Stochastic Processes that Generate Polygonal and Related Random Fields

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***Abstract*—A reversible, ergodic, Markov process taking values in the space of polygonally segmented images is constructed. The stationary distribution of this process can be made to correspond to a Gibbs-type distribution for polygonal random fields as introduced by Arak and Surgailis and a few variants thereof, such as those arising in Bayesian analysis of random fields. Extensions to generalized polygonal random fields are presented where the segmentation boundaries are not necessarily straight line segments.**

***Index Terms*—Polygonal random fields, generalized polygonal random fields, reversible Markov process, interacting particle system, Monte Carlo simulation of random fields.**

I. Introduction

In a remarkable series of papers, Arak and Surgailis [1]–[3] studied a class of Markov random fields called polygonal random fields (PRF's) whose realizations can be construed as polygonally segmented images. An important aspect of this work is the specification of an interacting particle system on the line with certain birth, death, branching, and annihilation mechanisms, whose trace in the space–time domain gives a realization of the PRF. Since PRF's provide a convenient model for polygonally segmented images, it is important to be able to construct a reversible Markov process taking values in the space of possible PRF realizations such that its sample at any given time gives a PRF realization with the desired statistics. This is needed, e.g., for Bayesian reconstruction of a polygonally segmented image by Monte Carlo methods. Motivated by this, Clifford [4], Clifford and Middleton [5], and Judish [6] proposed schemes for constructing such processes. Their algorithms proceed by modifying at each step the present realization of the PRF on a randomly chosen rectangular subdomain, so as to achieve the desired Gibbs distribution. These algorithms, however, are strewn with many analytic and computational difficulties, discussed at length in [6]. Our aim here is to provide a simpler alternative scheme which explicitly uses the Arak–Surgailis particle dynamics. This scheme also leads to an important generalization to Markov random fields exhibiting polygonal-like segmentations, but with curved (as opposed to straight-line) boundaries. We call these generalized polygonal random fields (GPRF's).

The paper is organized as follows: The next section describes the notation and the Arak–Surgailis framework. The Arak–Surgailis particle system is described next in Section III. Section IV describes our construction of a process taking values in PRF realizations. Section V describes the extension to GPRF's.

II. Preliminaries

Let $T \subset \mathcal{R}^2$ be a bounded, open, convex domain. Parameterize the straight lines in $\mathcal{R}^2$ by $(p, \alpha) \in \mathcal{R} \times [0, \pi)$ where p is the signed length of the perpendicular to line ℓ from the origin and α the angle it makes with the horizontal axis. Let $\mathcal{L}_T$ denote the set of all straight lines in $\mathcal{R}^2$ that intersect T and $\mathcal{L}^0_{T,n}$ the set of n-tuples of distinct elements of $\mathcal{L}_T$. Let J be a prescribed finite set of "colors." Define

$$\Omega_T(\ell)_n = \{\omega \colon T \to J \mid \partial\omega \triangleq$$

the set of points of discontinuity of ω, satisfies:

$$\partial\omega \subset \bigcup_{i=1}^{n} \ell_i \bigcap T \quad \text{where } (\ell)_n \triangleq \{[\ell_1, \cdots, \ell_n] \in \mathcal{L}^0_{T,n}\}.$$

To avoid any ambiguity in the definition of $\omega \in \Omega_T(\ell)_n$ on $\partial\omega$, we further impose the condition

$$\omega(z) = \inf_S \limsup_{\varepsilon \downarrow 0} \{\omega(z') \mid z' \in T \setminus S, \|z' - z\| < \varepsilon\}$$

where the infimum is over all $S \subset T$ of Lebesgue measure zero, with respect to an arbitrary but fixed ordering of J. Let

$$\Omega_T = \bigcup_{n=0}^{\infty} \bigcup_{(\ell)_n \in \mathcal{L}^0_{T,n}} \Omega_T(\ell)_n.$$

This is the space of "polygonally segmented images," topologized as follows: A local base for the topology at

$$\omega \in \bigcup_{(\ell)_n \in \mathcal{L}^0_{T,n}} \Omega_T(\ell)_n$$

is given by sets of the type

$$\left\{\omega' \in \Omega_T \mid \omega' \in \bigcup_{(\ell)_n \in \mathcal{L}^0_{T,n}} \Omega_T(\ell)_n,\ \partial\omega' \in (\partial\omega)^\varepsilon,\right.$$
$$\left. \omega' = \omega \text{ on } ((\partial\omega)^\varepsilon)^c \right\}$$

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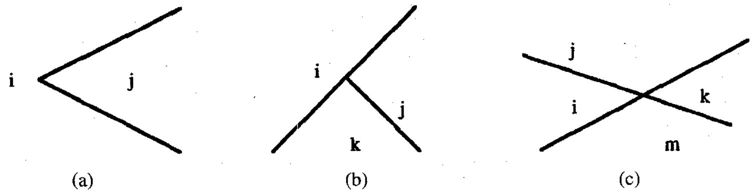


Fig. 1. Nodes of different kind. (a) Corner. (b) T-junction. (c) X-junction.

where we use the notation

$$A^{\varepsilon} \triangleq \{x \in T \mid \inf_{y \in A} \|x - y\| < \varepsilon\}$$

for $A \subset T$, $\varepsilon > 0$. We endow Ω_T with the corresponding Borel σ-field $\mathcal{B}_T$.

Let $\mu = \mu(d\ell)$ be a finite, nonatomic, nonnegative measure on $\mathcal{L}_T$. Define the set of "admissible potentials"

$$\Phi_{T,\mu} = \Bigg\{ F\colon \Omega_T \to \mathcal{R} \cup \{\infty\} \mid Z_{T,F,\mu} \triangleq \sum_{n=0}^{\infty} \int_{\mathcal{L}^0_{T,n}} \frac{1}{n!} \mu(d\ell_1) \cdots \mu(d\ell_n) \times \sum_{\omega \in \Omega_T(\ell)_n} e^{-F(\omega)} < \infty \Bigg\}.$$

A polygonal random field on T corresponding to measure μ and potential $F \in \Phi_{T,\mu}$ is the probability measure $P_T = P_{T,F,\mu}$ on $(\Omega_T, \mathcal{B}_T)$ given by

$$P_T(A) = \sum_{n=0}^{\infty} \int_{\mathcal{L}^0_{T,n}} \frac{1}{n!} \mu(d\ell_1) \cdots \mu(d\ell_n) \times \sum_{\omega \in \Omega_T(\ell)_n \cap A} e^{-F(\omega)}/Z_T, \quad A \in \mathcal{B}_T.$$

Remark 2.1: In [2], Arak and Surgailis give a somewhat more general definition allowing for μ that are not nonatomic. But the specific μ that they use later on in [2] is nonatomic. We shall be using the same choice of μ.

Recall our parametrization of $\ell \in \mathcal{L}_T$. A random sequence of lines ℓ_j, $j \geq 1$, $\ell_j \approx (p_j, \alpha_j)$, is said to be a Poisson line process with intensity $\mu(d\ell)$ if (p_j, α_j), $j \geq 1$, is a Poisson point process on $\mathcal{R} \times [0, \pi)$ with intensity $\mu(dp, d\alpha)$. It is stationary if and only if $\mu(d\ell) = \mu(dp, d\alpha)$ is of the form $dp\nu(d\alpha)$ for a bounded nonnegative measure ν on $[0, \pi)$. Motivated by image processing applications, we shall be interested in stationary isotropic PRF's, i.e., those PRF's whose satistics is invariant under Euclidean motions and reflections. Therefore, we take (as in [2]) $\nu(d\alpha) = d\alpha$.

The next step is to choose $F(\cdot)$. Given $\omega \in \Omega_T$, let a "node" of ω refer to any point in T that belongs to more than one distinct line segment of $\partial\omega$. Fig. 1 describes three kinds of nodes (i, j, k, m stand for colors in J).

Let $n_2(i,j)(\omega), n_3(i;j,k)(\omega), n_4(i,j,k,m)(\omega)$ denote the number of such corners, T-junctions, and X-junctions, respectively.

Define $F(\cdot)$ by

$$\begin{aligned} -F(\omega) = {} & \frac{1}{2} \sum_{i \neq j} n_2(i,j)(\omega) \log b(i,j) \\ & + \frac{1}{2} \sum_{i \neq k \neq j} n_3(i;j,k)(\omega) \log c(i;j,k) \\ & + \sum_{i,j,k,m} c'(i,j,k,m) n_4(i,j,k,m)(\omega) \log d(i,j,k,m) \\ & + \frac{1}{2} \sum_{i,j} \sum_{[\ell] \in \partial\omega(i,j)} (\log a(i,j) - \lambda e(i,j) L([\ell])) \\ & - \sum_i f(i) |T_i(\omega)| \end{aligned} \tag{1}$$

where "log" denotes the natural logarithm (with $\log 0 \triangleq -\infty$) and

i) $\lambda = \int_0^{\pi} |\sin\beta| d\beta$,
ii) $c'(i,j,k,m) = \frac{1}{8}$ if i,j,k,m are distinct, $= \frac{1}{4}$ if $i \neq k$, $j = m$ or $i = k$, $j \neq m$, $= \frac{1}{2}$ if $i = k, j = m$,
iii) $T_i(\omega) = \{z \in T \mid w(z) = i\} \setminus \partial\omega$, $|T_i(\omega)|$ its area,
iv) $a(i,j)$, $b(i,j)$, $c(i,j,k)$, $c'(i,j,k,m)$, $d(i,j,k,m)$, $e(i,j)$ are nonnegative weights satisfying [2, conditions (5.5)–(5.8), (5.12)–(5.18)], recalled in the Appendix. These conditions involve a symmetric transition matrix $[[p_{ij}]]_{i,j \in J}$, $p_{ij} = p_{ji}$, on J.
v) $[\ell]$ denotes a line segment belonging to line ℓ and $\partial\omega(i,j)$ the set of all (i,j)-segments, i.e., line segments in $\partial\omega$ that separate colors i and j in ω. $L(\cdots)$ denotes "the length of…."
vi) For $\omega \in \Omega_T(\ell)_n$, $\ell = [\ell_1, \cdots, \ell_n]$, the set $\partial\omega \bigcap \ell_i$, when nonempty, is a single line segment for each i.

We set $F(\omega) = -\infty$ if $\partial\omega$ contains a node of any type other than those described in Fig. 1. This is not a serious restriction because other kinds of nodes (such as more than two line segments meeting or crossing at a point) are structurally unstable, i.e., become qualitatively different under arbitrarily small perturbations.

For $S \subset T$ open, let $\pi_S(\omega) \in \Omega_S$ for $\omega \in \Omega_T$ denote the restriction of ω to S and let $\mathcal{B}_S$ denote the sub-σ-field of $\mathcal{B}$ generated by the map $\pi_S\colon \Omega_T \to \Omega_S$. A measurable map $X\colon \Omega_T \to R \cup \{\infty\}$ is said to be additive if, whenever $T = S \bigcup V$, S, V open, $X = X_S + X_V$ for some X_S, $X_V\colon \Omega_T \to R \bigcup \{\infty\}$ which are, respectively, $\mathcal{B}_S$, $\mathcal{B}_V$-measurable. (This decomposition need not be unique.) With this definition, the potential $F(\cdot)$ above is seen to be additive.

The polygonal random field P_T is said to be Markov if for S, V as above and any $A \subset \mathcal{B}_S$, $P_T(A/\mathcal{B}_V) = P_T(A/\mathcal{B}_{V \cap S})$. Let $\mathcal{G}_0$ denote the set of bounded convex open sets in $\mathcal{R}^2$.

Theorem 2.1 [2, Theorem 5.1]: For the above choice of μ and F as in (2.1), the probability measures P_T, $T \in \mathcal{G}_0$, define a consistent family of isotropic Markov PRF's.

The next result characterizes the conditional distribution under P_T.

Theorem 2.2 [2, Lemma 8.3]: For $A \in \mathcal{B}$, $U \subset T$ open

$$P_T(A/\mathcal{B}_U)(\omega) = Z_{T \setminus U}(A/\pi_U(\omega))/Z_{T \setminus U}(\pi(\omega)).$$

Here for $\xi \in \Omega_U$

$$Z_{T \setminus U}(A/\xi) = \sum_{m=0}^{\infty} \int_{\mathcal{L}^0_{T \setminus U, m}} \frac{1}{m!} d^m \mu(\ell) \Sigma^\xi I_A(\omega) e^{-F(\omega)}$$

where the sum Σ^ξ is over all $\omega \in \Omega_T$ satisfying $\pi_U(\omega) = \xi$ and $\omega \in \Omega_T((\ell)_m \cup \tilde{L}(\xi))$, $\tilde{L}(\xi)$ being the set of lines that constitute ξ.

$$Z_{T \setminus U}(\pi_U(\omega)) = Z_{T \setminus U}(\Omega_T/\pi_U(\omega))$$

is the normalizing factor.

The proof of [2, Theorem 2.1] uses the realization of these PRF's via an interacting particle system with prescribed dynamics. We describe this in the next section. In conclusion, we mention that the definition of isotropy in [2] does not include reflection symmetry, but this can be easily incorporated without altering the proof of Theorem 2.1.

III. Dynamics of the Particle System

We start with further notation and definitions from [2]. We consider a T which is a bounded convex polygon. Without any loss of generality, suppose that

$$T \subset [0,1]^2 \triangleq \{(t,y) \mid 0 \le t, y \le 1\}$$

with no side parallel to the y-axis. This can always be achieved by redefining the axes and scaling—see, e.g., Fig. 2. In particular, $\bar{T}$ has one point each on the lines $(0,y)$ and $(1,y)$. Let $\partial T = \partial T^+ \cup \partial T^-$ where $\partial T^\pm = \{(t, y_t^\pm),\ 0 \le t \le 1\}$ are the upper and lower parts of $\partial T =$ the boundary of T, so that $T = \{(t,y) \mid 0 < t < 1,\ y_t^- < y < y_t^+\}$. We interpret the t-axis as the time axis. By a particle we mean a quadruple $z = (y, v, i, j)$ where $y \in [0,1]$ is its position, $v \in \mathcal{R}$ its velocity, and $i, j \in J$, $i \ne j$, are the "environments" above and below the particle, respectively. Call such a particle an (i,j)-particle. A system of particles is a finite collection

$$z = (z_1, \cdots, z_n), \quad z_r = (y_r, v_r, k_r^+, k_r^-)$$

of particles such that $(y_r, v_r) \ne (y_s, v_s)$ for $s \ne r$. The system is said to be ordered if for $1 \le r < n$, either $y_r < y_{r+1}$ or $y_r = y_{r+1}$, $v_r < v_{r+1}$. Any system of n particles can be ordered by a permutation of its indices. An ordered system is said to be consistent if $k_r^+ = k_{r+1}^-$, $1 \le r < n$. Let $X^{(n)}$, $n \ge 1$, denote the set of ordered consistent systems of n particles and for $t \in [0,\ 1]$, $X_t^{(n)} \subset X^{(n)}$ its subset consisting of these systems for which for $1 \le r \le n$, either

$$y_t^- < y_r < y_t^+$$

or

$$y_r = y_t^-, \quad v_r > v_t^-$$

or

$$y_r = y_t^+, \quad v_r < v_t^+$$

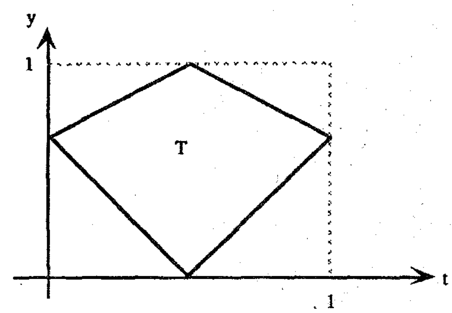


Fig. 2. Redefinition of axes.

holds, where

$$v_t^\pm = \lim_{s \downarrow t} \frac{dy_s^\pm}{ds}$$

is the tangent to $\partial T^\pm$ at t. Set $X^{(0)} = X_t^{(0)} = J$ and

$$X = \bigcup_{n=0}^{\infty} X^{(n)}$$

$$X_t = \bigcup_{n=0}^{\infty} X_t^{(n)}.$$

For any $x = [z_1, \cdots, z_n] \in X_t^{(n)}$, $n \ge 1$, define its environment as the right-continuous function $w(\cdot, x) : (y_t^-, y_t^+) \to J$ such that

$$w(y,x) = \begin{cases} k_r^+ = k_{r+1}^-, & \text{if } y \in (y_r, y_{r+1}), 1 \le r \le n \\ k_1^-, & \text{if } y \in (y_t^-, y_1) \\ k_n^+, & \text{if } y \in (y_n, y_t^+) \end{cases}$$

for $y \ne y_1, \cdots, y_n$. For $x = k \in X_t^{(0)}$, set $w(y,x) = k$, $y \in (y_t^-, y_t^+)$. The evolution of the particle system as a Markov process taking values in X_t at time t is described by i)–x) below:

i) The initial distribution of $x(t)$ at $t = 0$ is concentrated on $X_0^{(0)} = J$ with $P(x(0) = j) = 1/|J|$. Let $x(t) = x \in X_t^{(n)}$ be the value of $x(t)$ at time $t \in [0,1)$. In a small time interval $(t, t+\Delta t) \subset [0,1]$, the following changes can occur.

ii) With probability $p_{ij} q(v_t^+, du)\Delta t + o(\Delta t)$, a new particle (y, v, i, j) is born at ∂T^+ with $j = k_n^+$, $v \in du$, $v < v_t^+$, where

$$q(v, du) = |u - v| du dt/(1+u^2)^{3\setminus 2}.$$

iii) With probability $p_{ij} q(v_t^-, du)\Delta t + o(\Delta t)$ a new particle (y, v, i, j) is born at ∂T^- with $i = k_1^-$, $v \in du$, $v > v_t^-$.

iv) With probability $p_{ij}^2 b(i,j)|u' - u''| V(du')\, V(du'') dy \Delta t + o(\Delta t)$ two new particles (y, v', i, j) and (y, v'', i, j) are born with $y \in dy \subset (y_t^-, y_t^+)$, $i = w(y,x)$, $v' \in du'$, $v'' \in du''$, $v' > v''$

$$V(du) = |\{\alpha \in (0,\pi) \mid \cot(\alpha) \in du\}|/\sqrt{1+u^2}.$$

v) With probability $p_{ij} b(i,j) q(v, du)\Delta t + o(\Delta t)$, one of the particles z_r, $1 \le r \le n$, $z_r = (y, v, i, j)$, turns into the particle (y, v', i, j) with $v' \in du$.

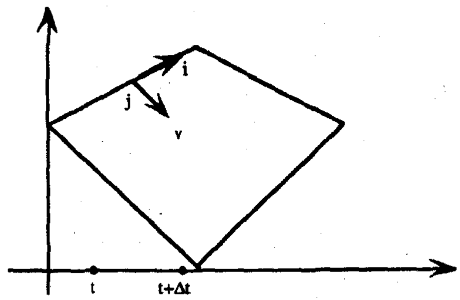


Fig. 3. Birth of particles.

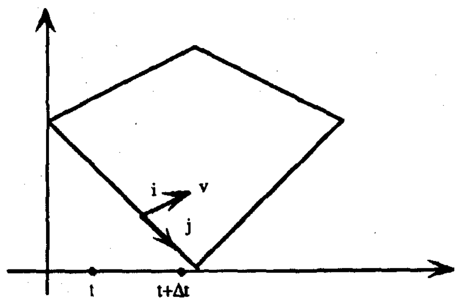


Fig. 4. Birth of particles.

vi) With probability $p_{ij}p_{kj}cq(v, du)\Delta t + o(\Delta t)$, one of the particles $z_r, 1 \leq r \leq n$, $z_r = (y, v, i, j)$, turns into either two new particles (y, v, i, k), (y, v', k, j) with $v' \in du$, $v' < v$, $c = c(i; j, k)$ or into two new particles (y, v, k, j), (y, v', i, k) with $v' \in du$, $v' > v$, $c = c(j; i, k)$.

vii) With probability

$$1 - \{q(v, \{u > v\}) + q(v, \{u < v\}) + \int_{y_t^-}^{y_t^+} f(w(y, x))dy + \sum_{r=1}^{n}(q(v_r, \{u < v_r\})e(k_r^+, k_r^-) + q(v_r, \{u > v_r\})e(k_r^-, k_r^+))\}\Delta t + o(\Delta t)$$

none of the changes in ii)–vi) occur and

$$x(t + \Delta t) = [z_1', \cdots, z_n'] \in X_{t+\Delta t}^{(n)}$$

where

$$z_r' = (x_r + v_r\Delta t, v_r, k_r^+, k_r^-),\ 1 \leq r \leq n$$

In ii)–vii) above, Δt is assumed to be so small that the particles do not hit ∂T or collide. If $z_r = (y, v, i, j)$ and $z_{r+1} = (y, u, j, k)$ collide at (t, y) with $u > v$, then:

viii) If $i = k$, with probability $b(i, j)$ both die, or, with probability $d(i, j, i, m)p_{im}^2$, they turn into two new particles (y, v, m, i), (y, u, i, m) for $i \neq m \in J$.

ix) If $i \neq k$, then:

ixa) With probability $c(k; i, j)p_{ik}$, they merge into a single particle (y, u, i, k),

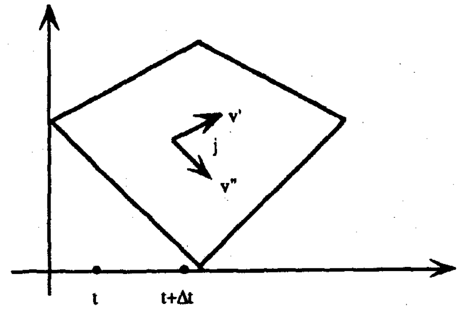


Fig. 5. Birth of particles.

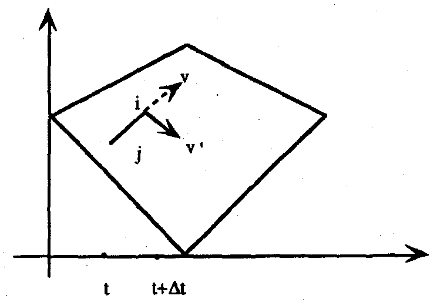


Fig. 6. Sequence of events after birth of particles.

ixb) With probability $c(i; j, k)p_{ik}$, they merge into a single particle (y, v, i, k),

ixc) With probability $d(i, j, k, m)p_{im}p_{km}$ they turn into two particles (y, u, i, m), (y, v, m, k), $m \neq i, k, m \in J$.

x) If one of the particles (say, z_n) reaches ∂T at time t, it dies and the process $x(\cdot)$ jumps from $x(t-) = x$ to $x(t) = x' = [z_1, \cdots, z_{n-1}] \in X_t^{(n-1)}$.

Figs. 3–9 illustrate the events ii)–vi), viii), and ix), respectively.

There exists a Markov process $x(\cdot)$ evolving as per i)–x) above and to each trajectory $x(\cdot)$ thereof there corresponds a unique polygonally segmented image given by

$$w(t, y) = w_{x(\cdot)}(t, y) = w(y, x(t)), \quad (t, y) \in T \setminus \partial\omega.$$

Let Q_T be the probability measure induced by this random element of Ω_T on $(\Omega_T, \mathcal{B}_T)$.

Theorem 3.1 [2, Lemma 6.1]: $Q_T = P_{T,F,\mu}$.

IV. Process of Polygonal Random Fields

Our aim is to construct an Ω_T-valued reversible ergodic process such that at each time t it yields a PRF with a prescribed additive potential H satisfying

$$\{\omega \mid F(\omega) = \infty\} \subset \{\omega \mid H(\omega) = \infty\}.$$

We shall consider the specific case of T = a rectangle. The case $H = F$ is the simplest and we consider it first. In accordance with Fig. 2, draw T as shown in Fig. 10. We have marked its corners as a, b, c, d, while e, f are midpoints of ad, bc, respectively. The unit vector α is directed along the

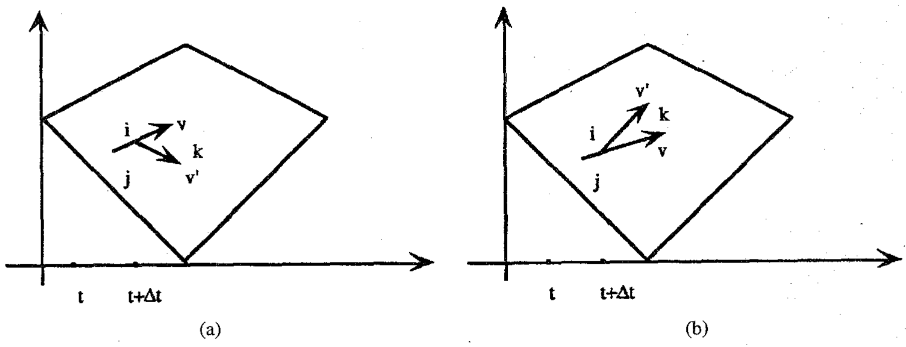


Fig. 7. Sequence of events after birth of particles. (a) $v < v'$. (b) $v' > v$.

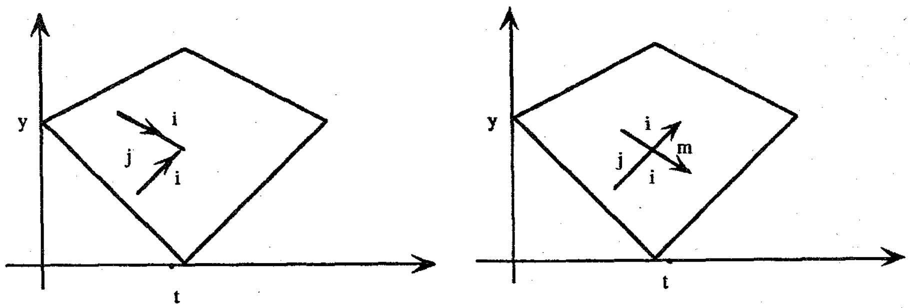


Fig. 8. Diagrammatic description of events.

perpendicular from the origin to ab and θ is the angle it makes with the positive t-axis. Let $\hat{T}$ (respectively, $\hat{\hat{T}}$) denote the one-sided (respectively, two-sided) infinite "cylinder" obtained from T by dropping cd (respectively, ab and cd) and extending ad, bc indefinitely. (See Fig. 11.) Construct a system of particles evolving as in Section II on $\hat{T}$, except that it is now allowed to go on indefinitely, i.e., $x(t)$ is now defined for $t \in [0, \infty)$. Define a rectangle-valued process $T(t), t \in \mathcal{R}$ by $T(0) = T$, $T(t) = T + \alpha t$. Define an Ω_T-valued process ξ_t, $t \geq 0$, by

$$\xi_t(s, y) = \omega(s + t\cos\theta, y + t\sin\theta), \quad (s, y) \in T,\ t \geq 0.$$

Call an Ω_T-valued process $\{\Gamma_t, t \geq 0\}$ R-reversible if for any $t_0 > 0$, $\{\Gamma_t, t \in [0, t_0]\}$ and $\{R(\Gamma_{t_0 - t}), t \in [0, t_0]\}$ agree in law, where R: $\Omega_T \rightarrow \Omega_T$ is the map that maps $\omega \in \Omega_T$ to its reflection across the line ef in Fig. 2.

Lemma 4.1: $\xi_t, t \geq 0$, is a stationary R-reversible Markov process.

This is immediate from the isotropy and Markov property of P_T. In particular, R-reversibility allows us to symmetrically define ξ_t for $t \leq 0$. Thus we consider $x(t)$ and ξ_t as being defined for $t \in \mathcal{R}$.

Theorem 4.1: $\xi_t, t \in \mathcal{R}$ is ergodic.

Proof: Let $t > 0$ be such that $\overline{T(t)} \cap \overline{T(0)} = \phi$. Let $C =$ the convex hull of $T(0)$ and $T(t)$ and let $\omega', \omega'' \in \Omega_T$. For any $\bar{\omega} \in \Omega_{T(t)}$, we can always introduce an appropriate number of births, deaths, branching, etc., in $C \setminus (\overline{T(0) \bigcup T(t)})$ to construct a valid trajectory of $x(\cdot)$ that restricts to ω' (respectively, ω'') on $\Omega_{T(0)}$ and to $\bar{\omega}$ on $\Omega_{T(t)}$. Let φ_t: $\Omega_{T(0)} \rightarrow \Omega_{T(t)}$ denote the map

$$(s, y) \rightarrow (s + t\cos\theta, s + t\sin\theta)$$

and let

$$A' = \{\omega \in \Omega_C | \omega|_{\Omega_{T(t)}} \in \varphi_t(A)\}, \quad A \in B.$$

Then

$$P(\xi_t \in A/\xi_0 = \omega') = Z_{C \setminus T(0)}(A'/\pi_{T(0)}(\omega'))/Z_{C \setminus T(0)}(\pi_{T(0)}(\omega'))$$

in the notation of Theorem 2.2, with a similar expression for $P(\xi_t \in A/\xi_0 = \omega'')$. From the explicit expressions for the right hand side derived from Theorem 2.2, it follows that the probability measures $P(\xi_t \in d\omega/\xi_0 = \omega')$, $P(\xi_t \in d\omega/\xi_0 = \omega'')$ are mutually absolutely continuous. Thus if $\{\xi_t\}$ has two invariant probability measures, they must be mutually absolutely continuous. Since distinct ergodic measures must be mutually singular, the claim follows. □

The next two lemmas establish some additional properties of $\{\xi_t\}$. Let T_1, T_2, T_3 denote the open rectangles $abf''e''$, $e'f'cd$, $e''f''cd$, respectively, in Fig. 10. For $S, U \subset T$ open, say that $w_1 \in \Omega_S$, $\omega_2 \in \Omega_U$ are compatible if they are the restrictions to S, U respectively of some $\omega \in \Omega_T$. For $\omega_1 \in \Omega_T$, the trace of ω_1 on $e''f''$, denoted by $\mathrm{tr}\,(\omega_1)$, is an alternating sequence of colors, points of $e''f''$, and scalars, say $i_1, x_1, v_1, i_2, x_2, v_2, \cdots, i_n, x_n, v_n, i_{n+1}$, with the following interpretation: Under ω_1, if we move from e'' to f'' along $e''f''$ looking at the immediate neighborhood in T_1, we first encounter a patch of color i_1, till at x_1 a trajectory from ω_1

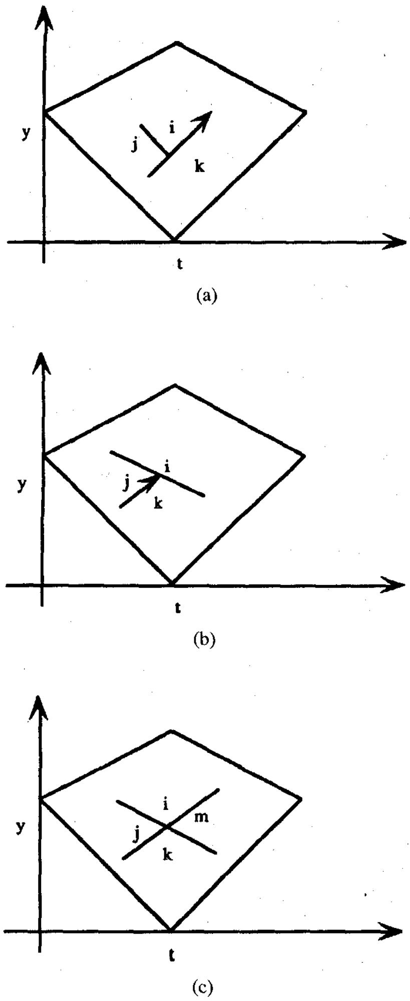


Fig. 9. Diagrammatic description of events.

hits $e''f''$ with velocity v_1. This is followed by a patch of color i_2 and so on. Clearly, $i_k \neq i_{k+1}$, $1 \leq k \leq n$. (Situations such as $x_1 = e''$ are also possible and can be handled analogously.) Let ω_1, $\omega_2 \in \Omega_{T_1}$ with

$$\operatorname{tr}(\omega_k) = (i_1^k, x_1^k, v_1^k, \cdots, i_{m_k+1}^k), \quad k = 1, 2.$$

Let $d(\operatorname{tr}(\omega_1), \operatorname{tr}(\omega_2)) = \infty$ if either $m_1 \neq m_2$ or $m_1 = m_2$ but $i_k^1 \neq i_k^2$ for some k, and $= \max_i(|x_i^1 - x_i^2|, |v_i^1 - v_i^2|)$ otherwise. It is clear that if $\omega_n \to \omega$ in Ω_{T_1}, $\operatorname{tr}(\omega_n) \to \operatorname{tr}(\omega)$ w.r.t. the metric 'd'.

Recall our definition of nodes. We call these interior nodes to distinguish them from boundary nodes which are points on the boundary where a particle is born or dies. For $\omega \in \Omega_T$, let separation of ω, denoted by $\operatorname{sep}(\omega)$, be the minimum of the distances between any two nodes of either variety, between a node and any line segment in $\partial\omega \cup \partial T$ that does not contain it, the angles between any line segment in $\partial\omega$ and the y-axis, or the angles beween any two distinct line segments in $\partial\omega \cup \partial T$ that meet at a point. Let $N(\omega) =$ the number of distinct line segments in ω.

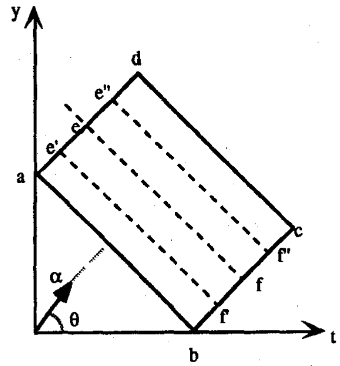


Fig. 10. Redrawing of domain T in accordance with Fig. 2.

Lemma 4.2: $\{\xi_t\}$ is a Feller process.

Before proving this result, we first reduce it to another equivalent claim. Note that it suffices to show that for $f \in C_b(\Omega_{T_2})$

$$\int f(\pi_{T_2}(\omega))dP_T(d\omega/\pi_{T_1}(\omega) = \omega')$$

depends continuously on ω'. By Theorem 2.2, this equals

$$\left(\sum_{m=0}^{\infty} \int_{\mathcal{L}^0_{T \setminus T_1, m}} \frac{1}{m!} d^m \mu(\ell) \Sigma_{\omega'} f(\pi_{T_2}(\omega)) e^{-F(\omega)}\right) \Big/ \left(\sum_{m=0}^{\infty} \int_{\mathcal{L}^0_{T \setminus T_1, m}} \frac{1}{m!} d^m \mu(\ell) \Sigma_{\omega'} e^{-F(\omega)}\right)$$

when $\Sigma_{\omega'}$ denotes the summation over ω in $\Omega_T((\ell)_n \cup \tilde{L}(\omega'))$ compatible with ω'. By the additivity of F, this is seen to equal

$$\left(\sum_{m=0}^{\infty} \int_{\mathcal{L}^0_{T \setminus T_1, m}} \frac{1}{m!} d^m \mu(\ell) \Sigma' f(\pi_{T_2}(\omega)) e^{-F(\pi_{T_2}(\omega))}\right) \Big/ \left(\sum_{m=0}^{\infty} \int_{\mathcal{L}^0_{T \setminus T_1, m}} \frac{1}{m!} d^m \mu(\ell) \Sigma' e^{-F(\pi_{T_2}(\omega))}\right) \quad (\dagger)$$

where Σ' denotes summation over $\pi_{T_2}(\omega)$ in $\Omega_{T_2}((\ell)_n)$ compatible with ω'. Then it suffices to prove that the last expression above depends continuously on ω'.

Proof: (Sketch) Let $\eta, \epsilon, \epsilon', \delta > 0, N \geq 1, \tilde{\omega} \in \Omega_{T_1}$ and

$$D = \{\omega \in \Omega_{T_3} \mid \operatorname{sep}(\omega) \geq \delta, N(\omega) \leq N, \tilde{\omega}, \omega \text{ are compatible}\}.$$

It is easy to see that D is relatively sequentially compact in our topology on Ω_{T_3}. Keep $\tilde{\omega}$ fixed henceforth and let $\bar{\omega} \in \Omega_{T_1}$ be such that $d(\operatorname{tr}(\tilde{\omega}), \operatorname{tr}(\bar{\omega})) < \epsilon$. Pick $\delta > 0$ small enough and $N \geq 1$ large enough such that

$$P(\pi_{T_2}(\omega) \in D / \operatorname{tr}(\pi_{T_1}(\omega)) = \operatorname{tr}(\tilde{\omega})) > 1 - \eta.$$

Given $\omega' \in D$, construct $\omega'' \in \Omega_{T_3}$ compatible with $\bar{\omega}$ as follows: Let

$$\operatorname{tr}(\tilde{\omega}) = (i_1, x_1, r_1, \cdots, i_{n+1})$$
$$\operatorname{tr}(\bar{\omega}) = (i_1, \bar{x}_1, \bar{v}_1, \cdots, i_{n+1}).$$

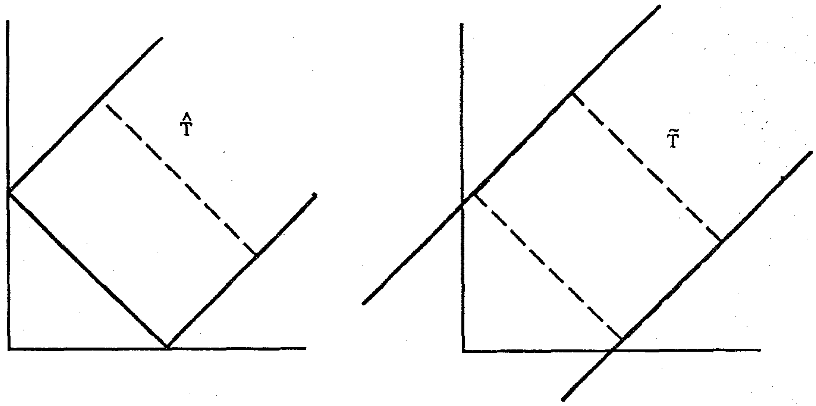


Fig. 11. Redrawing of domain T in accordance with Fig. 2.

For each k, $1 \leq k \leq n$, start a particle at $\bar{x}_k$ with velocity $\bar{v}_k$ and environment (i_k, i_{k+1}).

Let births of the type depicted in Figs. 3–5 take place for ω'' in exactly the same manner as for ω'. The particle in ω'' that started at $(\bar{x}_i, \bar{v}_i)$ undergoes the same sequence of events of the type depicted in Figs. 6 and 7 as the particle in ω' that started at (x_i, v_i), with exactly the same times of occurence and same angles at the nodes generated thereby. The same also holds for the corresponding pairs of newly born particles (*a la* Figs. 3–5) in ω' and ω''. Furthermore, events of the type depicted in Figs. 8 and 9 are in one-to-one correspondence in ω', ω'' and occur in the same order. Finally, ω'' satisfies: if $\bar{\omega}'$ (respectively, $\bar{\omega}''$) denotes $\tilde{\omega} \cup \omega'|_{T_2}$ (respectively, $\bar{\omega} \cup \omega''|_{T_2}$), then

$$|\exp(-F(\omega')) - \exp(-F(\omega''))| < \epsilon', |f(\bar{\omega}') - f(\bar{\omega}'')| < \epsilon'. \quad (*)$$

Of course, all this may not be possible, but for prescribed ϵ', δ and N, it is possible for $\bar{\omega}$ in a sufficiently small neighborhood of $\tilde{\omega}$. Let h denote the map $\omega' \to \omega''$ and let $D' = h(D)$. Then $h: D \to D'$ is seen to be a continuous bijection. Now $(*)$ and $(\dagger)$ together lead to

$$|P_T(\pi_{T_3}(\omega) \in D'/\pi_{T_1}(\omega) = \bar{\omega}) - P_T(\pi_{T_3}(\omega) \in D/\pi_{T_1}(\omega) = \tilde{\omega})| < \eta/2$$

for sufficiently small ϵ' and (correspondingly small) ϵ. Hence

$$P_T(\pi_{T_3}(\omega) \in D'/\pi_{T_3}(\bar{\omega}) = \bar{\omega}) \geq 1 - \eta/2.$$

Using $(*)$, $(\dagger)$ once more, we have, for $\bar{\omega}$ in a sufficiently small neighborhood of $\tilde{\omega}$

$$\begin{aligned} &\left| \int f(\pi_{T_2}(\omega)) P_T(d\omega/\pi_{T_1}(\omega) = \bar{\omega}) - \int f(\pi_{T_2}(\omega)) P_T(d\omega/\pi_{T_1}(\omega) = \tilde{\omega}) \right| \\ &\leq 2\eta K + \left| \int_D f(\pi_{T_2}(\omega)) P_T(d\omega/\pi_{T_1}(\omega) = \tilde{\omega}) - \int_{D'} f(\pi_{T_2}(\omega)) P_T(d\omega/\pi_{T_1}(\omega) = \bar{\omega}) \right| < 3\eta K \end{aligned}$$

where $K > 0$ is a bound on $|f(\cdot)|$. The claim follows. $\square$

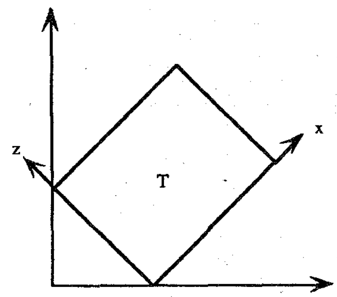


Fig. 12. Parameterization of T.

Pick $t > 0$ large enough so that $\overline{T(0)} \cap \overline{T(t)} = \phi$. Let $\varphi_t: \Omega_{T(0)} \to \Omega_{T(t)}$ denote the map

$$(s, y) \to (s + t\cos\theta, y + t\sin\theta)$$

as before.

Lemma 4.3: For any open set $O \subset \Omega_T$ and $\omega \in \Omega_T$

$$P(\xi_t \in O/\xi_0 = \omega) > 0.$$

Proof: It suffices to consider $O =$ an open neighborhood of $\bar{\omega} \in \Omega_T$. Let $C =$ the convex hull of $T(0)$ and $T(t)$. By introducing an appropriate number of births, deaths, branching, etc., in $C \setminus (\overline{T(0) \cup T(t)})$, we can always construct a valid trajectory η of $\{x(\cdot)\}$ that restricts to ω on $T(0)$ and $\bar{\omega} \circ \varphi_t^{-1}$ on $T(t)$. Then from the particle dynamics described in the preceding section, it is clear that for any open set $A \subset \Omega_C$ containing η

$$P_C(\pi_{T(t)}(\tilde{\omega}) \in A/\pi_{T(0)}(\tilde{\omega}) = \omega) > 0.$$

Pick A such that $\pi_{T(t)}(A) \subset$ the image of O under φ_t, to conclude. $\square$

Thus we have an Ω_T-valued R-reversible ergodic process $\{\xi_t\}$ with invariant distribution P_T. One may also consider its discrete time version, i.e., a process $\xi_n^\Delta = \xi_{\Delta n}$, $n = 0, \pm 1, \cdots$, for some $\Delta > 0$. This will be a discrete-time R-reversible ergodic process with invariant measure P_T.

We shall now use $\{\xi_t\}$ as a basis for constructing a discrete-time reversible ergodic process $\{Z_n\}$ such that for fixed n,

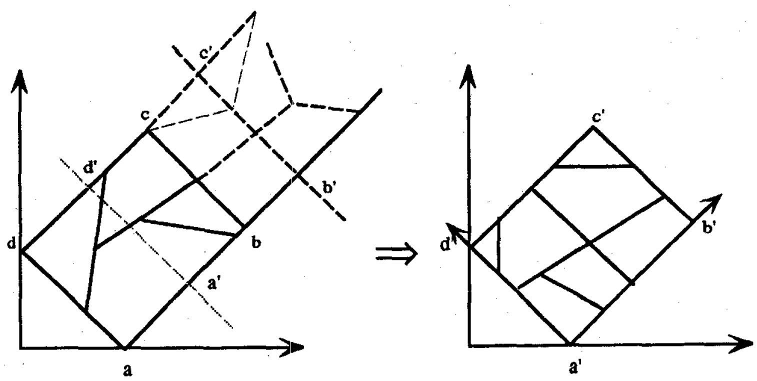


Fig. 13. Step in describing state transition for process ξ_t.

Z_n is a PRF with potential H. For this purpose, introduce the following convention: Parametrize T as

$$T = \{(x, z) \mid 0 < x < b_1,\ 0 < z < b_2\}$$

where b_1, b_2 are the lengths of the sides of T (see Fig. 12). Let

$$T_p = \{(x, z) \mid 0 < x < b_1/2,\ 0 < z < b_2\}$$

and

$$T_s = \{(x, z) \mid b_1/2 < x < b_1,\ 0 < z < b_2\}.$$

Given $\omega \in \Omega_T$ define $\omega_p \in \Omega_{T_p}$ and $\omega_s \in \Omega_{T_s}$ as the restrictions of ω to T_p, T_s, respectively, which we refer to as the prefix and the suffix of ω. Given $\omega, \omega' \in \Omega_T$, we say that ω, ω' are neighbors if and only if either $\omega'_p = \omega_s$ or $\omega'_s = \omega_p$. This is clearly a symmetric relation. Let

$$N(\omega) \triangleq \{\text{ neighbors of } \omega\} \subset \Omega_T.$$

Suppose $Z_n = \omega$ for some n. The state transition at time n is effected as follows: First pick one element from the set $\{p, s\}$ with equal probability. Suppose you get s. Let an independent copy of the process $x(\cdot)$ evolve conditioned on $x(\cdot)|_T = \omega$. Let $\tilde{\omega} = x(\cdot)$ restricted to $T(b_1t/2)$. Thus $\tilde{\omega} \in \Omega_{T(b_1t/2)}$. Let $\omega' = \varphi_s^{-1}(\tilde{\omega}) \in \Omega_T$, where $s = b_1t/2$. Set $Z_{n+1} = \omega'$ with probability $\exp[-(G(\omega') - G(\omega))^+)$ and $= \omega$ with probability $1 - \exp(-(G(\omega') - G(\omega))^+)$ where $G = H - F$. Note that $\omega'_p = \omega_s$ and thus $\omega' \in N(\omega)$ (Fig. 13). If one picks p instead of s in the first step, the procedure is similar except that one evolves $x(\cdot)$ in reversed time, leading to $\omega'_s = \omega_p$. Then $\{Z_n\}$ is an Ω_T-valued Markov process whose transition probability is given by

$$\begin{aligned}
&P(Z_{n+1} \in [\tilde{\omega}, \tilde{\omega} + d\tilde{\omega}]/Z_n = \bar{\omega}) \\
&= \frac{1}{2}(P_T(\pi_{T_s}(\omega) \in [\tilde{\omega}, \tilde{\omega} + d\tilde{\omega}]/\pi_{T_p}(\omega) \\
&= \pi_{T_p}(\bar{\omega})) + P_T(\pi_{T_p}(\omega) \in [\tilde{\omega}, \tilde{\omega} + d\tilde{\omega}]/\pi_{T_s}(\omega) \\
&= \pi_{T_s}(\bar{\omega}))) \exp(-(G(\tilde{\omega}) - G(\bar{\omega}))^+) d\tilde{\omega}
\end{aligned}$$

for $\tilde{\omega} \neq \bar{\omega}$ and

$$P(Z_{n+1} = \bar{\omega}/Z_n = \bar{\omega}) = 1 - P(Z_{n+1} \neq \bar{\omega}/z_n = \bar{\omega})$$

where the rightmost quantity is obtained by integrating the right-hand side of the preceding equation over $\{\tilde{\omega} \mid \tilde{\omega} \neq \bar{\omega}\}$.

Theorem 4.2: $\{Z_n\}$ is a reversible ergodic process with invariant measure $P_{T,H,\mu}$.

Proof: Let $\xi_n = \xi_n^\Delta$, $n = 0, \pm 1, \pm 2, \cdots$, for $\Delta = (b_1 \cos\theta)/2$. Let $\nu^+(\omega, d\omega'), \nu^-(\omega, d\omega'), \tilde{\nu}(\omega, d\omega')$ denote the transition probability measures for $\{\xi_n\}$, $\{\xi_{-n}\}$, $\{Z_n\}$, respectively. Then

$$\begin{aligned}
\tilde{\nu}(\omega, d\omega_1) = &\frac{1}{2} e^{-(G(\omega_1) - G(\omega))^+} (\nu^+(\omega, d\omega_1) + \nu^-(\omega, d\omega_1)) \\
&+ g(\omega)\delta_\omega(d\omega_1)
\end{aligned}$$

where $\delta_\omega(\cdot)$ is the Dirac measure at ω and

$$\begin{aligned}
1 - g(\omega) = &\frac{1}{2} \int_{\Omega_T} e^{-(G(\omega_1) - G(\omega))^+} \\
&\cdot (\nu^+(\omega, d\omega_1) + \nu^-(\omega, d\omega_1)).
\end{aligned}$$

Let $\eta = P_{T,F,\mu} (= P_T)$ and $\tilde{\eta} = P_{T,H,\mu}$. Then

$$\tilde{\eta}(d\omega) = e^{-G(\omega)} \eta(d\omega).$$

We need to show that

$$\tilde{\eta}(d\omega)\tilde{\nu}(\omega, d\omega') = \tilde{\eta}(d\omega')\tilde{\nu}(\omega', d\omega).$$

The left-hand and the right-hand sides, respectively, equal

$$\begin{aligned}
&\frac{1}{2}\eta(d\omega)e^{-G(\omega)}e^{-(G(\omega') - G(\omega))^+}(\nu^+(\omega, d\omega') + \nu^-(\omega, d\omega')) \\
&\qquad + \eta(d\omega)e^{-G(\omega)}g(\omega)\delta_\omega(d\omega')
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\eta(d\omega')e^{-G(\omega')}e^{-(G(\omega) - G(\omega'))^+}(\nu^+(\omega', d\omega) + \nu^-(\omega', d\omega)) \\
&\qquad + \eta(d\omega')e^{-G(\omega')}g(\omega')\delta_{\omega'}(d\omega).
\end{aligned}$$

It is easily checked that $\eta(d\omega)\exp(-G(\omega))g(\omega)\delta_\omega(d\omega')$ and $\eta(d\omega')\exp(-G(\omega'))g(\omega')\delta_{\omega'}(d\omega)$ represent the same measure concentrated on the diagonal $\{\omega = \omega'\}$. Thus we only need to verify that the first terms of the above expressions match. Consider the case $G(\omega') \geq G(\omega)$. (The reverse case follows by a symmetric argument.) Then we are reduced to verifying

$$\begin{aligned}
&\eta(d\omega)(\nu^+(\omega, d\omega') + \nu^-(\omega, d\omega')) \\
&\qquad = \eta(d\omega')(\nu^+(\omega', d\omega) + \nu^-(\omega', d\omega)).
\end{aligned}$$

Since η is the invariant measure for $\{\xi_n\}$, we have

$$\eta(d\omega)\nu^+(\omega, d\omega') = \eta(d\omega')\nu^-(\omega', d\omega).$$

This completes the proof of the fact that $\{Z_n\}$ is stationary-reversible when the law of Z_0 is $\tilde{\eta}$. Ergodicity follows by arguments analogous to those used for proving Theorem 3.1. □

Examples of H:

1) Consider the PRF given by P_T observed at points $\{t_1, \cdots, t_n\} \subset T$ through a channel with distortion and noise, modeled as follows: We have observations $y_i = f(\omega(t_i)) + \beta_i$, $1 \le i \le n$, for some function $f: \Omega_T \to R$ and i.i.d. $N(0, \sigma^2)$ random viarables $\beta_1, \cdots, \beta_n$. The posterior distribution of the PRF given these observations then corresponds to a PRF with distribution $P_{T,H,\mu}$ where [5], [6]

$$H(\omega) = F(\omega) + \sum_{i=1}^{n} f^2(\omega(t_i))/2\sigma^2 - \sum_{i=1}^{n} y_i f(\omega(t_i))/\sigma^2.$$

2) An alternative model of observations is [5]: We observe an inhomogeneous Poisson point process on T generated by ω with spatial intensity $f(\omega(t))$ at point t. The posterior distribution now corresponds to

$$H(\omega) = F(\omega) + \int_T f(\omega(t))\,dt - \int_T \log f(\omega(t))\Lambda\,(dt)$$

where Λ is the counting measure for the observed point process [5].

3) We may take $H = F + G$ where $G(\omega) =$ the sum of angles (in absolute value) between any two straight-line segments in $\partial\omega$ that meet each other. This is in the spirit of the "total turn" considered in [7].

Note that each H above is additive and thus $P_{T,H,\mu}$ is a Markov random field by the arguments of [2, sec. 8].

The process $\{Z_n\}$ proposed above has much simpler dynamics compared to the processes proposed in [4]–[6]. In the next section, we consider a variant that permits segmentations with curved boundaries.

V. Extensions to GPRF

This section extends some of the foregoing to "Generalized Polygonal Random Fields" (GPRF) which have polygonal-like realizations, but with curved boundaries. We begin with some preliminaries.

To each $x \in \mathcal{R}^2$, attach a set C_x of non-self-intersecting C^1 curves through x satisfying

1) Each $c \in C_x$ admits a parametrization $t \in \mathcal{R} \to z_c(t) = [x_c(t), y_c(t)]$ such that $x_c(\cdot), y_c(\cdot) \in C^1$, $z_c(0) = x$. We write $c \sim z_c(\cdot)$. Without loss of generality, we may and do assume that $\dot{x}_c(t)^2 + \dot{y}_c(t)^2 = 1\ \forall t$. Also, $\{z_c(\cdot) \mid c \in C_x\}$ is assumed to be compact under the topology of uniform convergence on compacts.
2) For any bounded open $A \subset \mathcal{R}^2$ with $x \in A$
$$\sup\{|t| \mid z_c(t) \in A\} < \infty.$$
3) If $c \in C_x$ and c' is obtained by rotating c around x, then $c' \in C_x$. (This operation will be called rotation.)
4) If $c \in C_x$, then $c' \in C_x$ for $c' \sim z_c(\tau + \cdot) - z_c(\tau) + x, \tau \in \mathcal{R}$. (This operation will be called time shift.)
5) If $c \in C_x$, then $c' \in C_x$, when $z_{c'}(t) = z_c(-t)$. (This operation will be called time reversal.)
6) If $\theta \in \mathcal{R}^2$ denotes the origin
$$C_x = \{c \mid z_c(\cdot) = x + z_{c'}(\cdot) \text{ for some } c' \in C_\theta\}.$$
7) If $c \in C_x, c' \in C_y$ satisfy $z_c(t) = z_{c'}(\tau + t)$ for $t \in (a, b)$, for some $a < b$ and $\tau \in \mathcal{R}$, then $z_c(\cdot) = z_{c'}(\tau + \cdot)$.

For $A \subset \mathcal{R}^2$, set $C_A = \bigcup_{x \in A} C_x$.

Remark 5.1: If for $c \in C_x$, $z_c(\cdot)$ is viewed as the trajectory of a particle starting at x, 2) implies that the particle exits from any finite domain in finite time. 6) says the C_x is obtained from C_θ by translation, so it suffices to prescribe C_θ. 7) says that if two trajectories agree on a nonempty open interval, one must be a time shift of the other. By 3)–5), C_x is closed under rotation, time shift, and time reversal.

Example 5.1: Let $\hat{C}_\theta$ be a finite collection of curves $c_1, \cdots, c_n$ passing through θ such that $z_{c_i}(t) = [t, f_i(t)]$ where $t \to f_i(t)$ are periodic with a common period τ and no piece of any one of the curves or any of its rotations or translations coincides with any other of these curves on some interval. Let $C_\theta = \{$all curves obtained from $\hat{C}_\theta$ by rotation, time shift, or time reversal$\}$. C_x, $x \in \mathcal{R}^2$ are now automatically specified through 6).

Typically one expects to obtain C_θ from a "core" $\hat{C}_\theta$ by the above procedure. As we shall be interested in C_T for a rectangle T, the above example may often provide a sufficiently rich class in applications for suitable choices of $n, \{c_1, \cdots, c_n\}$ and with $\tau \gg$ diameter (T). It has the advantage of easy parametrization.

Let ζ_θ denote a probability measure on C_θ which is invariant under rotation, time shift, and time reversal. The existence of a probability measure that is invariant under rotation and time shift is guaranteed by elementary ergodic theory. It may be rendered invariant under time reversal by taking its image under time reversal and then taking the average of the two. We assume that support $(\zeta_\theta) = C_\theta$. (If not, it is equivalent to considering a smaller C_θ, viz., support (ζ_θ).) Let ζ_x denote the probability measure on C_x obtained as the image of ζ_θ under the map $c \in C_\theta \to x + c \in C_x$.

Let $T \subset \mathcal{R}^2$ be a prescribed rectangle as before. By a "raw image" on T, we mean T endowed with a finite collection of finite curves, each of them a segment of some element of C_T. We shall now construct a probability measure on $I_R =$ the set of all raw images on T. This is done in the following steps:

Procedure 1:

i) Generate random points in T according to a Poisson point process with intensity λ.
ii) From each point x obtained above, pick a random curve
$$c \sim z_c(\cdot) = [x_c(\cdot), y_c(\cdot)] \in C_x$$
according to ζ_x.

iii) Initiate a particle at each x with trajectory $t \to z_c(t)$, $t \geq 0$, and with extinction time exponentially distributed with mean 1. Extinction times of distinct particles are independent.

iv) Draw the traces of their trajectories till the extinction time or the first time they hit ∂T, whichever occurs first, thus obtaining a finite segment of the corresponding curve.

This clearly gives an isotropic probability measure on I_R, viz., the law of the raw image generated by the above procedure.

Given a raw image $\gamma \in I_R$, let $D(\gamma)$ denote the set of curve segments that constitutes γ and $G(\gamma)$ their union. Let $A \subset T$ be a connected component of $T \setminus G(\gamma)$. Then $\partial A \subset G(\gamma) \cup \partial T$. We can write $\partial A = \partial_1 A + \partial_2 A$ where $\partial_1 A$ is that part of ∂A which is also a part of the boundary of some other connected component of $T \setminus G(\gamma)$ or of ∂T, and $\partial_2 A = \partial A \setminus \partial_1 A$. Let $A' =$ interior of $A \cup \partial_2 A$. Then $\partial A' = \partial_1 A$. A set A' thus obtained will be called a piece of γ. In Fig. 14, for example, if A is the interior of the region bounded by the contour $abcd$ with the curve ef removed, then A' is the entire interior of the same region. Let $G_b(\gamma) \subset G(\gamma)$ denote the union of all $\partial A'$ such that A' is a piece of γ. For each $c \in D(\gamma)$ parametrized as, say, $c = \{z(t) \mid a \leq t \leq b\}$, define $b(c) \subset c$ as follows. If $c \cap G_b(\gamma) \neq \phi$, $b(c) = \{z(t) \mid a' \leq t \leq b'\}$ with $a \leq a'$, $b \geq b'$, such that $b(c)$ is the minimal such set containing $c \cap G_b(\gamma)$. If $c \cap G_b(\gamma) = \phi$, $b(c) = \phi$. Define the "trimming operator" Tr: $I_R \to I_R$ to be the map that maps $\gamma \in I_R$ to its "trimmed version" $\gamma' \in I_R$ obtained by replacing each $c \in D(\gamma)$ by $b(c)$. Fig. 15 shows a raw image and its trimmed version. We shall denote by I_T the set of trimmed images, i.e., the range of Tr. By a proper image (or simply an image) we mean a map ω: $T \to J \cup \{j^*\}$, J being a finite set of colors as before and $j^* \notin J$ another distinguished color, such that the following hold: These exist $\gamma(\omega) \in I_T$ such that ω is constant and equal to an element of J on each piece of $\gamma(\omega)$, $\omega = j^*$ on $\cup_{c \in D(\gamma(\omega))} b(c)$. Thus $\partial\omega = G_b(\gamma(\omega))$, where $\partial\omega =$ the set of points of discontinuities of ω. Let I denote the set of images. Note that unlike in the case of PRF's, we are allowing "internal" discontinuities that lie in the interior of a piece and not on its boundary. (For example, if $\gamma(\omega)$ is as in Fig. 15(b), then ω will have the same color on either side of the segment ab, but a different color on it.) Conversely, given $\gamma \in I_T$, define

$$\Omega(\gamma) = \{\omega \in I \mid \partial\omega = G(\gamma)\}$$

and $\lambda(\gamma) = |\Omega(\gamma)|$. In the foregoing, we have a procedure for generating a random $\gamma \in I_T$ (viz., generate a random element of I_R by Procedure 1 and trim it). Given this γ, we may generate a random $\omega \in I$ by picking any element of $\Omega(\gamma)$ with equal probability ($=1/\lambda(\gamma)$). Let $P_T =$ the probability measure on I induced by the random sample thereof generated as above, where we endow I with the Borel σ-field corresponding to its topology defined analogously as for Ω_T. We call P_T a Generalized Polygonal Random Field (GPRF) on T.

Define on C_T an equivalence relation "$\approx$" by: $c \approx c'$ if c' is a time shift of c. Let $\hat{C}_T$ denote the set of equivalence classes thus obtained and

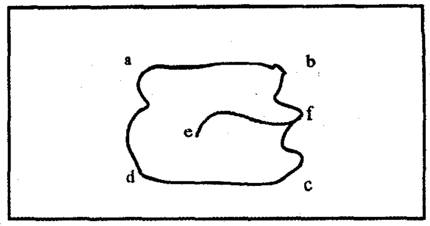


Fig. 14. The set $A' =$ the region bounded by the contour $abcd$ is a piece of γ.

$$\mathcal{C}_n = \{\mathcal{C} \subset \hat{C}_T \mid |\mathcal{C}| = n\}, \quad n = 0, 1, 2, \cdots.$$

Let μ_n denote the probability measure on $\mathcal{C}_n$ induced by steps i) and iii) of Procedure 1, conditioned on n curves being picked by the procedure. Probability of the latter event is

$$(\bar{\lambda}|T|)^n \exp(-\bar{\lambda}|T|)/n!.$$

Clearly, μ_n is isotropic for each n. For $\eta \in \mathcal{C}_n$, $n \geq 0$, let

$$I_T(\eta) = \{\omega \in I \mid G_b(\gamma(\omega)) \subset \eta, G_b(\gamma(\omega)) \not\subset \eta' \text{ for any proper subset } \eta' \text{ of } \eta\}.$$

Note that this is a finite set. As before, $L(\omega) =$ the total length of $\partial\omega$ for $\omega \in I$.

Theorem 5.1: The GPRF P_T obtained above is an isotropic Markov random field given by

$$P_T(A) = \left(\sum_{n=0}^{\infty} \frac{(\bar{\lambda}|T|)^n}{n!} e^{-\bar{\lambda}|T|} \int_{\mathcal{C}_n} \mu_n(d\eta) \times \sum_{I_T(\eta) \cap A} e^{-(L(\omega)+\log[\lambda(\gamma(\omega))])}\right) \Big/ Z_T$$

where Z_T is the normalizing constant.

Proof: Isotropy of P_T follows from its construction. Now the probability that Procedure 1 picks n curves $c_1, \cdots, c_n$ in $[\eta, \eta + d\eta] \subset \mathcal{C}_n$ and the independent system of particles planted one each on these survives for larger than $\ell_1, \cdots, \ell_n$ (respectively) time units (call this entire event Q) is

$$\frac{(\bar{\lambda}|T|)^n}{n!} e^{-\bar{\lambda}|T|} \mu_n(d\eta) e^{-(\ell_1 + \cdots + \ell_n)}.$$

The traces left by these particles need not, however, lead to a legal element of I_T. Hence the probability of obtaining an element $\gamma(\omega) \in I_T(\eta)$ thus is the probability of Q conditioned on the particle traces constituting an element of I_T. This is

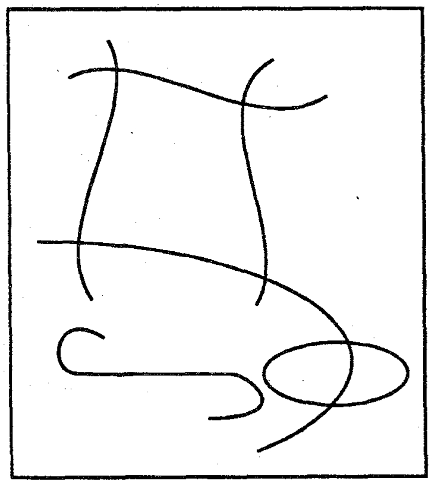
$$\sum_{n=0}^{\infty} \frac{(\bar{\lambda}|T|)^n}{n!} e^{-\bar{\lambda}(T)} \mu_n(d\eta) I\{\gamma(\omega) \in I_T(\eta)\} e^{-L(\omega)} / Z_T$$

with

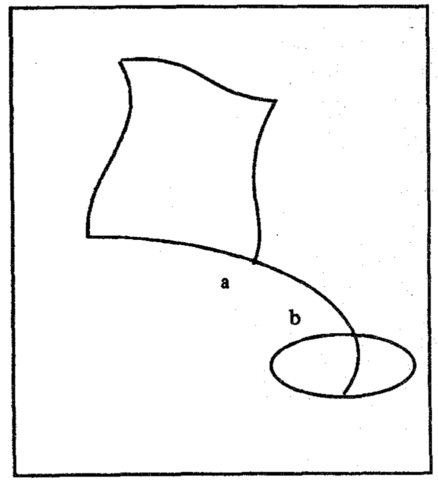
$$Z_T = \sum_{n=0}^{\infty} \frac{(\bar{\lambda}|T|)^n}{n!} e^{-\bar{\lambda}|T|} \int_{\mathcal{C}_n} \mu_n(d\eta) I\{\gamma(\omega) \in I_T(\eta)\} e^{-L(\omega)}.$$

Given $\gamma(\omega)$, a candidate ω is picked by choosing a coloring with probability

$$1/\lambda(\gamma(\omega)) = \exp(-\log \lambda(\gamma(\omega))).$$



(a)



(b)

Fig. 15. Raw image and its trimmed version. (a) Raw. (b) Trimmed.

This completes the derivation of (5.1). The Markov property can be proved as follows: Let $T = S \cup U$, S, U open. Consider $\gamma \in I_T$ generated by Procedure 1. Then this procedure implies that the conditional statistics of $\pi_U(\gamma)$ given $\pi_S(\gamma) = \xi$ (say) may be simulated as follows: Generate random points in $U \setminus S$ according to a Poisson point process with intensity $\bar{\lambda}$ and follow ii)–iv) of Procedure 1. Also, extend those curves in ξ that hit $\partial S \cap U$ and can be extended into $U \setminus S$, as in iii) of Procedure 1. Trim the resulting image. Accept it if it restricts to ξ on S, otherwise reject and repeat the procedure. Now it is clear that in the above, one could replace T by U, S by $U \cap S$, and ξ by its restriction to $U \cap S$ to obtain the same statistics for $\pi_U(\gamma)$. This is because $T = U \cup S$ and thus any curve in γ that straddles both S and U must pass through $S \cap U$. Markov property follows. (A more formal proof could be given along the lines of [2, sec. 8].) Now to prove that it is preserved in our passage from I_T to I, we only need verify that the "potential" $\log \lambda(\gamma(\omega))$ is additive. The event of picking a random coloring of $\gamma(\omega) \in I_T$ can be viewed as taking place in two steps: First one picks a coloring for the restriction of $\gamma(\omega)$ to S (denoted $\gamma_S(\omega) \in I_S$) according to uniform probability $1/\lambda(\gamma_S(\omega))$. Let $\gamma_U(\omega) \in I_U$, $\gamma_{S\cap U}(\omega) \in I_{S\cap U}$ denote the restrictions of $\gamma(\omega)$ to U and $S \cap U$, resectively, and $\lambda(\gamma_U(\omega)/\beta)$ the number of possible colorings of $\gamma_U(\omega)$ compatible with the coloring of $\gamma_{S\cap U}(\omega)$ given by $\beta =$ the coloring it inherited from $\gamma_S(\omega)$. The second step is to color $\gamma_U(\omega)$ by picking a random coloring from those compatible with $\gamma_{S\cap U}(\omega) = \beta$ with equal probability, i.e., with probability $\lambda(\gamma_U(\omega)/\beta)^{-1}$. Then

$$1/\lambda(\gamma(\omega)) = (1/\lambda(\gamma_S(\omega)))(1/\lambda(\gamma_U(\omega)/\beta))$$

and

$$\log \lambda(\gamma(\omega)) = \log \lambda(\gamma_S(\omega)) + \log \lambda(\gamma_U(\omega)/\beta).$$

Thus $\log \lambda(\gamma(\omega))$ is additive. □

Remark 5.2: An important limitation of the above theorem, in contrast to the corresponding result for PRF's, is that we do not claim the family P_T, $T \in \mathcal{G}_0$, to be consistent. In order to achieve consistency, it is clear that one will have to allow the particle trajectories that hit the boundary re-enter if they do so before the extinction time. But then a curve may contribute to the image more than one segment separated in space (i.e., with strictly positive distance from each other). The Markov property cannot hold in such a situation.

The next task is to generate an I-valued reversible ergodic Markov process $\{Z_n\}$ whose law at any time instant is

$$\tilde{P}_T(d\omega) = \alpha P_T(d\omega) \exp(-G(\omega))$$

for an additve $G: I \to [0, \infty]$, α being the normalization constant. We mimick closely the earlier procedure for the PRF's, as described below: Define T_p, T_s and the prefix ω_p and suffix ω_s of an image $\omega \in I$ the same way as we did for the PRF's.

Procedure 2:

Let $Z_n = \omega$.

i) Pick one element of $\{p, s\}$ with equal probability (say, s).

ii) Construct $\omega' \in I$ as follows:

a) Set $\omega'_p = \omega_s$ (see Fig. 16).

b) In T_s, pick m (say) points according to a Poisson point process with rate $\bar{\lambda}$. At each point, pick a random curve as in Procedure 1, ii).

c) From each point picked in b) and each point on ef where a trajectory from T_p hits ef, start a particle with exponential lifetime and unit speed along the corresponding curve. (In the latter case, the motion should be toward the interior of T_s). Trace its trajectory till extinction or till it hits ∂T, whichever comes first.

d) Trim the resultant raw image.

e) If the trimmed image does not restrict to $\gamma(\omega_s)$ on T_p, then reject those trajectories that led to the alterations of the trimmed image on T_p and replace

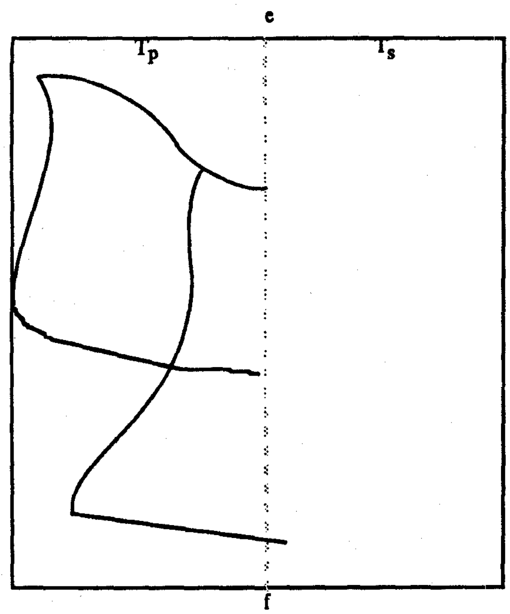


Fig. 16. Step in the construction of reversible process for polygon with curved boundaries.

them by new independently generated trajectories from the same initial points. Trim again.

f) Repeat till the trimmed image is consistant with $\gamma(\omega_s)$ on T_p.

g) Color the trimmed image on T_s by sampling uniformly from all colorings thereof that are compatible with the coloring on T_p. The resultant image on T is the desired ω'. (In practice, this step calls for a good graph coloring heuristic.)

iii) Set $Z_{n+1} = \omega'$ with probability $\exp(-(G(\omega') - G(\omega))^+)$ and $= \omega$ with the remaining probability.

Theorem 5.2: $\{Z_n\}$ is a reversible ergodic Markov process with stationary distribution $\hat{P}_T$.

This can be proved by adapting the proofs of the corresponding results for PRF's. We omit the lengthy details. As for PRF, we may choose G so as to incorporate an observation-dependent term for Bayesian analysis or to incorporate extra "costs" such as the "total turn" discussed in [7].

A "greedy" heuristic for step (Procedure iig) above is as follows: identify the "uncolored" image with a planar graph by identifying each piece of it with a node, with two nodes connected by an edge if and only if the corresponding pieces are adjacent (i.e., their boundaries intersect). Rank the nodes in the decreasing order of their degrees. Color the top node (i.e., the corresponding piece) by an element of J picked with uniform probability. Color the nodes in decreasing order, picking a color at each step uniformly from the admissible colors at that node. If a node is encountered for which there is no color left admissible, restart the whole procedure. Repeat till a complete coloring is found. (A color is "admissible" if it has not been already used to color a neighboring node.)

Appendix

We summarize here from [2] the conditions on coefficients featuring in the definition of F. Here $[[p_{ij}]], i,j \in J$, is a stochastic matrix with $p_{ij} = p_{ji}, p_{ii} = 0, i,j \in J$.

$$p_{ij} = a(i,j) = a(j,i), b(i,j) = b(j,i). \tag{A1}$$

$$c(i;j,k) = c(i;k,j). \tag{A2}$$

$$d(i,j,k,m) = d(j,k,m,i) = d(k,j,i,m). \tag{A3}$$

$$b(i,j) + \sum_{k \neq j} d(i,j,k,j) p_{jk}^2 = 1 \tag{A4}$$

$$(c(i;j,k) + c(j;i,k)) p_{ij} + \sum_{m \neq i,j} d(i,k,j,m) p_{jm} p_{im} = 1 \tag{A5}$$

$$\sum_{k \neq i,j} (c(i;j,k) - c(j;i,k)) p_{ik} p_{jk} = 0. \tag{A6}$$

$$e(i,j) = b(i,j) p_{ij} + \sum_{k \neq i,j} c(i;j,k) p_{ij} p_{jk}. \tag{A7}$$

$$f(i) = c_1 \sum_{j \neq i} p_{ij}^2 b(i,j),$$

with

$$c_i = \int \int_{\mathcal{R}^2} |u - v| V(du) V(dv). \tag{A8}$$

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