $\Sigma_1 < 0$ and the result follows. ■

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