

The Duality Between Estimation and Control

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This paper is dedicated by Sanjoy Mitter to Alain Bensoussan as a token of his friendship and intellectual collaboration over many years.

1 Introduction

In his book, *Filtrage Optimal des Systèmes Linéaires*, Alain Bensoussan presented a variational view of optimal filtering for linear infinite-dimensional stochastic differential systems. The viewpoint he presented is related to the work of Bryson and Frazier [1] where the Kalman Filter was viewed in terms of the solution of a linear optimal control problem with a quadratic cost criterion. Implicit in this view is the duality between estimation and control as reflected in the duality of the concepts of controllability and observability. It has been an open question as to whether this duality can be extended to the non-linear situation in a clear conceptual and mathematically precise way. A hint that this might be possible is contained in my joint work with

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Wendell Fleming [2] where we presented a stochastic control view of non-linear filtering using a logarithmic transformation due originally to Hopf. This logarithmic transformation allows one to transform the robust form of the Zakai equation (originally due to Martin Clark [3]; see also the work of Mark Davis [4]) into a Bellman equation where the observations $Y(\cdot)$ appear as a parameter. This Bellman equation has the interpretation as the value function of an appropriate stochastic control problem. Our motivation at that time was to prove a theorem on the existence of solutions to the Zakai equation with unbounded observations. A physical or system-theoretic interpretation of the stochastic control problem was not given in that paper. The main contribution of this paper is to show that the duality between filtering and control is exactly the variational duality between Relative Entropy and Free Energy which is at the heart of the variational characterization of Gibbs measures [5]. This duality plays an important role in the work of Donsker and Varadhan on large deviations and is a result in the duality between conjugate convex functions (cf. Deuschel–Stroock [6]).

Although I examine a variational approach to Non-linear Filtering in this paper, this research has implications for Bayesian Estimation and places Maximum Entropy Estimation in the correct contextual framework. Therefore this line of inquiry is relevant to Image Analysis where attributes of images are modelled as Markov random fields.

There has recently been considerable activity on the stability of non-linear filters with respect to incorrectly chosen initial density but where the observation path is fixed. To date, the situation of stability of the filter where other probabilistic parameters are varied has not been examined. The ideas of this paper indicate why relative entropy is a natural Lyapunov function for studying stochastic stability of diffusion (or conditional diffusion) processes. It is well known that there is a close relationship between Hamilton–Jacobi equations and Lyapunov Functions via the value function of an optimal control problem. These ideas were generalized to an input–output setting by J.C. Willems in his work on Dissipative Systems [7]. We suggest that there is a similar relationship between the Bellman equation for Stochastic control problems and stability of stochastic dynamical systems, using the Davis–Varaiya theory of Partially Observed Stochastic Control [8]. This leads to a definition of a Stochastic Dissipative System which I believe has important connections to the recent work on Nonequilibrium Statistical Mechanics [9]. In some sense, I am hinting at the development of a Non-equilibrium Statistical Mechanics where the fundamental objects are not states (probability

measures) but information states (conditional probability measures).

In this paper, I emphasize the conceptual ideas and not the technical details which are of considerable importance. A detailed version of this work will be presented in my forthcoming paper with Nigel Newton [10].

2 Gibbs Measures (Variational Characterization)

To set the stage, consider the finite situation. Let S be a finite set, the set of sites and let E be a finite set, the state set and let $\Omega = E^S$. Consider the Hamiltonian describing a system

$$(2.1) \quad H(\omega) = \sum_{ACS} \Phi_A(\omega),$$

where $\Phi_A : \Omega \rightarrow \mathbb{R}$ is a potential function.

Let

$$(2.2) \quad \nu(\omega) = Z^{-1} \exp[-H(\omega)], \quad \omega \in \Omega$$

where the partition function

$$Z = \sum_{\omega \in \Omega} \exp[-H(\omega)].$$

$\nu(\omega)$ is the Gibbs measure corresponding to the Hamiltonian H . For a probability measure μ on Ω , let

$$(2.3) \quad \mu(H) = \sum_{\omega \in \Omega} \mu(\omega) H(\omega)$$

denote the average energy of the system.

Let

$$(2.4) \quad \mathcal{H}(\mu) = - \sum_{\omega \in \Omega} \mu(\omega) \log(\mu(\omega))$$

denote the Entropy of the system. Then the Free energy corresponding to μ is given by

$$(2.5) \quad F(\mu) = \mu(H) - \mathcal{H}(\mu).$$

We then have

Proposition 2.1. *For all probability measures μ on Ω*

$$(2.6) \quad F(\mu) = \mu(H) - \mathcal{H}(\mu) \geq -\log Z,$$

with equality iff $\mu = \nu$.

The proof relies on Jensen's inequality and the strict convexity of the function $\varphi(x) = x \log x$ on $[0, \infty)$. ■

Let (Ω, \mathcal{F}) be a measurable space and let $\mathcal{P}(\Omega)$ denote the set of all probability measures on (Ω, \mathcal{F}) . For $\mu \in \mathcal{P}(\Omega)$, the *relative entropy* is a map $\mathcal{H}(\cdot | \mu) : \mathcal{P}(\Omega) \rightarrow \overline{\mathbb{R}}$ is defined as

$$(2.7) \quad \mathcal{H}(\nu | \mu) = \iint_{\Omega} \left(\log \frac{d\nu}{d\mu} \right) d\nu$$

if ν is absolutely continuous w.r.t. μ , and $\frac{d\nu}{d\mu}$ is the Radon–Nikodym Derivative of ν with respect to μ . $\mathcal{H}(\nu | \mu)$ is said to be the Relative Entropy of ν w.r.t. μ . The following properties of Relative Entropy are well known.

Proposition 2.2.

- (i) $\mathcal{H}(\nu | \mu) \geq 0$
- (ii) $\mathcal{H}(\nu | \mu) = 0 \Leftrightarrow \nu = \mu$
- (iii) $\mathcal{H}(\nu | \mu)$ is a convex function of μ and ν . ■

We now present a generalization of Proposition 2.1 which exhibits the Fenchel-Duality relationship between Free Energy and Relative Entropy (cf. [6] and [11]).

Using the notation of this section, let $\mu \in \mathcal{P}(\Omega)$ and $\Phi : \Omega \rightarrow \mathbb{R}$ a measurable function. The Free energy of Φ w.r.t. μ is defined by

$$(2.8) \quad F(\Phi) = \log \left(\int e^{\Phi} d\mu \right) \in (-\infty, \infty].$$

We make the assumption that Φ bounded below and $e^{\Phi} \in L^1(\mu)$. Let \mathcal{O} be this class of Φ 's.

We then have

Proposition 2.3. (i) For every $\nu \in \mathcal{P}(\Omega)$

$$(2.9) \quad \mathcal{H}(\nu \mid \mu) = \sup_{\Phi \in \mathcal{O}} \left[\int_{\Omega} \Phi d\nu - F(\Phi) \right]$$

$$(2.10) \quad F(\Phi) = \sup_{\mu \in \mathcal{P}(\Omega)} \left[\int_{\Omega} \Phi d\mu - \mathcal{H}(\nu \mid \mu) : H(\nu \mid \mu) < +\infty \right].$$

Moreover if $\Phi e^{\Phi} \in L^1(\mu)$ then the supremum in (2.10) is attained at ν^* given by

$$\frac{d\nu^*}{d\mu} = \frac{e^{\Phi}}{\int e^{\Phi} d\mu}$$

Note that ν^* is a Gibbs measure corresponding to the potential Φ .

3 Bayesian Estimation and Gibbs Measures

In this section, we discuss how the ideas of the previous section apply to Bayesian Estimation. In the process we give an Information Theoretic view of Bayesian Estimation.

Let (Ω, \mathcal{F}, P) be a probability space, $(\mathbf{X}, \mathcal{X})$ and $(\mathbf{Y}, \mathcal{Y})$ measurable spaces, and let

$$\begin{aligned} X &: \Omega \rightarrow \mathbf{X} \\ \text{and } Y &: \Omega \rightarrow \mathbf{Y} \end{aligned}$$

measurable mappings that induce probability measures P_X , P_Y and P_{XY} on \mathcal{X} , \mathcal{Y} and $\mathcal{X} \times \mathcal{Y}$, respectively. We assume

(H1) there exists a σ -finite (reference) measure, λ_Y , on \mathcal{Y} such that $P_{XY} \ll P_X \otimes \lambda_Y$.

Let L be the associated Radon–Nikodym derivative. Let $H : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbb{R} \cup \{+\infty\}$ be any measurable function such that

$$(3.1) \quad H(X, y) = \begin{cases} -\log(L(X, y)) & \text{a.s. if } y \in \overline{\mathbf{Y}} \\ 0 & \text{otherwise,} \end{cases}$$

where \bar{Y} is the set of all y such that L is integrable w.r.t. P_x . We think of H as the Hamiltonian and we define the Gibbs measure

$$(3.2) \quad \Lambda(x, y) = \frac{\exp(-H(x, y))}{\int_{\mathbf{X}} \exp(-H(\tilde{x}, y)) dP_X(\tilde{x})}.$$

Then, for any bounded, measurable $\Phi : \mathbb{X} \rightarrow \mathbb{R}$, the function

$$\int_{\mathbf{X}} \Phi(x) \Lambda(x, \cdot) dP_X(x) : \mathbf{Y} \rightarrow \mathbb{R}$$

is measurable; and

$$\int_{\mathbf{X}} \Phi(x) \Lambda(x, Y) dP_X(x) = E(\Phi(X)|Y) \quad \text{a.s.}$$

In particular, $P_{X|Y} : \mathcal{X} \rightarrow \mathbf{Y} \rightarrow [0, 1]$, defined by

$$(3.3) \quad P_{X|Y}(A, y) = \int_A \Lambda(x, y) dP_X(x),$$

is a *regular conditional probability* for X given Y . Equations (3.1)–(3.3) constitute an ‘outcome-by-outcome’ abstract Bayes’ formula, yielding a posterior probability measure for X for each outcome of Y .

Let $\mathcal{P}(\mathcal{X})$ be the set of probability measures on $(\mathbf{X}, \mathcal{X})$ and, for $\tilde{P}_X \in \mathcal{P}(\mathcal{X})$, let $\mathcal{H}(\tilde{P}_X | P_X)$ be the *relative entropy*,

$$(3.4) \quad \mathcal{H}(\tilde{P}_X | P_X) = \begin{cases} \int_{\mathbf{X}} \log \left(\frac{d\tilde{P}_X}{dP_X}(x) \right) d\tilde{P}_X(x) & \text{if } \tilde{P}_X \ll P_X \text{ and } \log \left(\frac{d\tilde{P}_X}{dP_X} \right) \in L_1(\tilde{P}_X) \\ +\infty & \text{otherwise,} \end{cases}$$

and let $F(\tilde{P}_X, y)$ be the *free energy* of \tilde{P}_X relative to $(P_X, H(\cdot, y))$,

$$(3.5) \quad F(\tilde{P}_X, y) = \begin{cases} \mathcal{H}(\tilde{P}_X | P_X) + \int_{\mathbf{X}} H(x, y) d\tilde{P}_X(x) & \text{if } H(\cdot, y) \in L_1(\tilde{P}_X) \\ +\infty & \text{otherwise.} \end{cases}$$

Theorem 3.1. *Suppose that (H1) is satisfied, L is a version of $\frac{dP_{XY}}{d(P_X \otimes \lambda_Y)}$, and H and $P_{X|Y}$ are as defined in (3.2) and (3.3). Then for any y such that*

$$(3.6) \quad \int_{\mathbf{X}} L(x, y) \log(L(x, y)) dP_X(x) < \infty,$$

$P_{X|Y}(\cdot, y)$ is the unique element of $\mathcal{P}(\mathcal{X})$ with the following property:

$$(3.7) \quad F(P_{X|Y}(\cdot, y), y) = -\log \left(\int_{\mathbf{X}} \exp(-H(x, y)) dP_X(x) \right)$$

$$(3.8) \quad = \min_{\tilde{P}_X \in \mathcal{P}(\mathcal{X})} F(\tilde{P}_X, y).$$

The fact that $\mathcal{H}(\cdot | P_X)$ is strictly convex on the subset of $\mathcal{P}(\mathcal{X})$ for which it is finite establishes the uniqueness of $P_{X|Y}(\cdot, y)$. \blacksquare

Remark. If the *mutual information* between X and Y is finite,

$$(3.9) \quad \int_{\mathbf{X} \times \mathbf{Y}} \log \left(\frac{dP_{XY}}{d(P_X \otimes P_Y)} \right) dP_{XY} < \infty,$$

then there exists a version of L for which (3.6) is satisfied for all y .

The following is an information-theoretic interpretation of Theorem 3.1. Let

$$A = \left\{ x \in \mathbf{X} : \int_{\mathbf{Y}} L(x, y) d\lambda_Y(y) = 1 \right\}$$

$$\text{and } \tilde{H}(x, y) = \begin{cases} H(x, y) & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $A \in \mathcal{X}$, $P_X(A) = 1$ and $P_{Y|X} : \mathcal{Y} \times \mathbf{X} \rightarrow [0, 1]$, defined by

$$P_{Y|X}(B, x) = \int_B \exp(-\tilde{H}(x, y)) d\lambda_Y(y),$$

is a regular conditional probability for Y given X . Let

$$(3.10) \quad \begin{aligned} I_Y(y) &= -\log \left(\int_{\mathbf{X}} \frac{dP_{Y|X}}{d\lambda_Y}(y, x) dP_X(x) \right) \\ &= -\log \left(\int_{\mathbf{X}} \exp(-H(x, y)) dP_X(x) \right) \end{aligned}$$

$$(3.11) \quad \begin{aligned} \text{and } I_{Y|X}(y, x) &= -\log \left(\frac{dP_{Y|X}}{d\lambda_Y}(y, x) \right) \\ &= \tilde{H}(x, y), \end{aligned}$$

be, respectively, the information and the (regular) X -conditional information in the observation ‘ $Y = y$ ’, both relative to the reference measure λ_Y . Then, for all $y \in \mathbf{Y}$

$$H(X, y) = I_{Y|X}(y, X) \quad \text{a.s.}$$

and Theorem 3.1 shows that for all $\tilde{P}_X \in \mathcal{P}(\mathcal{X})$

$$(3.12) \quad \mathcal{H}(\tilde{P}_X | P_X) + \int_{\mathbf{X}} I_{Y|X}(y, x) d\tilde{P}_X(x) \geq I_Y(y),$$

with equality if and only if $\tilde{P}_X = P_{X|Y}(\cdot, y)$.

4 Non-Linear Filtering

The variational representation of Bayes’ formula of the last section is developed further here for the special case where the observations are of the following ‘signal plus white noise’ variety:

$$(4.1) \quad Y_t = \int_0^t h_s(X) ds + V_t \quad \text{for } 0 \leq t \leq T.$$

Here, $(h_t(X) \in \mathbb{R}^d, 0 \leq t \leq T)$ is the ‘signal’ process depending on the quantity to be estimated, X , and $(V_t, 0 \leq t \leq T)$ is a d -dimensional Brownian motion (noise) process, independent of X . The abstract space $(\mathbf{Y}, \mathcal{Y})$ now becomes the Borel space $(C_0([0, T]; \mathbb{R}^d), \mathcal{B}_T)$ of continuous functions from $[0, T]$ to \mathbb{R}^d with initial value 0. We continue to use the notation \mathbf{Y} and \mathcal{Y} .

It is well known that, if h satisfies

$$(H2) \quad \mathbf{E} \int_0^T \|h_t(X)\|^2 dt < \infty,$$

then (H1) is satisfied when λ_Y is Wiener measure, and the Radon–Nikodym derivative takes the form:

$$(4.2) \quad \frac{dP_{XY}}{d(P_X \otimes \lambda_Y)}(X, Y) = \exp \left(\int_0^T h'_t(X) dY_t - \frac{1}{2} \int_0^T \|h_t(X)\|^2 dt \right) \quad \text{a.s.}$$

Let $(\mathcal{F}_t, 0 \leq t \leq T)$ be a filtration on (Ω, \mathcal{F}, P) , to which the process $(h_t(X), V_t)$ is adapted, and we assume that

(H3) $(h_t(X), \mathcal{F}_t; 0 \leq t \leq T)$ is a semimartingale;

then we can ‘integrate by parts’ in (4.2) and define L as any measurable function such that, for each y ,

$$(4.3) \quad L(X, y) = \exp \left(y'_T h_0(X) + \int_0^T (y_T - y_t)' dh_t(X) - \frac{1}{2} \int_0^T \|h_t(X)\|^2 dt \right) \quad \text{a.s.}$$

and for each $y \in Y$

$$H(X, y) = -y'_T h_0(X) - \int_0^T (y_T - y_t)' dh_t(X) + \frac{1}{2} \int_0^T \|h_t(X)\|^2 dt \quad \text{a.s.}$$

Theorem 3.1 thus shows that, for each y , the regular conditional probability for X given the observation $(Y_t, 0 \leq t \leq T)$ is the only probability measure on $(\mathbf{X}, \mathcal{X})$ with the property that

$$(4.4) \quad \begin{aligned} F(P_{X|Y}(\cdot, y), y) &= \min_{\tilde{P}_X \in \mathcal{P}(\mathcal{X})} F(\tilde{P}_X, y) \\ &= -\log(\mathbf{E} \exp(-H(X, Y))), \end{aligned}$$

where

$$F(\tilde{P}_X, y) = \mathcal{H}(\tilde{P}_X | P_X) + \int_{\mathbf{X}} H(x, y) d\tilde{P}_X(x).$$

Consider now the further specialization, in which X is an \mathbb{R}^n -valued diffusion process satisfying the following Itô equation:

$$(4.5) \quad \begin{aligned} X_t &= X_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s, \quad 0 \leq t < \infty, \\ X_0 &\sim \mu. \end{aligned}$$

Here, X_0 is an \mathbb{R}^n -valued random variable with distribution μ , $(W_t, \mathcal{F}_t; 0 \leq t < \infty)$ is a n -dimensional Brownian motion, and X_0 , W and V (of (4.1)) are independent. The abstract space \mathbf{X} of Section 2 now becomes the ‘path space’ $C([0, \infty); \mathbb{R}^n)$, and \mathcal{X} is the σ -field generated by the co-ordinate process on \mathbf{X} . We impose conditions on the coefficients b and σ such that (4.5) has a *strong* solution $\Phi : \mathbb{R}^n \times C_0([0, \infty); \mathbb{R}^n) \rightarrow C([0, \infty); \mathbb{R}^n)$. In particular, this means that $(X_t = \Phi_t(X_0, W), \mathcal{F}_t; 0 \leq t < \infty)$ is a continuous (\mathcal{F}_t) -semimartingale satisfying (4.5). The observation, Y , is now $h(X_t)$ for some

measurable $h : \mathbb{R}^n \rightarrow \mathbb{R}^d$. Under appropriate hypotheses on b, σ, h there exists a continuous, regular conditional probability distribution for X given Y , $P_{X|Y}$, and this is the only probability measure on the path space $(\mathbf{X}, \mathcal{X})$ with the property (4.4) for the Hamiltonian

(4.6)

$$H(X, y) = -y_T' h(X_0) - \int_0^T (y_T - y_t)' dh(X_t) + \frac{1}{2} \int_0^T \|h(X_t)\|^2 dt \quad \text{a.s.}$$

There is a *dynamic programming* interpretation of the optimization problem (4.4) as the following argument shows. Let $P_{X|X_0} : \mathcal{X} \times \mathbb{R}^n \rightarrow [0, 1]$ be a regular conditional probability for $(X_t, 0 \leq t < \infty)$ given X_0 , e.g., let

$$P_{X|X_0}(A, z) = \mathbf{E} \mathbf{1}_{\Phi^{-1}(A)}(z, W),$$

where Φ is the strong solution of (4.5), $P_{X|X_0}$ is also a regular conditional probability for $(X_t, s \leq t < \infty)$ given X_s . Let $\Lambda : [0, T] \times \mathbf{X} \times \mathbf{Y} \rightarrow \mathbb{R}^+$ be any measurable function such that, for each x, y , $\Lambda(\cdot, x, y)$ is continuous, and, for each s, y ,

$$\Lambda(s, X, y) = \exp \left((y_s - y_0)' h(X_0) + \int_0^s (y_s - y_t)' dh(X_t) - \frac{1}{2} \int_0^s \|h(X_t)\|^2 dt \right) \quad \text{a.s.},$$

and, for some $0 \leq s \leq T$, let

$$L_s(x, y) = \Lambda(s, x, y) \int_{\mathbf{X}} L(T - s, \tilde{x}, S_s y) dP_{X|X_0}(\tilde{x}, x_s),$$

where S_s is the ‘shift’ operator:

$$(S_s y)_t = y_{s+t};$$

then

$$L_s(X, y) = \mathbf{E}(L(X, y) \mid X_t, 0 \leq t \leq s).$$

Let $(\mathcal{X}_t, 0 \leq t < \infty)$ be the following filtration on $(\mathbf{X}, \mathcal{X})$

$$\mathcal{X}_t = \sigma(\chi_s, 0 \leq s \leq t) \quad \text{for } 0 \leq t < \infty,$$

where χ is the coordinate function on \mathbf{X} , and suppose that $A \in \chi_s$; then

$$\begin{aligned} \int_A L(x, y) dP_X(x) &= \int_A L_s(x, y) dP_X(x) \\ &= \int_A \exp(-L)s(x, y) dP_X(x), \end{aligned}$$

where

$$(4.7) \quad H_s = -\log(L_s).$$

We thus have the following Bayes' formula for the restriction of $P_{X|Y}$ to \mathcal{X}_s , $P_{X^s|Y}$, (the *nonlinear path interpolator* for $(X_t, 0 \leq t \leq s)$)

$$(4.8) \quad \begin{aligned} P_{X^s|Y}(A, y) &= P_{X|Y}(A, y) \\ &= \frac{\int_A \exp(-H_s(x, y)) dP_X(x)}{\int_{\mathbf{X}} \exp(-H_s(x, y)) dP_X(x)} \quad \text{for } A \in \mathcal{X}_s, \end{aligned}$$

and by Theorem 3.1, $P_{X^s|Y}$ is the only probability measure on \mathcal{X}_s with the property

$$(4.9) \quad \begin{aligned} F_s(P_{X^s|Y}(\cdot, y), y) &= \min_{\tilde{P}_{X^s} \in \mathcal{P}(\mathcal{X}_s)} F_s(\tilde{P}_{X^s}, y) \\ &= -\log \left(\int_{\mathbf{X}} \exp(-H_s(x, y)) dP_{X^s}(x) \right) \\ &= -\log \left(\int_{\mathbf{X}} \exp(-H(x, y)) dP_X(x) \right), \end{aligned}$$

where P_{X^s} is the restriction of P_X to \mathcal{X}_s , H is the Hamiltonian of the path estimator, (4.6),

$$(4.10) \quad \begin{aligned} F_s(\tilde{P}_{X^s}, y) &= \mathcal{H}(\tilde{P}_{X^s} | P_{X^s}) + \int_{\mathbf{X}} H_s(x, y) d\tilde{P}_{X^s}(x) \\ &= \left(\mathcal{H}(\tilde{P}_{X^s} | P_{X^s}) + \int_{\mathbf{X}} -\log(L(s, x, y)) d\tilde{P}_{X^s}(x) \right) \\ &\quad + \int_{\mathbb{R}^n} -\log \left(\int_{\mathbf{X}} L(T-s, \tilde{x}, S_s y) dP_{X|X_0}(\tilde{x}, z) \right) d\tilde{\nu}(z), \end{aligned}$$

and $\tilde{\nu}$ is the distribution of X_s under \tilde{P}_{X^s} .

The first term on the right-hand side of (4.10) is the free energy of \tilde{P}_{X^s} for the problem of estimating $(X_t, 0 \leq t \leq s)$ given $(Y_t, 0 \leq t \leq s)$; the second term is the *minimum* free energy for the problem of estimating $(X_t, s \leq t \leq T)$ given $(Y_t - Y_s, s \leq t \leq T)$ when the initial distribution is the Dirac measure at the point z , averaged over $\tilde{\nu}$ (the terminal distribution associated with \tilde{P}_{X^s}). Thus (4.10) is a dynamic programming equation for

the path estimator (4.4), the integrand of the second term on the right-hand side being the value function:

$$v(z, s) = -\log \left(\int_{\mathbf{X}} L(T-s, \tilde{x}, S_s y) dP_{X|X_0}(\tilde{x}, z) \right).$$

This minimum free energy is achieved by the *posterior* regular conditional probability distribution for $(X_t, s \leq t \leq T)$, i.e., the regular conditional probability given that $X_s = z$ and that $Y = y$. Unlike the prior regular conditional probabilities, these are not stationary. This is because of the non-constancy of y and the ‘finite observation horizon’, T . It turns out that they can be constructed by a Girsanov transformation, which relates the path estimation problem to a problem in stochastic optimal control.

Let $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$ be a measurable function satisfying a linear growth condition. We consider the ‘controlled’ Itô equation

$$(4.11) \quad X_t^u = \phi + \int_0^t (b(X_s^u) + \sigma(X_s^u)u(X_s^u, s)) ds + \int_0^t \sigma(X_s^u) d\tilde{W}_s$$

where $\phi \in \mathbb{R}^n$ is a non-random initial condition and $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$ is a measurable *feedback control function* satisfying a linear growth condition. The aim is to find a u such that the following cost is minimized:

$$(4.12) \quad J(u, y) = \tilde{\mathbf{E}} \int_0^T \left(\frac{1}{2} \|u(X_t^u, t)\|^2 + \frac{1}{2} \|h(X_t^u)\|^2 - y_T' h(\phi) \right. \\ \left. - (y_T - y_t)' (\mathcal{L}h(X_t^u) + \text{div}(X_t^u)\sigma(X_t^u)u(X_t^u, t)) \right) dt$$

where $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{P}, \tilde{V}, X^u)$ is a weak solution of (4.9), $\tilde{\mathbf{E}}$ is expectation with respect to \tilde{P} , $y \in C_0([0, T]; \mathbb{R}^d)$

$$\text{div} = \left[\frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \cdots \frac{\partial}{\partial z_n} \right]$$

$$\text{and } \mathcal{L} = \sum_{i=1}^n b_i \frac{\partial}{\partial z_i} + \frac{1}{2} \sum_{j=1}^n a_{i,j} \frac{\partial^2}{\partial z_i \partial z_j}.$$

Equation (4.11) has a unique weak solution; i.e., all weak solutions to (4.11) have the same distribution on $(C([0, T]; \mathbb{R}^n), \mathcal{B}_T)$, P_X^u . That a weak solution *exists* follows from the following argument. Let \tilde{P} be a measure on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ of the path estimator, defined by

$$\frac{d\tilde{P}}{dP} = \exp \left(\int_0^T u'(X_t, t) dV_t - \frac{1}{2} \int_0^T \|u(X_t, t)\|^2 dt \right).$$

This defines a *probability* measure. Under \bar{P} , the process $(\bar{W}_t, 0 \leq t \leq T)$, defined by

$$\bar{W}_t = W_t - \int_0^t u(X_s, s) ds,$$

is a Brownian motion and so $(\Omega, \mathcal{F}, (\mathcal{F}_t), \bar{P}, \bar{W}, \Phi(\phi, W))$ is a weak solution to (4.9). We note that, for this solution,

$$J(u, y) = F(P_X^u, y),$$

where F is the free energy functional of the path estimator.

The following is the Hamilton–Jacobi–Bellman equation for the above stochastic optimal control problem

$$\begin{aligned} \frac{\partial v}{\partial t} + \mathcal{L}v + \frac{1}{2}\|h\|^2 - y'_T h(\phi) - (y_T - y_t)' \mathcal{L}h \\ + \inf_{\theta \in \mathbb{R}^n} \left\{ \frac{1}{2}\|\theta\|^2 - [(y_T - y_t)' \operatorname{div} h] \theta + (\operatorname{div} \nu) \theta \right\} = 0 \end{aligned}$$

$$\theta(\cdot, T) = 0.$$

The circle has now been closed and I have now shown how my previous work with Fleming has a natural Information Theoretical interpretation. It would be interesting to make a connection with the work on Maximum A Posteriori Probability Filters via the variational representation of conditional distributions I have obtained here (cf. the work of Mortensen, Hijab and Zeitouni). Finally, the variational interpretation has implications in obtaining lower bounds for estimation error.

5 On Stochastic Dissipativeness

Consider a partially observed stochastic control problem

$$(5.1) \quad dX_t = b(t, X_t, u_t) dt + dW_t, \quad X_t \in \mathbb{R}^n$$

and $u_t \in \mathbb{R}$ and where the last m -components of X_t form a vector Y_t which is observed and where the control is feedback control

$$u_t = u(t, Y_{[0,t]})$$

leading to the controlled equation

$$dX_t = f(t, X_t, u(t, Y_{[0,t]})) + dW_t.$$

We are required to choose the control $u(\cdot)$ to minimize

$$(5.2) \quad J(u) = E \left\{ \int_0^T c(t, X_t, u_t) dt + \gamma(X_T) \right\}.$$

where $c > 0$. Henceforth c_s^u denotes $c(s, X_s, u)$.

Let V_t^u denote the minimum expected future cost given that the law u is used in $[0, t]$ and given the σ -field of observations \mathcal{F}_t^Y . Now the Principle of Optimality states that for $0 \leq t < t+h \leq T$ and $u \in U$, V_t^u satisfies

$$\begin{aligned} V_t^u &\leq E^u \left\{ \int_t^{t+h} c_s^u ds \mid \mathcal{F}_t^y \right\} + E^u \{ V_{t+h}^u \mid \mathcal{F}_t^u \} \quad \text{a.s.} \\ V_T^u &= E^u \{ \gamma \mid \mathcal{F}_T^y \} \quad \text{a.s.} \end{aligned}$$

where E^u represents expectation with respect to P^u where P^u is the transformed measure corresponding to the Girsanov Functional

$$L_{st}^u = \exp \left\{ \int_s^t f(\tau, X_\tau, u_\tau) dX_\tau - \frac{1}{2} \int_s^t \|f(\tau, X_\tau, u_\tau)\|^2 d\tau \right\}.$$

For $u \in U$, define the process $(W_t^u, \mathcal{F}_t^y, P^u)$ by

$$W_t^u = E^u \left\{ \int_0^t c_s^u ds \mid \mathcal{F}_t^y \right\} + V_t^u.$$

Then $(W_t^u, \mathcal{F}_t^y, P^u)$ is a sub-martingale and u is optimal $\Leftrightarrow W_t^u$ is a martingale. This implies that $(V_t^u, \mathcal{F}_t^y, P^u)$ is a positive supermartingale for optimal u .

Now we think of c_s^u as a supply rate and V_t^u as a storage function and we say that (5.1) is dissipative w.r.t. the supply rate c_s^u if for all admissible controls u and all finite intervals, there exists a V_t^u (as defined previously) which is a positive supermartingale. The implication of these ideas to stability questions in non-linear filtering will be explored elsewhere [10].

References

- [1] A.E. Bryson and M. Frazier, Smoothing for linear and non-linear dynamic systems. TDR 63-119, Tech. Reft., Aero System Division. Wright Patterson Air Force Base, Ohio, pp. 353–364.
- [2] W. Fleming and S.K. Mitter, Optimal control and pathwise nonlinear filtering of non-degenerate diffusions, *Stochastics*, **8**(1) (1982), 63–77.
- [3] J.M.C. Clark, The design of robust approximations to the stochastic differential equations in nonlinear filtering, in: *Communication Systems and Random Process Theory*, J. Skwirzaynski, eds., Sijthoff Noordhoff, 1978.
- [4] M.H.A. Davis, On a multiplicative transformation arising in non-linear filtering, *Z. Wahrschein. verrw. Geb.*, **54** (1981), 125–139.
- [5] H.O. Georgii, *Gibbs Measures and Phase Transitions*, De Gruyter, Holland.
- [6] J.D. Deuschel and D.W. Stroock, *Large Deviations*, Academic Press, New York, 1989.
- [7] J.C. Willems, Dissipative dynamical systems, part I: general theory, *Arch. Rat. Mech. and Anal.*, **45**(5) (1972), 321–351.
- [8] M.H.A. Davis and P. Varaiya, Dynamic programming conditions for partially observable stochastic systems, *SIAM J. on Control*, **11**(2) (1973), 226–261.
- [9] J.R. Dorfman, *An Introduction to Chaos in Nonequilibrium Statistical Mechanics*, Cambridge University Press, Cambridge, UK, 1999.
- [10] S.K. Mitter and N. Newton, Forthcoming papers.
- [11] P. Dai Pra, L. Meneghini, and W.J. Runggaldier, Connections between stochastic control and dynamical games, *Math. Control Signals Systems*, **9** (1996), 303–326.