Filtering and Stochastic Control: A Historical Perspective

Sanjoy K. Mitter

In this article we attempt to give a historical account of the main ideas leading to the development of non-linear filtering and stochastic control as we know it today.

The article contains six sections. In the next section we present a development of linear filtering theory, beginning with Wiener-Kolmogoroff filtering and ending with Kalman filtering. The method of development is the innovations method as originally proposed by Bode and Shannon and later presented in its modern form by Kailath. The third section is concerned with the Linear-Quadratic-Gaussian problem of stochastic control. Here we give a discussion of the separation theorem which states that for this problem the optimal stochastic control can be constructed by solving separately a state estimation problem and a deterministic optimal control problem. Many of the ideas presented here generalize to the non-linear situation. The fourth section gives a reasonably detailed discussion of non-linear filtering, again from the innovations viewpoint. Finally, the fifth and sixth sections are concerned with optimal stochastic control. The general method of discussing these problems is Dynamic Programming.

We have chosen to develop the subject in continuous time. In order to obtain correct results for nonlinear stochastic problems in continuous time it is essential that the modern language and theory of stochastic processes and stochastic differential equations be used. The book of Wong [5] is the preferred text. Some of this language is summarized in the third section.

Wiener and Kalman Filtering

In order to introduce the main ideas of non-linear filtering we first consider linear filtering theory. A rather comprehensive survey of linear filtering theory was undertaken by Kailath in [1] and therefore we shall only expose those ideas which generalize to the non-linear situation. Suppose we have a signal process (z_t) and an orthogonal increment process (w_t), the noise process and we have the observation equation

$$y_t = \int_0^t z_s ds + w_t \ . \tag{1}$$

Note that if w_t is Brownian motion then this represents the observation

$$\dot{y}_t = z_t + \eta_t \tag{2}$$

where η_t is the formal (distributional) derivative of Brownian motion and hence it is white noise. We make the following assumptions.

(A1) (w_t) has stationary orthogonal increments

(A2) (z_t) is a second-order q.m. continuous process

(A3) For $\forall s$ and $t > s$

$$(w_t - w_s) \perp H_s^{w,z}$$

where $H_s^{w,z}$ is the Hilbert space spanned by $(w_\tau, z_\tau \mid \tau \le s)$.

The last assumption is a causality requirement but includes situations where the signal z_s may be influenced by past observations as would typically arise in feedback control problems. A slightly stronger assumption is

(A3)′ $H^w \perp H^z$

which states that the signal and noise are uncorrelated, a situation which often arises in communication problems. The situation which Wiener considered corresponds to (2), where he assumed that (z_t) is a stationary, second-order, q.m. continuous process.

The *filtering* problem is to obtain the best linear estimate $\hat{z}_t$ of z_t based on the past observations $(y_s \mid s \le t)$. There are two other problems of interest, namely, *prediction*, when we are interested in the best linear estimate $\hat{z}_r$, $r > t$ based on observations $(y_s \mid s \le t)$ and *smoothing*, where we require obtaining the best linear estimate $\hat{z}_r$, $r < t$ based on observations $(y_s \mid s \le t)$. Abstractly, the solution to the problem of filtering corresponds to explicitly computing

$$\hat{z}_t = P_t^y(z_t) \tag{3}$$

where P_t^y is the projection operator onto the Hilbert space H_t^y. We proceed to outline the solution using a method originally proposed by Bode and Shannon [2] and later presented in modern form by Kailath [3]. For a textbook account see Davis [4] and Wong [5], which we largely follow.

Let us operate under the assumption (A3)′, although all the results are true under the weaker assumption (A3). The key to obtaining a solution is the introduction of the innovations process

$$\nu_t = y_t - \int_0^t \hat{z}_s ds \tag{4}$$

The following facts about the innovations process can be proved:

The author is with the Department of Electrical Engineering and Computer Science and Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139. *This research has been supported by the Army Research Office under grant number DAAL03-92-G-0115 (through the Center for Intelligent Control Systems).*

0272-1708/96/$05.00©1996IEEE

(F1) ν_t is an orthogonal increment process.

(F2) $\forall s, \forall t > s$

$$\nu_t - \nu_s \perp H_s^y \quad \text{and}$$

$$\text{cov}(\nu_t) = \text{cov}(w_t)$$

(F3) $H_t^y = H_t^\nu$.

The name "innovations" originates in the fact that the optimum filter extracts the maximal probabilistic information from the observations in the sense that what remains is essentially equivalent to the noise present in the observation. Furthermore, (F3) states that the innovations process contains the same information as the observations. This can be proved by showing that the linear transformation relating the observations and innovations is causal and causally invertible. As we shall see later, these results are true in a much more general context. To proceed further, we need a concrete representation of vectors residing in the Hilbert space H_t^y. The important result is that every vector $Y \in H_t^y$ can be represented as

$$Y = \int_0^t \beta(s) dy_s \tag{5}$$

where β is a deterministic square integrable function and the above integral is a stochastic integral. For an account of stochastic integrals see the book of Wong [*loc. cit.*]. Now using the Projection Theorem, (5), and (F1)-(F3) we can obtain a representation theorem for the estimate $\hat{z}_t$ as:

$$\hat{z}_t = \int_0^t \left(\frac{\partial}{\partial s} E(z_t \nu_s') d\nu_s \right). \tag{6}$$

What we have done so far is quite general. As we have mentioned, Wiener assumed that (z_s) was a stationary q.m. second-order process, and he obtained a linear integral representation for the estimate where the kernel of the integral operator was obtained as a solution to an integral equation, the Wiener-Hopf equation. As Wiener himself remarked, effective solution to the Wiener-Hopf equation using the method of spectral factorization (see, for example, Youla [6]) could only be obtained when (z_s) had a rational spectral density. In his fundamental work Kalman ([7, 8, 9]) made this explicit by introducing a Gauss-Markov diffusion model for the signal

$$\begin{cases} dx_t = Fx_t dt + Gd\beta_s \\ z_t = Hx_t \end{cases} \tag{7}$$

where x_t is an n-vector-valued Gaussian random process, w_t is m-dimensional Brownian motion, z_t is a p-vector-valued Gaussian random process, and F, G, and H are matrices of appropriate order. We note that (7) is actually an integral equation

$$x_t = x_0 + \int_0^t Fx_s ds + \int_0^t Gd\beta_s \tag{7'}$$

where the last integral is a stochastic integral. The Gauss-Markov assumption is no loss of generality since in Wiener's work the best linear estimate was sought for signals modeled as second-order random processes. The filtering problem now is to compute the best estimate (which is provably linear)

$$\hat{x}_t = P_t(x_t). \tag{8}$$

Moreover, in this new setup no assumption of stationarity is needed. Indeed the matrices F, G, and H may depend on time. The derivation of the Kalman filter can now proceed as follows. First note that

$$\hat{x}_t = \int_0^t \frac{\partial}{\partial s} E(x_t \nu_s') d\nu_s . \tag{9}$$

(See Equation (6).)

Now we can show that

$$\hat{x}_t - \hat{x}_0 - \int_0^t F\hat{x}_s ds = \int_0^t K(s) d\nu_s \tag{10}$$

where $K(s)$ is a square integrable matrix-valued function. This is analogous to the representation theorem given by (5).

Equation (10) can be written in differential form as

$$d\hat{x}_t = F\hat{x}_t dt + K(t) d\nu_t \tag{11}$$

and let us assume that $\hat{x}_0 = 0$. The structure of Equation (11) shows that the Kalman Filter incorporates a model of the signal and a correction term, which is an optimally weighted error $= K(t)(dy_t - \hat{z}_t dt)$ (see Figure 1).

It remains to find an explicit expression for $K(t)$. Here we see an interplay between filtering theory and linear systems theory. The solution of (11) can be written as

$$\hat{x}_t = \int_0^t \Phi(t,s) K(s) d\nu_s \tag{12}$$

where $\Phi(t, s)$ is the transition matrix corresponding to F. From (9) and (12)

$$\Phi(t,s) K(s) = \frac{\partial}{\partial s} E(x_t \nu_s')$$

and hence

$$K(t) = \frac{\partial}{\partial s} E(x_t \nu_s')|_{s=t} .$$

Some further calculations using the fact that $x_t \perp H_s^w$ shows that

$$K(t) = P(t)H' ,$$

where $P(t) = E(\tilde{x}_t \tilde{x}_t')$, $\tilde{x}_t = x_t - \hat{x}_t$. Finally, using the representation of solutions of the linear stochastic differential equations (7) and using (11) we can write a linear stochastic differential equation for $\tilde{x}_t$ and write down a representation for $P(t) = E(\tilde{x}_t \tilde{x}_t')$ as

$$P(t) = \psi(t,o)P(0)\psi'(t,0) + \int_0^t \psi(t,s)GG'\psi'(t,s)ds + \int_0^t \psi(t,s)P(s)H'HP(s)\psi'(t,s)ds \tag{13'}$$

where $\psi(t,s)$ is the transition matrix corresponding to $(F - PH'H)$. There is again a role of linear systems theory evident here. Differentiating w.r. to t, we get a matrix differential equation for $P(t)$, the matrix Ricatti equation

$$\begin{cases} \dfrac{dP}{dt} = GG' - P(t)H'HP(t) + FP(t) + P(t)F' \\ P(0) = \mathrm{cov}(x_0) = P_0 \, . \end{cases} \tag{14}$$

Note that $K(t) = P(t)H'$ is deterministic and does not depend on the observation process y_t, and hence can be pre-computed. The approach to the solution of the Wiener Filtering Problem consists in studying the equilibrium behavior of $P(t)$ as $t \to \infty$. There is again a beautiful interplay between the infinite time behavior of the filter and the structural properties of Equation (7). One can prove that if the pair (F, G) is stabilizable and (H, F) is detectable then $P(t) \to \bar{P}$ as $t \to \infty$ where $\bar{P}$ is the unique non-negative solution to the algebraic Ricatti equation corresponding to (14) and that $F - \bar{P}H'H$ is a stability matrix. Thus the filter is stable, in the sense that the error covariance converges to the optimal error covariance for the stationary problem even if F is not a stability matrix. For the linear systems concepts introduced here and the proof of the above results the reader may consult Wonham [10].

The Linear Quadratic Gaussian (LQG) Problem and the Separation Principle

At about the same time that the theory of filtering using linear stochastic differential equations (Gauss-Markov Processes) was being developed, an analogous development for the optimal control of linear dynamical systems with a quadratic cost function was taking place. This work was inspired by the development of Dynamic Programming by Bellman [11] and the ideas of Caratheodory related to Hamilton-Jacobi Theory [12] and was developed by Merriam [13] and Kalman [14]. For textbook accounts see Brockett [15], Wonham [10], and Bryson and Ho [16]. An extension of the quadratic cost optimal control problem for linear dynamical systems in the presence of additive white process noise perturbations leads us to consider the quadratic cost problem for linear stochastic dynamical systems.

The general situation here is that we are given a linear stochastic dynamical system

$$dx_t = Fx_t \, dt + Bu_t \, dt + G \, d\beta_t \tag{15}$$

and the observation equation

$$dy_t = Hx_t \, dt + dw_t. \tag{16}$$

Here (β_t) and (w_t) are taken to be independent vector Brownian motions and u_t is a control variable which needs to be chosen based on the information available to the controller so as to minimize the cost function

$$J(u) = E\left\{ \int_0^T (x_t'Qx_t + u_t'Ru_t)dt + x_T'Mx_T \right\} \tag{17}$$

where $Q \geq 0$ is symmetric and $R > 0$ is symmetric. In the general partial observation case the control u_t is required to be a function of the past of the observation, i.e., of $(y_s \mid o \leq s \leq t)$. Historically, these problems in somewhat specialized situations were first examined and solved by Florentin [17,18], by Joseph [19] in discrete-time, and by Kushner [20]. The definitive treatment of this problem is due to Wonham [21]. See also the important paper of Lindquist [22].

When stochastic disturbances are present, there is a fundamental difference between open-loop control (that is, where the control is not a function of the observations) and feedback control (where control is a function of the past of the observations). In general, feedback control will lead to a lower cost than open-loop control. Furthermore, the only general methodology for handling these problems is Dynamic Programming. To approach the partially observable-stochastic control problem involving linear stochastic dynamics and a quadratic cost function, we first consider the corresponding fully observable stochastic control problem by setting $w_t \equiv 0$, $\forall t \in [0, T]$ and $H \equiv I$. In this case, the problem can be solved using Dynamic Programming and certain ideas of stochastic calculus which we now describe. We set up the necessary language, which will be useful later.

All stochastic processes will be defined on a fixed probability space (Ω, F, P) and a finite time interval $[0, T]$, on which there is defined an increasing family of σ-fields $\{F_t, 0 \leq t \leq T\}$. It is assumed that each process $\{x_t\}$ is *adapted* to F_t—i.e., x_t is F_t-measurable for all t. The σ-field generated by $\{x_s, 0 \leq s \leq t\}$ is denoted by $F_t^x = \sigma\{x_s, 0 \leq s \leq t\}$. (x_t, F_t) is a *martingale* if it is a *supermartingale* and a *sub-martingale*. $E[x_t \mid F_s] = x_s$, x_t is a *supermartingale* if $E[x_t \mid F_s] \leq x_s$ and a *submartingale* if $E[x_t \mid F_s] \geq x_s$. The process (x_t, F_t) is a *semimartingale* if it has a decomposition $x_t = x_0 + a_t + m_t$, where (m_t, F_t) is a martingale and $\{a_t\}$ is a process of bounded variation. Given two square integrable martingales (m_t, F_t) and (n_t, F_t), one can define the *predictable quadratic covariation* $(\langle m, n\rangle_t, F_t)$ to be the unique "predictable process of integrable variation" such that $(m_t n_t - \langle m, n\rangle, F_t)$ is a martingale. For the purposes of this article, however, the only necessary facts concerning $\langle m, n\rangle$ are that (a) $\langle m, n\rangle_i = 0$ if $m_t n_t$ is a martingale; and (b) if β is a standard Brownian motion process, then

$$\langle \beta, \beta \rangle_t = t \quad \text{and} \quad \left\langle \int_0^t \eta_s^1 d\beta_s, \int_0^t \eta_s^2 d\beta_s \right\rangle_t = \int_0^t \eta_s^1 \eta_s^2 ds \, .$$

Finally, we need to use the Ito differential rule. Suppose that x_t is a vector diffusion process given by

$$x_t = x_0 + \int_0^t f(x_s)ds + \int_0^t G(x_s)d\beta_s \tag{18}$$

where $x_t \in \mathbf{R}^n$, $\beta_t \in \mathbf{R}^m$ is a vector of independent Brownian motions and f and g are vector and matrix-valued functions, suitably smooth.

In the above the last integral is a stochastic integral which is a generalization of the Wiener integral we have encountered before and is defined through an appropriate approximation process and a quadratic mean-limiting process (see, for example, Wong [5]). This cannot be defined as a Lebesgue-Stieltjes integral because the trajectories of Brownian motion are not of bounded variation almost surely. Now, if ψ is a twice continuously differentiable function of x, then

$$\psi(x_t) = \psi(x_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial \psi}{\partial x^i}(x_s) dx_s + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x_s) a^{ij}(x_s) ds \tag{19}$$

where $A(x) = \left(a^{ij}(x)\right)_{i,j=1}^{n} = G(x)G'(x)$.

Note that in contrast to ordinary calculus we have a second-order term in the formula arising from the variance properties of Brownian motion. This is the Ito differential rule.

Let us now return to the fully observable stochastic control problem. Associated with the linear stochastic differential equation

$$\begin{cases} dx_s = Fx_s ds + G d\beta_s & s \in [t, T] \\ x_t = x(\text{fixed}) \end{cases} \tag{20}$$

and L is the operator

$$(Lf)(t,x) = f_t + f_x' F x + \frac{1}{2} \operatorname{tr}(G' f_{xx} G).$$

Then an application of Ito's differential rule shows that for V smooth

$$V(t,x) = -E_{t,x}\left[\int_t^T LV(s, x_s) ds - x_T' M x_T\right] \tag{21}$$

(Dynkin's formula).

This formula is valid in a much more general context. Now returning to the control problem let us consider admissible controls (feedback) in the class

$$u(t, x_t) = K(t) x_t \tag{22}$$

where $K(t)$ is a piecewise continuous matrix-valued function. This is quite a general class of control laws, and it can be shown that there is nothing to be gained by considering more general linear non-anticipative control laws. Consider an optimal control $u_t^0 = K_0(t) x_t$. Then

$$V^0(t,x) = E_{t,x}\left[\int_t^T \left(x_s^{0\prime} Q x_s^0 + u_s^{0\prime} R u_s^0\right) ds + x_T^{0\prime} M x_T^0\right]$$
$$= E_{t,x}\left[\int_t^T x_s^{0\prime} S^0(t) x_s^0 ds + x_T^{0\prime} M x_T^0\right]$$

where x_s^0 is the solution of

$$\begin{cases} dx_s = \left(F + BK_0(t)\right) x_s + G d\beta_s \\ x_t = x, \end{cases}$$
$$S^0(t) = S_{K_0}(t) = Q + K_0'(t) R K_0$$

For a feedback matrix $K(t)$, let

$$(L_K f)(t,x) = f_t + f_x'\left(F + BK(t)\right)x + \frac{1}{2}\operatorname{tr}(G' f_{xx} G)$$

and let $S_K(t) = Q + K(t)RK(t)$.

Then applying Bellman's Principle of Optimality we obtain

$$0 = L_{K_0} V^0(t,x) + x' S_{K_0} x \le L_K V^0(t,x) + x' S_K x, \tag{23}$$

where K_0 is the optimal feedback gain and K is any admissible feedback gain. The above equation can be explicitly written as

$$\min_K \begin{pmatrix} V_t + \frac{1}{2}\operatorname{tr}(G' V_{xx} G) + V_x\left(F - BK(t)\right)x \\ + x'\left(Q + K'(t) R K(t)\right)x \end{pmatrix} = 0. \tag{24}$$

Note that in contrast to the deterministic situation ($\beta \equiv 0$), there is a second-order operator in the above equation. This equation can be solved for K_0, V_0 using essentially the same method as in the deterministic case. The result is that the optimal control $u^0(t)$ is given by

$$u^0(t) = -R^{-1} B' P(t) x_t$$

where $P(t)$ is a symmetric non-negative solution of the matrix Riccati equation

$$\begin{cases} \frac{dS}{dt} + S(t)F + F'S(t) + Q - S(t)BR^{-1}B'S(t) = 0 \\ S(T) = M \end{cases} \tag{25}$$

and the optimal cost function is

$$J(u_0) = \int_0^T \operatorname{tr}\left(G' P(s) G\right) ds + m_0' P(0) m_0 + tr\left(\Sigma_0 P(0)\right) \tag{26}$$

where $E(x_0) = m_0$ and $\operatorname{cov}(x_0) = \Sigma_0$.

It is interesting to note that the optimal control is the same as in the deterministic case, but not the expression for the optimal cost function. Indeed the deterministic situation can be recovered by setting $C = 0$ and $P\Sigma_0 = 0$. This result, however, crucially depends on the quadratic nature of the cost functions and the linearity of the dynamics.

In proving optimality we have restricted ourselves to control laws which are linear. One can prove the same results by considering non-linear control laws which are Lipschitz function of x (see Wonham, Loc. at).

Let us now return to the partially observable problem. The key idea here is to introduce the idea of an information state (cf. Davis and Varaiya [23]) and reduce the partially observable problem to the fully observable problem. Now the information state is the conditional distribution

$P^u(x_t \mid y_s, 0 \le s \le t)$ where the superscript denotes the dependence on the control u. In our case this conditional distribution is conditionally Gaussian and given by the Kalman filter

$$d\hat{x}_t = F\hat{x}_t dt + K(t)d\nu_t + B(t)u_t \tag{27}$$

where $K(t)$ is given by (13). Furthermore, the innovations process ν_t as given by (4) satisfies (F1), (F2), and (F3) even in this case. In fact ν_t is a Brownian motion adapted to F_t^y, the σ-field generated by $(y_s \mid 0 \le s \le t)$. This is true for control laws which are Lipschitz functions of the past of y. If we restrict ourselves to this class, then Wonham showed that the admissible control laws are $u_t = \varphi(\hat{x}_t)$ where φ is Lipschitz. The issue of the choice of admissible control laws is a subtle one because of questions of existence and uniqueness of non-linear stochastic differential equations. For a detailed discussion cf. Lindquist [loc. cit.]. Now by writing $x_t = \hat{x}_t + \tilde{x}_t$, where $\tilde{x}_t$ is the error process and using the fact that $\tilde{x}_t \perp H_t^{y,u}$, the cost function given by (17) can be rewritten as:

$$J(n) = E\left[\int_0^T (\hat{x}_t' Q\hat{x}_t + u_t' R u_t)dt + \hat{x}_T M \hat{x}_t\right] + \int_0^T \mathrm{Tr}\left[P(t)Q\right]dt + Tr\left(P(T)M\right). \tag{28}$$

Now it can be shown using the arguments of the fully observable case that the optimal control is given by

$$u^0(t) = -R^{-1}B'\Lambda(t)\hat{x}_t \tag{29}$$

where $\Lambda(t)$ is a symmetric non-negative solution of Equation (25) and $\hat{x}_t$ is given by (27). Now $\Lambda(t)$ is the same as in the deterministic optimal control problem, and we have the separation theorem which states that the partially observable stochastic control separates into the solution of a deterministic optimal control problem and a Kalman filtering problem.

We do not go into a detailed discussion of the relationship between the separation principle and the certainty equivalence principle here (cf. Witsenhausen, [24]). It should be mentioned that the certainty equivalence principle was discussed in the economics literature in the late '50s (cf. Holt et al. [25]). For an illuminating discussion on the distinctions between open-loop stochastic control, feedback control, and open-loop feedback control, see Dreyfus [26].

Nonlinear Filtering

To develop the theory of non-linear filtering we follow the scheme of development of linear filtering theory. It is interesting that using the theory of martingales the generalization to the non-linear filtering case is very natural. The ideas that we use were first introduced and developed by Frost-Kailath [27] and in somewhat definitive form by Fujisaki-Kallianpur-Kunita [28]. The historical development proceeded in a somewhat different manner and we shall discuss this in a later part of this section.

Our basic model is the observation equation

$$y_t = \int_0^t z_s ds + w_t \tag{30}$$

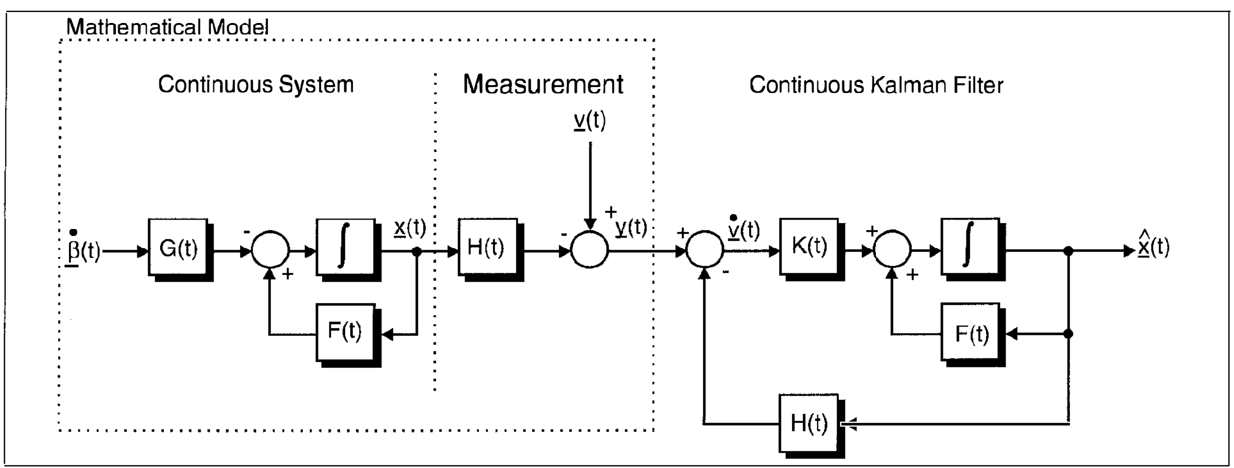


Fig. 1. System model and continuous Kalman filter.

with the assumptions

(H1) y_t is real-valued process

(H2) w_t is standard Brownian motion

(H3) $E\left[\int_0^T z_s^2 ds\right] < \infty$

(H4) (z_t) is independent of w_t.

These assumptions are similar to (A1), (A2), (A3)′ in the linear situation.

Consider the innovations process

$$\left.\begin{aligned} \nu_t &= y_t - \int_0^t \hat{z}_s ds \\ \text{where} \quad \hat{z}^s &= E\left(z_s | F_s^y\right) \end{aligned}\right\} \tag{31}$$

It can now be shown that:

The process $\left(\nu_t, F_t^y\right)$ is standard Brownian motion and F_s^y and $\sigma(\nu^u - \nu_t \mid 0 \le s \le t < u \le T)$ are independent. This result is proved by showing that ν_t is a square integrable martingale with continuous sample paths with quadratic variation t, and the result follows from the Levy characterization of Brownian motion.

Now analogous to (F3) in the linear case one can prove that

$$\mathcal{F}_t^y = \mathcal{F}_t^\nu ; \tag{32}$$

that is, the innovations contains the same information as the observation. This rather delicate result was proved by Allinger-Mitter [29].

Now combining this with the representation of square integrable martingales as stochastic integrals due to Kunita and Watanabe, we obtain the following:

Every square-integrable martingale $\left(m_t, F_t^y\right)$ can be represented as

$$m_t = E(m_0) + \int_0^t \eta_s ds \tag{33}$$

$$\int_0^T E\left(\eta_s^2\right) ds < \infty \quad \text{and} \quad \eta_t \quad \text{is adapted to} \quad F_t^\nu .$$

It should be remarked that Fujisaki-Kallianpur-Kunita in their important paper proved the same result without (32) holding but with η_t adapted to $\mathcal{F}_t^y$.

To proceed further let us assume that

$$z_t = h(x_t) \tag{34}$$

and x_t satisfies a stochastic differential equation

$$x_t = x_0 + \int_0^t f(x_s) ds + \int_0^t G(x_s) d\beta_s ,$$

which is the same as Equation (18).

Suppose we want to obtain the estimate

$$\pi_t(\varphi) \overset{\Delta}{=} E\left(\varphi(x_t) | \mathcal{F}_t^y\right). \tag{35}$$

We want to obtain a recursive equation for $\pi_t(\varphi)$. We need some preliminaries.

Let L be the second-order elliptic operator defined by

$$L\psi = \sum_{i=1}^n f^i(x) \frac{\partial^i \psi}{\partial x^i} + \sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 \psi}{\partial x^i \partial x^j} \tag{36}$$

$$\text{and } A(x) = \left(a^{ij}(x)\right)_{i,j=1}^n = G(x) G'(x) .$$

Then we can write Ito's differential rule (19) as:

$$\psi(x_t) - \psi(x_0) - \int_0^t L\psi(x_s) ds = \int_0^t \left(\nabla \psi(x_s)\right)' G(x_s) d\beta_s \tag{37}$$

where ∇ is the gradient operator, and the last term $M_t^\psi = \int_0^t (\nabla\psi(x_s)) G(x_s) d\beta_s$ is a $\mathcal{F}_t^\beta$-martingale (being a stochastic integral).

To obtain the recursive equation for $\pi_t(\varphi)$, one shows that $M_t^\varphi = \pi_t(\varphi) - \pi_0(\varphi) - \int_0^t \pi_s(L\varphi)$ is a square integrable $\mathcal{F}_t^y$, hence $\mathcal{F}_t^\nu$ martingale. Therefore from the representation theorem $M_t^\varphi = \int_0^t \eta_s d\nu_s$, where η_s is square integrable and adapted to $\mathcal{F}_t^\eta$. Therefore

$$\pi_t(\varphi) = \pi_0(\varphi) + \int^t \pi_s(L\varphi) ds + \int_0^t \eta_s d\nu_s . \tag{38}$$

It remains to identify η_s. This can be obtained as follows:

By the Ito differential rule (37)

$$\varphi(x_t) = \varphi(x_0) + \int_0^t L\varphi(x_s) ds + M_t^\varphi .$$

Also

$$y_t = y_0 + \int_0^t h(x_s) ds + w_t .$$

Now, using the Ito-differential rule for semi-martingales,

$$\begin{aligned} \varphi(x_t) y_t &= \varphi(x_0) y_0 + \int_0^t y_s d\varphi(x_s) + \int_0^t \varphi(x_s) dy_s + \left\langle M^\varphi, w \right\rangle_t \\ &= \varphi(x_0) y_0 + \int_0^t y_s \left(L\varphi(x_s) ds + dM_s^\varphi\right) \\ &\quad + \int_0^t \varphi(x_s)\left(h(x_s) ds + dw_s\right) \end{aligned} \tag{39}$$

(since $<M^{\varphi}, w>_t = 0$ from the independence of (x_t) and (w_t).

From the innovations representation

$$y_t = y_0 + \int_0^t \pi_s(h)ds + \nu_t \ . \tag{40}$$

Therefore

$$\begin{aligned}\pi_t(\varphi)y_t &= \pi_0(\varphi)y_0 + \int_0^t \pi_s(\varphi)(\pi_s(h)ds + d\nu_s) \\ &\quad + \int_0^t y_s(\pi_s(L\varphi)ds + \eta_s d\nu_s) - \langle N, \nu \rangle_t \\ &\qquad \text{where } N_t = \int_0^t \eta_s d\nu_s \\ &= \pi_0(\varphi)y_0 + \int_0^t \pi_s(\varphi)(\pi_s(h)ds + d\nu_s) \\ &\quad + \int_0^t y_s(\pi_s(L\varphi)ds + \eta_s d\nu_s) + \int_0^t \eta_s ds \ . \end{aligned} \tag{41}$$

Now noting that

$$E\left(\varphi(x_t)y_t - \pi_t(\varphi)y_t \mid \mathcal{F}_s^y\right) = 0 \ ,$$

from (40) and (41) we get

$$\eta_t = \pi_t(h\varphi) - \pi_t(\varphi)\pi_t(h)$$

and hence we get from (38):

$$\pi_t(\varphi) = \pi_0(\varphi) + \int_0^t \pi_s(L\varphi)ds + \int_0^t (\pi_s(h\varphi) - \pi_s(h)\pi_s(\varphi))d\nu_s \ . \tag{42}$$

This is one of the fundamental equations of non-linear filtering. If the conditional distribution π_t has a density given by $\tilde{p}(t,x)$, then $\tilde{p}$ satisfies the stochastic partial differential equation

$$d\tilde{p}(t,x) = L^* \tilde{p}(t,x)dt + \tilde{p}(t,x)[h(x) - \pi_t(h)]d\nu_t \tag{43}$$

where $\pi_t(h) = \int h(x)\tilde{p}(t,x)dx$. The question of existence of a conditional density can be discussed using the Malliavin calculus [46]. Equation (43) in this form, where the Ito calculus is involved, was first derived by Kushner [30]. The difficulty in deriving a solution for a conditional statistic $\hat{x}_t \stackrel{\Delta}{=} \pi_t(x) = \int x\tilde{p}(t,x)dx$ is the so-called closure problem. The equation is

$$d\hat{x}_t = \pi_t(f)dt + (\pi_t(hx)) - \pi_t(h)\hat{x}d\nu_t \ . \tag{44}$$

Note that computation of $\hat{x}_t$ requires computing

$$\pi_t(f) = \int f(x)\tilde{p}(t,x)dx, \quad \pi_t(hx) = \int h(x)x\tilde{p}(t,x)dx$$

and $\pi_t(h) \stackrel{\Delta}{=} \int h(x)\tilde{p}(t,x)$, and this requires solving stochastic differential equations for each of these above quantities, which in their turn involve higher moments. Hence non-linear filters are almost always infinite-dimensional. There are only a few known examples where the filter is known to be finite-dimensional. The first is the linear-gaussian situation leading to the Kalman filter which we have treated in an earlier section. The second is the finite-state case, first considered in an important paper by Wonham [31]. Let x_t be a finite-state Markov process taking values $S = (s_1, \dots, s_N)$. Let $p_t = (p_t^1, \dots, p_t^N)$ be the probability vector where $p_t^i = \text{Prob}(x_t = s_i)$. Then the evolution of p_t is given by the forward Kolmogoroff equation

$$\frac{dp_t}{dt} = Ap_t \ .$$

If we denote by $\tilde{p}_t = \text{Prob}(x_t = s_i \mid \mathcal{F}_t^y)$

$$B = \text{diag}\ (h(s_1), \dots, h(s_N)) \quad \text{and} \quad b' = (h(s_1), \dots, h(s_N)),$$

then $\tilde{p}_t$ satisfies

$$d\tilde{p}_t = A\tilde{p}_t dt + \left[B - (b'\tilde{p}_t)I\right]\tilde{p}_t\left(dy_t - (b'\tilde{p}_t)dt\right) . \tag{45}$$

We shall discuss a further example leading to a finite-dimensional filter a little later. One of the difficulties with Equation (43) is that it is a non-linear stochastic partial differential equation. An important idea due to Zakai [32], Duncan [33], and Mortensen [34] is to write $\pi_t(\varphi)$ as

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)} \tag{46}$$

where $\rho_t(\varphi)$ satisfies

$$\rho_t(\varphi) = \rho_0(\varphi) + \int_0^t \rho_s(L\varphi)ds + \int_0^t \rho_s(h\varphi)dy_s \tag{47}$$

ρ_t is an un-normalized version of π_t. Note that this is a linear stochastic partial differential equation. This is intimately related to the Feynman-Kac formula for integrating linear parabolic equations with a potential term. For a discussion between the analogies between non-linear filtering and quantum mechanics see Mitter [35]. Recall that the original probability space is $(\Omega, \mathcal{F}, P)$ on which there is an increasing family of σ-fields $(\mathcal{F}_t)_{t\geq 0}$ and the process (x_t) is adapted to it. Define a new probability measure P_0 on $(\Omega, \mathcal{F})$ in terms of the Radon-Nikodym derivative

$$\frac{dP_0}{dP} = \exp\left(-\int_0^T h(x_s)dy_s - \frac{1}{2}\int_0^t h^2(x_s)ds\right) \stackrel{\Delta}{=} \Lambda_t \ . \tag{48}$$

Under P_0, (y_t) is standard Brownian motion, (x_t) and (y_t) are independent, and (x_t) has the same distribution under P_0 and P. Now,

$$\pi_t(\varphi)E\left(\varphi(x_t)|\mathcal{F}_t^y\right)=\frac{E_0\left(\varphi(x_t)\Lambda_t|\mathcal{F}_t^y\right)}{E_0\left(\Lambda_t|\mathcal{F}_t^y\right)}$$
$$\overset{\Delta}{=}\frac{\rho_t(\varphi)}{\rho_t(1)}. \tag{49}$$

Furthermore, we can prove that

$$\tilde{\Lambda}_t\overset{\Delta}{=}E_0\left(\Lambda_t|\mathcal{F}_t^y\right)=\exp\left(\int_0^t\pi_s(h)dy_s-\frac{1}{2}\int_0^t\pi_s(h)^2dy_s\right).$$

From (49) $\rho_t(\varphi)=\pi_t(\varphi)\tilde{\Lambda}_t$. Then using the Ito differential rule we get Equation (47). This derivation is due to Davis and Marcus [36], where the full details can be found. The measure transformation idea in stochastic differential equation is due to Girsanov (cf. Liptser and Shiryayev [37] and the references cited there).

Equation (47) is an Ito stochastic partial differential equation. There is a calculus, the so-called Stratanovich calculus, which in many ways is like ordinary calculus. The conditional density equation for non-linear filtering was derived using this calculus by Stratanovich [38]. For the relation between the two calculi see Wong [5]. This is an important modeling question. The Stratanovich form of Equation (47) is

$$\rho_t(\varphi)=\rho_0(\varphi)+\int_0^t\rho_s\left(L\varphi-\frac{1}{2}h^2\varphi\right)ds+\int_0^t\rho_s(h\varphi)\circ dy_s \tag{50}$$

where the last integral is a (symmetric) Stratanovich integral. It should be noted that geometry is preserved when we work with the Stratanovich form of the equation. The relation between (47) and (50) involves the Wong-Zakai correction (note that the generator L in Equation (50) has been replaced by $L-\frac{1}{2}h^2$). If ρ_t has a density $q(t, x)$ then $q(t, x)$ satisfies a linear stochastic partial differential equation

$$dq(t,x)=\left(L^*-\frac{1}{2}h^2\right)q(t,x)+h(x)q(t,x)\circ dy_t \tag{51}$$

It turns out that the Lie algebra generated by the operators $L^*-\frac{1}{2}h^2$ and h plays a role in the existence of a finite-dimensional filter. For a discussion of this see Brockett [39] and Mitter [40]. An example where a finite-dimensional filter exists is the following:

$$\begin{cases}dx_t=f(x_t)dt+d\beta_t, & x_t\in\mathbf{R}\\ dy_t=x_tdt+dw_t\end{cases}$$

and f satisfies the Riccati equation

$$\frac{df}{dx}+f^2=x^2$$

This example, first considered by Benes [41], is intimately related to the Kalman filter using a Gauge transformation $q(t, x) \mapsto \psi(x)q(t, x)$ where ψ is invertible (cf. Mitter, loc. cit.). On the other hand it can be shown that for the filtering problem

$$x_t=\beta_t$$
$$dy_t=x_t^3dt+dw_t$$

no finite-dimensional filter exists [42].

There are a number of other issues in non-linear filtering which we do not consider in this article. For discussions of pathwise non-linear filtering where the filter depends continuously on y see Clark [43] and Davis [44]. For the important problem of obtaining lower bounds on the mean-squared error see Bobrovsky-Zakai [45]. Results can be obtained when the signal (z_t) and the noise are correlated (cf. the review paper by Pardoux [46]).

Optimal Stochastic Control (Fully Observable Case)

The theory of optimal stochastic control in the fully observable case is quite similar to the theory we have sketched in the third section above, in connection with the linear quadratic stochastic control problem. The conceptual ideas here originated in the Dynamic Programming methodology developed by Bellman. An early work here is that of Howard [47], though not in the continuous-state continuous-time formulation. Important early papers related to this section are those of Florentin [17] and Fleming [48]. Many other people, notably Kushner, have contributed to this subject. For a textbook presentation where other references can be found, see Fleming-Rishel [49].

Consider the problem of minimizing

$$J(t,x,u)=E_{tx}\left[\int_t^T l(x(s),u(s))ds+\psi(x(T))\right] \tag{52}$$

where $x_s\in\mathbf{R}^n$ evolves according to the stochastic differential equation

$$\begin{cases}dx_s=f(x_s,u_s)ds+G(x_s,u_s)d\beta_s\\ x_t=x.\end{cases} \tag{53}$$

We define the value function $V(t, x)$ as:

$$V(t,x)=\operatorname*{Inf}_{u\in U}J(t,x,u) \tag{54}$$

By Bellman's Principle of Optimality,

$$V(t,x)=\inf_{u\in U}E_{tx}\left[\int_t^{t+h}\ell(x_s,u_s)ds+V(t+h,x_{t+h})\right] \tag{55}$$

$$t\leq t+h\leq T_1$$

Now, if we take constant controls v on the interval $[t, t+h]$, we clearly have

$$V(t,x) \le E_{tx}\int_t^{t+h} \ell(x_s,v)ds + E_{tx}V(t+h,x_{t+h}) \tag{56}$$

Now by Dynkin's formula (Ito Differential Rule)

$$E_{tx}\left[V(t+h,x_{t+h}) - V(t,x)\right] = E_{tx}\int_t^{t-h}\left(V_s(s,x_s) + L^v V(s,x_s)ds\right).$$

Dividing both sides by h and taking the limit as $h \to 0$, we obtain, $\forall v$

$$V_t(t,x) + L^v V(t,x) + \ell(x,v) \ge 0. \tag{57}$$

Now, if the class of admissible controls are taken to be Markov in the sense

$$u_s = g(s, x_s) \tag{58}$$

with g Lipschitz say, and

$$u_s^* = g^*(s,x_s)$$

is an optimal Markov control we get

$$V_t(t, x) + L^{g^*} V(t,x) + \ell(x, g^*(t,x)) = 0.$$

Therefore from (57) and (58) we get the fundamental Dynamic Programming equation

$$\min_v\left[V_t(t,x) + L^v V(t,x) + \ell(x,v)\right] = 0, \quad V(T,x) = \psi(x). \tag{60}$$

An optimal Markov control policy g^* is obtained by carrying out the minimization above pointwise. A solution $W(t, x)$ (classical) of the above equation allows one to verify that W is a value function.

Other than the linear quadratic problem discussed in the third section of this article, few explicit solutions of this equation are known. For controlled Markov processes with a finite state space equations (59) reduces to a non-linear system of ordinary differential equations.

Optimal Stochastic Control (Partially Observable Case)

We consider the following partially observable stochastic control problem

$$\begin{cases} dx_t = f(x_t,u_t)dt + G(x_t)d\beta_t \\ dy_t = h(x_t)dt + dw_t \end{cases} \tag{61}$$

and we are required to minimize

$$J(t,x) = E_{tx}\left[\int_t^T L(x_s,u_s)ds + \psi(x_T)\right]. \tag{62}$$

The controls u are required to be suitable functions of the past of y.

The conceptual idea to discuss this problem is similar to that used for the LQG problem. But there are severe technical difficulties which we ignore in this presentation. First we introduce the information state for this problem. For this purpose define the operator (see Equation (36))

$$L^u\varphi = \sum_{i=1}^n f^i(x,u)\frac{\partial^i\varphi}{\partial x^i} + \sum_{i,j} a^{ij}(x)\frac{\partial^2\varphi}{\partial x^i\partial x^j}. \tag{63}$$

Then the information state is given by (see Equation (51))

$$d\, q^u(t,x) = (L^u)^*\, q^u(t,x)\, dt + h(x)\, q^u(t,x),\, d\, y_t. \tag{64}$$

Note that $q^u(t, x)$ is the unnormalized conditional density corresponding to the non-linear filtering problem for (60).

The idea now is to rewrite the cost function given by (61) in terms of the information state $q^u(t,x)$. Formally this can be done and the resulting expression is

$$J(t,x) = E_{tx}\left[\int_t^T \int L(x,u)q^u(t,x)dx + \int \psi(x)q^u(t,x)dx\right]. \tag{65}$$

Equations (64) and (65) constitute the equivalent fully observable stochastic control problem. Note that the problem is essentially infinite-dimensional since the information state is infinite-dimensional. In principle we could write Dynamic Programming conditions for this problem, but other than the linear quadratic gaussian situation and the case of risk-sensitive control where the cost function is an exponential of a quadratic function (cf. Whittle [50], Bensoussan-Van Schuppen [51]), no explicit solution for these problems are known.

The partially observable stochastic control problem was probably first treated by Florentin [18]. There is important work here by Davis and Varaiya [52] and Fleming and Pardoux [53]. For detailed discussions see the research monograph by Borkar [54] and the references cited there.

Applications

The linear quadratic gaussian methodology has found wide applications in aerospace systems. It is also used as a design tool for the design of multi-variable control systems. The principal application of optimal non-linear stochastic control seems to be in the domain of finance. For these applications see the important book of Merton [55].

References

[1] T. Kailath, A View of Three Decades of Linear Filtering Theory, *IEEE Trans. on Inf. Th.*, IT-20, pp. 146-181, no. 2, March 1974.

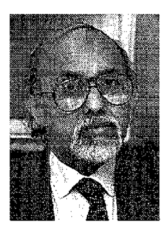
[2] H. W. Bode and C. E. Shannon, A Simplified Derivation of Linear Least Square Smoothing and Prediction Theory, *Proc. IRE*. Vol. 38, pp. 417-425, April 1950.

[3] T. Kailath, An Innovations Approach to Least Squares Estimation — Part I: Linear Filtering in Additive White Noise, *IEEE Trans. on Autonet Control*, Vol. AC-13, pp. 646-655, December 1968.

[4] M.H.A. Davis, *Linear Estimation and Stochastic Control*. Chapman Hall, London. 1977.

[5] E. Wong, *Stochastic Processes in Information and Dynamical Systems*, McGraw Hill, New York, 1971.

[6] D. C. Youla, On the Factorization of Rational Matrices, *IRE Trans. Inf. Th.*, IT-7, pp. 172-189, 1961.
[7] R. E. Kalman, A New Approach to Linear Filtering and Prediction Problems. *ASME Trans., Part D (j. of Basic Engineering)*, 82, pp. 35-45, 1960.
[8] R. E. Kalman and R. S. Bucy. New Results in Linear Filtering and Prediction Theory, *ASME Trans., Part D (J. of Basic Engineering)*, pp. 95-108. 1961.
[9] R. E. Kalman, New Methods of Wiener Filtering Theory, in *Proc. 1st Symposium, Engineering Applications of Random Function Theory and Probability*, J. L. Bogdanoff and F. Kozin, Eds., New York, Wiley, pp. 270-385, 1963.
[10] W. M. Wonham, *Linear Multivariable Control*, Springer-Verlag, New York 1985.
[11] R. E. Bellman. *Dynamic Programming*, Princeton Univ. Press, Princeton, N.J., 1957.
[12] C. Caratheodory, *Variationsrechnung und Partielle Differentialgleichungen Erster Ordnung*, Teubner, Leipzig, 1935.
[13] C. W. Merriam, *Optimization Theory and the Design of Feedback Control Systems*, McGraw Hill, New York, 1964.
[14] R. E. Kalman, Contributions to the Theory of Optimal Control, *Bol. Soc. Mat. Mex.*, Vol. 5, pp. 102-119, 1960.
[15] R. W. Brockett, *Finite Dimensional Linear Systems*, New York, Wiley, 1970.
[16] A. E. Bryson and Y. C. Ho, *Applied Optimal Control*, New York: Hemisphere Publishing Company, 1969 (first edition), 1979 (2nd edition).
[17] J. J. Florentin, Optimal Control of Continuous Time, Markov, Stochastic Systems, *J. of Electronics and Control*, 10, pp. 473-488, 1961.
[18] J. J. Florentin, Partial Observability and Optimal Control, *J. of Electronics and Control*, Vol. 13, pp. 263-279, 1962.
[19] P. D. Joseph and T. T. Tau, On Linear Control Theory, *AIEE Trans.*, 80 (II), pp. 193-196.
[20] H. J. Kushner, Optimal Stochastic Control, *IRE Trans. on Aut. Control*, pp. 120-122, October 1962.
[21] W. M. Wonham, Random Differential Equations in Control Theory, *Probabilistic Methods in Applied Mathematics*, Vol. 2. A. T. Bharucha-Reid, ed., Academic Press, New York, pp. 131-212, 1970.
[22] A. Lindquist, On Feedback Control of Linear Stochastic Systems, *SIAM J. Control*, Vol. 11, no. 2, pp. 323-343, 1973.
[23] M. H. A. Davis and P. P. Varaiya, Information States for Linear Stochastic Systems, manuscript.
[24] H. S. Witsenhausen, Separation of Estimation and Control for Discrete-time Systems, *Proc. IEEE*, 59, pp. 1557-1566, 1971.
[25] C. C. Holt, F. Modiglani, J. F. Muth and H. A. Simon, *Planning Production, Inventories and Work Force*, Prentice Hall, 1960.
[26] S. E. Dreyfus, Some Types of Optimal Control of Stochastic Systems, *SIAM J. Control*, Vol. 2, no. 1, pp. 120-134, 1964.
[27] P. Frost and T. Kailath, An Innovations Approach to Least Squares Estimation — Part III: Nonlinear Estimation in White Gaussian Noise, *IEEE Trans. on Automatic Control*, AC-16, pp. 217-226, June 1971.
[28] M. Fujisaki, G. Kallianpur, H. Kunita. Stochastic Differential Equations for the Nonlinear Filtering Problem, *Osaka J. of Math.*, 9, pp. 19-40, 1972.
[29] D. F. Allinger and S. K. Mitter, New Results on the Innovations Problem of Nonlinear Filtering, *Stochastics*, 4, pp. 339-348, 1981.
[30] H. J. Kushner, On Differential Equations satisfied by Conditional Probability Densities of Markov Processes, *SIAM J. Control*, Vol. 2, pp. 106-119, 1964.
[31] W. M. Wonham, Some Applications of Stochastic Differential Equations to Optimal Nonlinear Filtering, *SIAM J. Control*, 2, pp. 347-369, 1965.
[32] M. Zakai, On the Optimal Filtering of Diffusion Processes, *Z. Wahr. Verw. Geb.*, 11, pp. 230-243, 1969.
[33] T. E. Duncan, Probability Densities for Diffusion Processes with Applications to Nonlinear Filtering Theory and Detection Theory, Ph.D. dissertation, Stanford University, 1967.
[34] R. E. Mortenson, Doctoral Dissertation, Univ. of California, Berkeley, 1967.
[35] S. K. Mitter, On the Analogy Between Mathematical Problems of Nonlinear Filtering and Quantum Physics, *Ricerche di Automatice*, Vol. 10, no. 2, pp. 163-216, 1979.
[36] M. H. A. Davis and S. J. Marcus, An Introduction to Nonlinear Filtering, *Stochastic Systems: The Mathematics of Nonlinear Filtering and Identification and Applications*, eds. M. Hazewinkel and J. C. Willems, Reidel, Dordrecht, 1981.
[37] R. S. Liptser and A. N. Shiryayer, *Statistics of Random Processes I, General Theory*, Springer-Verlag, New York, 1977.
[38] R. L. Stratonovich, Conditional Markov Process Theory, *Theory Prob. Appl. (USSR)*,. Vol. 5, pp.156-178, 1960.
[39] R. W. Brockett, Remarks on Finite Dimensional Nonlinear Estimation, *Analyse des Systemes, Astaristique*, 75-76, 1980.
[40] S. K. Mitter, Filtering Theory and Quantum Fields, *Asterisque*, loc. cit.
[41] V. E. Benes, Exact Finite Dimensional Filters for Certain Diffusions with Nonlinear Drift, *Stochastics*, 5, pp. 65-92, 1981.
[42] M. Hazewinkel, S. I. Marcus, H. J. Sussman, Nonexistence of Exact Finite Dimensional Filters for Conditional Statistics of the Cubic Sensor Problem, *Systems and Control Letters*, 5, pp. 331-340, 1983.
[43] J. M. C. Clark, The Design of Robert Approximations to the Stochastic Differential Equations of Nonlinear Filtering, *Communication Systems and Random Process Theory*, J. Skwirzaynski ed., Sijthoff Noordhoff, 1978.
[44] M. H. A. Davis, On a Multiplicative Transformation Arising in Nonlinear Filtering, Z. Wahrschein-Verw. Geb., 54, pp. 125-139, 1981.
[45] B. Z. Bobrovsky and M. Zakai, A Lower Bound on the Estimation Error for Certain Diffusion Processes, *IEEE Trans. Inf. Th.*, IT-22, pp. 45-52, 1976.
[46] E. Pardoux, Filtrage Non Lineaire at Equations Aux Derivees Partielles Stochastique Associees, *Ecole d'Ete de Probabilites de Saint-Flour XIX*, ed. P. L. Hennequin, Spring Lecture Notes in Mathematics 1464, 1991.
[47] R. A. Howard, *Dynamic Programming and Markov Processes*, Wiley, New York, 1960.
[48] W. H. Fleming, Some Markovian Optimization Problems, *J. Math. and Mech.* 12 (1)m pp. 131-140, 1963.
[49] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, 1975.
[50] P. Whittle, *Risk Sensitive Optimal Control*, Wiley, New York, 1990.
[51] A. Bensoussan and J. H. Van Schuppen, Optimal Control of Partially Observable Stochastic Systems with an Exponential of Integral Performance Index, *SIAM J. on Control and Optimization*, 23 (4), 1985.
[52] M. H. A. Davis and P. P. Varaiya, Dynamic Programming Conditions for Partially Observable Stochastic Systems, *SIAM J. Cont. and Opt.*, 26, pp. 226-261, 1973.
[53] W. H. Fleming and E. Pardoux, Optimal Control for Partially Observed Diffusions, *SIAM J. on Control and Optimization*, 20, pp. 261-85, 1981.
[54] V. Borkar, *Optimal Control of Diffusion Processes*, Longman, 1989.
[55] R. Merton, *Continuous Time Finance*, Oxford: B. Blackwell, Cambridge, MA, 1992.



Sanjoy K. Mitter received his Ph.D. degree from the Imperial College of Science and Technology, University of London, in 1965. He is currently Professor of Electrical Engineering and Co-Director of the Laboratory for Information and Decision Systems at the Massachusetts Institute of Technology. He is also Director of the Center for Intelligent Control Systems, an inter-university (Brown-Harvard-MIT) center for research on the foundations of intelligent systems. Professor Mitter's research has spanned the broad areas of systems, communication, and control. Although his primary contributions have been on the theoretical foundations of the field, he has also contributed to significant engineering applications, notably in the control of interconnected power systems and automatic recognition and classification of electrocardiograms. His current research interests are theory of stochastic dynamical systems, nonlinear filtering, stochastic and adaptive control; mathematical physics and its relationship to system theory; image analysis and computer vision, and structure, function and organization of complex systems. He is a Fellow of the IEEE. In 1988 he was elected to the National Academy of Engineering.