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Duality of Linear Systems

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Dedicated to Jim Massey on the occasion of his 60th birthday

Abstract

In this technical note we present a duality theory of linear time-invariant finite-dimensional systems both in the context of linear input-output maps and in the context of linear behavioural systems. The dual input-output map is shown to correspond to the adjoint systems. A definition of an adjoint linear behaviour is presented and properties of the adjoint behaviour are investigated.

I Introduction

The objective of this technical note is to close a gap in the module theory of stationary finite-dimensional linear system as developed by R.E. Kalman (see e.g. KALMAN-FALB-ARBIB [3]). This is concerned with the duality of linear input-output maps and makes precise in what sense the dual of a linear input-output map gives rise to a dual linear state-space system and how reachability and observability of the original system corresponds to observability and reachability of the dual system.

We also relate these results to a theory of duality in the context of behavioural systems in the sense of WILLEMS [5]. This is motivated by applications to coding theory as evidenced in the work of FORNEY–TROTT [1] on duality for behavioral systems defined on compact abelian groups.

It is appropriate that this note be dedicated to Jim Massey. He more than anybody else has investigated the deep connections that exist between systems theory and coding (decoding) theory. We see ample demonstration of this in his early work on the Berlekamp-Massey algorithm and its connections to Partial Realization Theory, his joint work with Sain on inverses of linear sequential systems and in his recent joint work with his students Loeliger and Mittelholzer on the relationship between the behavioural theory of linear systems and coding theory.

II The Linear Dynamical System Σ

In this section we review the concepts associated with linear dynamic systems. The treatment is a minor extension of that in chapter ten of KALMAN, FALB and ARBIB [3] as in JOHNSTON [2]. All modules are assumed to be unitary left-modules, that is, the rings always will have a 1, which acts as an identity operator on the module, and the ring acts on module elements from the left.

Definition 2.1 *A discrete-time, linear, time-invarant (D.L.T.I.) system Σ over a ring R is a triple of R-module homomorphisms, $\Sigma = (F : X \to X,\ G : U \to X,\ H : X \to Y)$, where U (the input module) and Y (the output module) are finitely generated R-modules. X is the state module.*

The interpretation of this definition is that the triple Σ defines the dynamic equations

$$x_{t+1} = F \cdot x_t + G \cdot u_t \tag{2.1}$$

and

$$y_t = H \cdot x_t \tag{2.2}$$

where $u_t \in U$ denotes the input applied at time $t; x_t, x_{t+1} \in X$ denotes the states at times t and $t+1$ respectively, and $y_t \in Y$ denotes the output at time t. Note that in this formulation, an input u_t applied at time t has no effect on the output y_t at time t, although it does affect the output at time $t+1$. In other words, there is no "feed-through".

Secondly, each generator of U can be thought of as an input "port". If U has m generators over R, an input u_t can be specified by m elements of R, and so Σ can be pictured as having m input "ports." Similarly, each generator of Y can be thought of as an output port. This interpretation is strengthened if U and Y are free modules, i.e., of the form R^m.

When no confusion can arise, the dynamical system Σ will simply be denoted by the triple (F, G, H).

There are several important maps that can be derived from $\Sigma = (F, G, H)$. Equation 2.1 describes the operation of Σ under the application of an input at a single instant of time, and can be viewed as a map : $X \times U \to X$. This map can be extended to sequences of inputs of arbitrary but finite length:

Definition 2.2 *Let U^* denote the set of all finite length sequences of elements from U. That is,*

$$U^* = \{(u_0, u_1, \ldots, u_n) | u_i \in U; n \text{ arbitrary}\}$$

We will represent U^* by the finitely generated $R[z]$ module Ω, where

$$\Omega = U[z] = \{\text{all polynomials in } z \text{ with coefficients from } U\}. \tag{2.3}$$

Given an element $\omega \in \Omega$, we will interpret the coefficient of z^i as the input applied i instants of time before the present. Thus $z\omega$ will denote the input sequence ω followed by a zero input; the inputs are pictured as being applied up to and including time $= 0$, and the output resulting from this sequence of inputs appears at time $= 1$.

Definition 2.3 *Let $\Sigma = (F, G, H)$ be D.L.T.I. system over R. If*

$$\omega = \sum_{i=0}^{n-1} u_i z^{n-1-i} \in \Omega,$$

the map $\mathcal{R}^* : X \times \Omega \to X$ is defined by

$$\mathcal{R}^* : (x_o, \omega) \mapsto F^n \cdot x_0 + \sum_{i=0}^{n-1} F^{n-1-i} \cdot G \cdot u_i \tag{2.4}$$

$\mathcal{R}^*(x_0, \omega)$ is the state reached by starting Σ in state x_0, and applying the sequence of the n inputs represented by ω.

Definition 2.4 *$\bar{\mathcal{R}}_\Sigma : \Omega \to X$, the extended 0-state transition map of Σ, is defined by*

$$\bar{\mathcal{R}}_\Sigma : \omega \mapsto \mathcal{R}^*(0, \omega). \tag{2.5}$$

Definition 2.5 *Let $\Sigma = (F : X \to X, G : U \to X, H : X \to Y)$ be a D.L.T.I. system over R, and let $\bar{\mathcal{R}}_\Sigma : \Omega \to X$ be its extended 0-state transition map.*

Then X_r, the reachable set of X, is defined by

$$\begin{aligned} X_r &= im\, \bar{\mathcal{R}}_\Sigma \\ &= \bar{\mathcal{R}}_\Sigma(\Omega). \end{aligned} \tag{2.6}$$

Another map of interest derivable from 2.1 and 2.2 is the free response map which describes the output of Σ started in some state x and supplied with an all-zero input sequence. In general, this output sequence is semi-infinite.

Definition 2.6 *Let Y^{**} denote the set of all semi-infinite sequences of elements drawn from the R-module Y. We will identify Y^{**} with the $R[z]$-module Γ where*

$$\begin{aligned} \Gamma &= Y[[z^{-1}]] \\ &= \{\sum_{i=1}^{\infty} y_i z^{-i} | y_i \in Y\}. \end{aligned} \tag{2.7}$$

Γ is an $R[z]$-module under the following action of z : ordinary multiplication by z followed by discarding non-negative powers of z.

Definition 2.7 *Let $\Sigma = (F : X \to X, G : U \to X, H : X \to Y)$ be a D.L.T.I. system over R. Then $\bar{O}_\Sigma : Z \to \Gamma$, the free response map of Σ, is defined by*

$$\bar{O}_\Sigma : x \mapsto \sum_{i=1}^{\infty} (HF^{i-1} \cdot x) z^{-i} \tag{2.8}$$

Thus, $\bar{O}_\Sigma(x)$ represents the sequence $\{H(x), HF(x), \ldots, HF^i(x), \ldots\}$.

Definition 2.8 *Let $\Sigma = (F, G, H)$ be a D.L.T.I. system over R and let $\bar{\mathcal{R}}_\Sigma : \Omega \to X$, $\bar{O}_\Sigma : X \to \Gamma$ be the extended O-state transition map and free response map respectively.*

Then $f_\Sigma : \Omega \to \Gamma$, the input/output map of Σ, is defined by

$$f_\Sigma = \bar{O}_\Sigma \cdot \bar{\mathcal{R}}_\Sigma \tag{2.9}$$

Proposition 2.1 *$\Omega = U[z]$ and $\Gamma = Y[[z^{-1}]]$ are both $R[z]$-modules, as discussed above. X is also an $R[z]$-module with the action of z defined by*

$$\forall\, x \in X, \quad z \cdot x = F(x). \tag{2.10}$$

With this structure, $\bar{\mathcal{R}}_\Sigma : \Omega \to X$ and $\bar{O}_\Sigma : X \to \Gamma$ are both $R[z]$-homomorphisms, and hence so is $f_\Sigma : \Omega \to \Gamma$.

Proof. See Chapter 10 of KALMAN, FALB, ARBIB [3]. ∎

Thus, a D.L.T.I. system $\Sigma = (F : X \to X, G : U \to X, H : X \to X)$ over R induces an $R[z]$-homomorphism $f_\Sigma : U[z] \to Y[[z^{-1}]]$. The converse is also true, as outlined next.

2.1 Linear, Zero-state, Input/Output Maps

Definition 2.9 *Let U, Y be finitely-generated (F.G.) R-modules, and let Ω, Γ denote the $R[z]$-modules $U[z], Y[[z^{-1}]]$ as above. A linear, zero-state, input-output map f over R is simply an $R[z]$-homomorphism $f : \Omega \to \Gamma$.*

Proposition 2.2 *Let $f : \Omega \to \Gamma$ be linear, zero-state, input/put map over R. Then f induces a D.L.T.I. system $\Sigma_f = (F : X \to X, G : U \to \bar{X}, H : X \to Y)$.*

Proof. Take $X = \Omega \setminus \ker f$. Let $\mathcal{R}_f : \Omega \to X$ be the canonical surjection and let $\bar{O}_f : X \to \Gamma$ be the canonical injection induced by f. The action of F on X is taken to be the action of z on X, viewed as an R-module. G is defined by letting finding the images under $\bar{\mathcal{R}}_f$ of U's generators and extending this map by R-linearity. H is defined by finding the images in Γ of X's generators an R-module, retaining only the coefficients of z^{-1}, and extending this map by R-linearity. (For more details, see chapter 10 of KALMAN, FALB, ARBIB [3]). ∎

Definition 2.10 *Σ_f, the D.L.T.I. system induced by a linear input/output map $f : \Omega \to \Gamma$, will be called the canonical system induced by f.*

Notice that $\bar{\mathcal{R}}_f$ and $\bar{O}_f$ are precisely the extended 0-state transition map and free response map of Σ_f. The facts that these maps are surjective and injective respectively have several interesting interpretations and ramifications.

2.2 Reachability, Observability, and Realizability

Definition 2.11 *Let X be the state module of a D.L.T.I. system Σ. Σ is said to be completely reachable iff $X_r = X$, where $X_r = im\ \bar{\mathcal{R}}_\Sigma =$ the set of reachable states.*

Proposition 2.3 *The canonical system Σ_f induced by a linear input/put map is completely reachable. The proof follows from the fact that $\bar{\mathcal{R}}_f : \Omega \to X = \Omega \backslash \ker f$ is surjective.*

Definition 2.12 *Let X be the state module of a D.L.T.I. system Σ. A state $x \in X$ is said to be indistinguishable from 0 iff $\bar{\mathcal{O}}_\Sigma(x) = 0 \in \Gamma$. The set of all indistinguishable states is denoted X_i. A D.L.T.I. system is said to be completely distinguishable iff $X_i = (0)$.*

Proposition 2.4 *The set X_i of indistinguishable states in a D.L.T.I. system Σ is equal to the $R[z]$-submodule $\ker \bar{\mathcal{O}}_\Sigma$. The canonical system induced by a linear input/output map is completely distinguishable (since $\bar{H}_f$ is injective and so $X_i = \ker \bar{\mathcal{O}}_f = (0)$).*

Definition 2.13 *A D.L.T.I. system $\Sigma = (F : X \to X, G : U \to X, H : X \to Y)$ over R will be called a finite D.L.T.I. system iff X is a finitely generated R-module. (This will be the usual case and the adjective "finite" will be dropped if there is no cause for confusion).*

Definition 2.14 *Let $f : \Omega \to \Gamma$ be a linear input/output map over R. f is said to be realizable iff there exists a finite D.L.T.I. system Σ such that f_Σ (the input/output map of Σ) equal f. In this case, Σ is said to realize, or be a realization of, $f : \Omega \to \Gamma$. Note that if Σ_f, the canonical system induced by f, is a finite D.L.T.I. system, then f is realizable. In other words, if $X = \Omega \backslash \ker f$ is F.G. (finitely-generated) over R, then f is realizable, and furthermore, Σ_f is a canonical realization of f.*

III The Adjoint Linear System

In this section take the ring R to be a field $\mathcal{K}$. $\mathcal{K}$ will be either $\mathbf{R}$ (the real field) of $\mathbf{C}$ (complex field) or a finite field. The R-modules U, Y and X are finite-dimensional K-vector spaces with either the euclidean topology (in case $K = \mathbf{R}$ or $\mathbf{C}$) or the discrete-topology (in case K is a finite field).

Let $(Y[[z^{-1}]])'$ and $(U[z])'$ denote the algebraic duals of $Y[[z^{-1}]]$ and $U[z]$ respectively. Let $Q : Y[[z^{-1}]] \to K$ be K-linear and define the adjoint linear map

$$\left\{ \begin{array}{l} f'_\Sigma : (Y[z^{-1}]])' \to (U[z])' \text{ as} \\ f'_\Sigma \cdot Q = Q \cdot f_\Sigma \end{array} \right. \tag{3.1}$$

Now it is not a-priori clear that f'_Σ defines an input-output map, since the algebraic dual of $Y[[z^{-1}]]$ is not a space of polynomials. This may be seen by using the fact that the space of polynomials has a countable basis whereas the $Y[[z^{-1}]]'$ cannot have a countable basis. We first consider the case when $K = \mathbf{C}$ or $\mathbf{R}$. To surmount this difficulty we introduce a topology on the space of formal power series $\mathcal{F}$ with coefficients in Y. To simplify the exposition, take $Y = \mathbf{C}$ or $\mathbf{R}$ but the same remains true when Y is a finite-dimensional vector space.

We put on $\mathcal{F}$ the topology of convergence of each coefficient. This topology can be defined by the sequence of seminorms:

$$u = \sum_{p\in N} u_p z^p \rightsquigarrow \sup_{p\leq K} |u_p|, \quad K = 0, 1, 2, \cdots$$

This topology converts $\mathcal{F}$ into a Fréchet space (cf. TREVES [4], p. 91).

On the other hand the algebraic dual of a space of polynomials $\mathcal{P}$ with coefficients in $\mathbf{C}$ or $\mathbf{R}$ is a space of formal power series. In fact, there is a natural topology on the space of polynomials, the inductive limit topology defined in terms of Hausdorff finite dimensional topologies on the space of polynomials of degree $\leq K$, but the algebraic and topological duals coincide. Therfore for the sequel, when we refer to dual space we could think of both dual spaces as topological duals.

There is a natural duality between polynomials and formal power series which can be expressed by the bracket

$$< P, u >= \sum_{p\in \mathbf{N}} P_p u_p, \text{ where}$$

$$P = \sum_p P_p z^p \text{ and } u = \sum_p u_p z^p.$$

This is well-defined since all coefficients P_p, except possibly a finite number of them are equal to zero. We then have:

Theorem 3.1 *(TREVES [4], p. 228, Th. 22.1).*

(a) The map. $u \rightsquigarrow (P \rightsquigarrow < P, u >)$ is a vector space isomorphism of the Fréchet space of formal power series $\mathcal{F}$ onto dual of the space $\mathcal{P}$ of polynomials.

(b) The map. $P \rightsquigarrow (u \rightsquigarrow < P, u >)$ is a vector space isomorphism of $\mathcal{P}$ onto the dual of $\mathcal{F}$.

There is a natural $K[z^{-1}]$-module structure on $Y'[z^{-1}]$ and $U'[[z]]$, where $'$ denotes the dual space. Define multiplication of elements of $Y'[z^{-1}]$ by a polynomial as follows:

For

$$a(z^{-1}) = \sum_{i=0}^{n} a_i z^{-i}, a_i \in \mathbf{C}, a_i = 0, i > n, f = \sum_{j=0}^{m} f_j z^{-j}, f_j \in Y'$$

$$a(z^{-1}) \cdot f = \sum_{\ell=0}^{n+m} g_\ell z^{-\ell} = g$$

$$\text{where } g_\ell = \sum_{k=0}^{\ell} a_{\ell-k} f_k.$$

This multiplication is well-defined and $g \in Y'[z^{-1}]$. The module axioms are easily checked. As far as $U'[[z]]$ is concerned, define multiplication of elements of $U'[[z]]$ by a polynomial as follows:

For $a(z^{-1}) = \sum_{i=0}^{n} a_i z^{-i}$, $a_i \in \mathbf{C}$, and $\underline{f} = \sum_{j=1}^{\infty} f_{-j} z^j$ with $a_i = 0$ all other $i \in Z$ and $f_j = 0$, $\forall$ other $j \in Z$.

$$a(z^{-1}) \cdot \underline{f} = \sum_{\ell=1}^{\infty} g_{-\ell} z^{\ell} = g$$

where $g_{-\ell} = \sum_{k=0}^{\ell+n} a_{-\ell+k} f_{-k}, \ell \geq 1$ and $g_{-\ell} = 0$, for $\ell < 1$. Again the module axioms are easily verified. Let $(Y[[z^{-1}]])'$ and $(U[z])'$ denote the duals of $Y[[z^{-1}]]$ and $U[z]$ respectively.

Define the pairings

(i)

$$\begin{aligned} < \cdot, \cdot >_1 \; &: \; Y'[z^{-1}] \times Y[[z^{-1}]] \longrightarrow K \\ &: \; (f, \underline{y}) \longmapsto \sum_{i=o}^{n} f_i y_{i+1}, \end{aligned}$$

where

$$f = \sum_{i=0}^{n} f_i z^{-i} \text{ and } \underline{y} = \sum_{i=1}^{\infty} y_i z^{-i}$$

(ii)

$$\begin{aligned} < \cdot, \cdot >_2 \; &: \; U'[[z]] \times U[z] \longrightarrow K \\ &: \; (\underline{f}, u) \longmapsto \sum_{i=1}^{\infty} f_{-i} u_{-i+1} \end{aligned}$$

where

$$\underline{f} = \sum_{i=1}^{\infty} f_{-i} z^i, u = \sum_{i=0}^{n} u_{-i} z^i.$$

Theorem 3.2 *(a) The map*

$$\begin{aligned} \varphi \; &: \; Y'[z^{-1}] \longrightarrow (Y[[z^{-1}]])' \\ &\quad \underline{f} \rightsquigarrow (\underline{y} \rightsquigarrow < f, \underline{y} >_1) \end{aligned}$$

is a $K[z^{-1}]$-module isomorphism from $Y'[z^{-1}]$ to the dual of $Y[[z^{-1}]]$.

(b) The map

$$\begin{aligned} \psi \; &: \; U'[[z]] \longrightarrow (U[z])' \\ &: \; f \longmapsto (u \rightsquigarrow < \underline{f}, u >_2) \end{aligned}$$

is a $K[z^{-1}]$-module isomorphism.

Proof. We first prove that φ and ψ are injective, surjective and K-linear and hence algebraic isomorphisms. This can be carried out as in the proof of Theorem 3.1 by Treves. We make $(Y[[z^{-1}]])'$ into a $K[z^{-1}]$ module in the following way:

For $f \in Y'[z^{-1}]$, let $\varphi(f)$ be denoted as Q_f which belongs to $Y[[z^{-1}]]'$. Define multiplication of Q_f by $a(z^{-1}) \in K[z^{-1}]$ as

$$a(z^{-1}) \cdot Q_f = Q_g \text{ where } g = a(z^{-1}) \cdot f$$

Now we can easily check that the module axioms are satisfied. Using the K-linearity and the above definition of multiplication it is easily checked that φ is $K[z^{-1}]$-linear.

In a similar way we make $(U[z])'$ into a $K[z^{-1}]$-module. ∎

In order to state the next theorem we introduce the adjoint system

$$(\bar{\Sigma}) \begin{cases} \xi_{t-1} = F'\xi_t + H'\eta_t; \quad \xi(+\infty) = 0 \\ \chi_t = G'\xi_t \end{cases}$$

where $F' : X' \to X', G' : X' \to U'$ and $H' : X' \to Y'$ are the adjoint linear maps.

Let $f_{\bar{\Sigma}}$ represent the input-output map corresponding to $\bar{\Sigma}$.

Theorem 3.3 *There exists module isomorphism φ and ψ as in the previous theorem such that the following diagram commutes:*

$$\begin{array}{ccc} (Y[[z^{-1}]])' & \xrightarrow{f'_{\Sigma}} & (U[z])' \\ \uparrow \varphi & & \uparrow \psi \\ Y'[z^{-1}] & \xrightarrow{f_{\bar{\Sigma}}} & U'[[z]] \end{array}$$

where f'_{Σ} is defined as, $f'_{\Sigma} \cdot Q = Q \cdot f_{\Sigma}$ and $Q : Y[[z^{-1}]] \to K$ is K-linear, f'_{Σ} is $K[z^{-1}]$-linear, and

$$f_{\bar{\Sigma}} : f_n z^{-n} + f_{n-1}z^{-n+1} + \cdots + f_0 \longmapsto g_{-1}z^1 + g_{-2}z^2 + \cdots.$$

with $f_i \in Y'$ and $g_{-i} = G'F'^{i-1}H'f_0 + \cdots + G'F'^{n+i-1}H'f_n$.

Proof. The proof is constructed by showing that

$$f'_{\Sigma}[\varphi(f)](u) = \psi[f_{\bar{\Sigma}}(f)](u), u \in U[z] \tag{3.2}$$

where $f = f_n z^n + f_{n-1}z^{-n+1} + \cdots + f_0, f_i \in Y', i = 0, \cdots, n$.

Now $f'_{\Sigma}[\varphi(f)](u) = \phi(f)[f_{\Sigma}(u)]$.

By solving the recurrence relation corresponding to (Σ), we get

$$\begin{aligned} \varphi(f)[f_{\Sigma}(u)] &= f_0(HGu_0) + f_1(HFGu_0) + \cdots + f_n(HF^nGu_0) \\ &\quad + f_0(HFGu_{-1}) + f_1(HF^2Gu_{-1}) + \cdots + f_n(HF^{n+1}Gu_{-1}) \\ &\quad + \ldots \\ &\quad + f_0(HF^KGu_{-k}) + f_1(HF^{k+1}Gu_{-k}) + \cdots + f_n(HF^{n+k}Gu_{-k}) \\ &= (G'H'f_0)(u_0) + (G'F'H'f_1)(u_0) + \cdots + (G'F'^nHf_n)(u_0) \end{aligned}$$

$$
\begin{aligned}
&+ (G'F'H'f_0)(u_{-1}) + (G'F'^2H'f_1)(u_{-1}) + \cdots + (G'F'^{n+1}H'f_n)(u_1)\\
&+ \ldots\\
&+ (G'F^{k'}H'f_0)(u_{-k}) + (G'F^{k+1'}H'f_1)(u_{-1}) + \cdots +\\
&\cdots (G'F'^{n+k}H'f_n)(u_{-k})
\end{aligned} \tag{3.3}
$$

On the other hand if,

$$
\begin{aligned}
\underline{g} &= g_{-1}z^1 + g_{-2}z^2 + \cdots, \quad g_{-i} \in U'\\
\psi(\underline{g})(u) &= \sum_{i=1}^{\infty} g_{-i}(u_{-i+1})
\end{aligned} \tag{3.4}
$$

By solving the recurrence relation corresponding to $(\bar{\Sigma})$, we get for $\underline{g} = f_{\bar{\Sigma}}(f)$, that

$$
\begin{aligned}
&g_{-1} = G'H'f_0 + \cdots + (G'F'^nHf_n)\\
&\vdots\\
&g_{-i} = G'F'^{i-1}H'f_0 + \cdots + (G'F'^{n+i-1}H'f_n)\\
&\vdots
\end{aligned}
$$

Therefore $\phi[f_{\bar{\Sigma}}(f)](u)$ is precisely the right hand side of (3.1). ∎

<u>The Case where K-Finite Field.</u> Let K be a finite field with the discrete topology and U and Y finite-dimensional K-vector spaces with the discrete topology. Let the topologies on K and U, Y be generated by = norms $|\cdot|_K$, $|\cdot|_U$, $|\cdot|_Y$ where $|v| = 0 \Leftrightarrow v = 0$ and $|v| = 1$ iff $v \neq 0$. Here $v \in K$, U or Y. On $Y[[z^{-1}]]$ we put the product topology. Theorem 3.3 now remains true.

Remark 3.1 *We may proceed using realization theory as in Section 2 instead of starting with an explicit state-space realization. Thus given an input-output map $f_\Sigma : U[z] \to Y[[z^{-1}]]$ obtain a minimal (reachable and observable) realization via the canonical factorization*

$$
\begin{array}{ccc}
U[z] & \xrightarrow{f_\Sigma} & Y[[z^{-1}]]\\
\mathcal{R}\searrow & & \nearrow O\\
& U[z]/\ker f_\Sigma = X &
\end{array}
$$

where the reachability operator $\mathcal{R}$ and the observability operator 0 are defined by

$$
\begin{aligned}
\mathcal{R} &: U[z] \to X : u \mapsto [u] \quad ([\cdot] \text{ denotes equivalence})\\
O &: X \to Y[[z^{-1}]] : [u] \mapsto f_\Sigma(u).
\end{aligned}
$$

Let $F : X \to X, G : U \to X$ and $H : X \to Y$ be K-linear maps defining the corresponding minimal state space realization. Now define

$$
\begin{aligned}
&f_{\bar{\Sigma}} : Y'[z^{-1}] \longrightarrow U'[[z]] \text{ by}\\
&f'_\Sigma \cdot \varphi = \varphi \cdot f_{\bar{\Sigma}}.
\end{aligned}
$$

Then as in Theorem 3.3, we can check that the state-space system defined by

$$\begin{array}{l} F' : X' \to X' \\ H' : Y' \to X' \\ G' : X' \to U' \end{array}$$

realizes the map $f_{\bar{\Sigma}}$ (note the time-reversal). We can explicitly compute the reachability operator $\bar{\mathcal{R}}$ and the obvervability operator $\bar{\mathcal{O}}$ corresponding to $f_{\bar{\Sigma}}$. We have that (Σ) is reachable iff $(\bar{\Sigma})$ is observable and (Σ) is observable iff $(\bar{\Sigma})$ is reachable.

IV Duality for Linear Behavioural Systems

Let K be a field and let W be a K-vector space of finite dimension q. Let W^Z denote the K-vector space of biinfinite sequences taking values in W and let $\sigma : W^Z \to W^Z$ be the left shift defined by $(\sigma w)(t) = w(t+1)$. Note that the same symbol w will be used to denote an element of W^Z or W when no confusion arises.

Definition 4.1 *A linear behavior on W is any subspace $\mathcal{B} \subseteq W^Z$ which satisfies $\mathcal{B}$ is shift invariant, that is,*

$$\sigma\mathcal{B} = \mathcal{B} \tag{4.1}$$

$\mathcal{B}$ is complete, that is

$$\begin{array}{l} \forall\, w \in W^Z,\ \text{such that } w|_I \in \mathcal{B}|_I \quad \forall\, I \subseteq Z \text{ a finite interval} \\ \Rightarrow w \in \mathcal{B} \end{array} \tag{4.2}$$

Definition 4.2 *$\mathcal{B}$ is said to be N-memory if (4.2) holds with the restriction to intervals $I \subseteq Z$ such that length $(I) \leq N+1$.*

Definition 4.3 *Let $\mathcal{B}$ be a linear shift-invariant subspace of W^Z. The completion of $\mathcal{B}$ is defined as*

$$\bar{\mathcal{B}} = \{w \in W^Z |\ w|_I \in \mathcal{B}|_I\ \forall\, I \subseteq Z \text{ finite}\}.$$

It is easy to see that $\bar{B}$ is a linear behaviour and is in fact the smallest linear behavior containing $\mathcal{B}$.

For $\mathcal{B}$ a linear behavior and $I \subseteq Z$, let

$$\mathcal{B}_I = \{w \in \mathcal{B} |\ w|_{I^c} = 0\}, \tag{4.3}$$

where I^c denotes the complement of I in Z, and

$$\mathcal{B}_f = \sum_{I,\text{finite}} \mathcal{B}_I. \tag{4.4}$$

Clearly $\mathcal{B}_f$ is a linear shift-invariant subspace of $\mathcal{B}$ and $\bar{\mathcal{B}_f} \subseteq \mathcal{B}$.

Definition 4.4 *A linear behavior is said to be controllable if:*

$\forall\, w_1, w_2 \in \mathcal{B}, \exists\, w \in \mathcal{B}$ *and* $\exists\, s \in \mathbf{N}$ *such that*

$$w(t) = w_1(t)\ \forall\, t < 0 \text{ and } (\sigma^s w)(t) = w_2(t)\ \forall\, t \geq 0.$$

$\mathcal{B}$ is said to be $\mathbf{N}$*-controllable if in the above s can always be chosen equal to N.*

Let W' be the dual space of W and let $W'[z, z^{-1}]$ represent the space of polynomials in the indeterminates z and z^{-1}. In a manner similar to that of Section 2, $W'[z, z^{-1}]$ has the structure of $K[z, z^{-1}]$ module.

Consider the pairing,

$$(\ ,\ ) : W^{\mathbf{Z}} \times W'[z, z^{-1}] \to K, \tag{4.5}$$

defined by

$$(\tilde{w}, w') = \sum_{i=-\infty}^{\infty} \langle w_i, w_i' \rangle$$

where

$$\tilde{w} = (\cdots, w_{i-1}, w_i, w_{i+1}, \cdots) \text{ and } w' = \sum_{i=-N}^{N} w_i' z^i$$

and $\langle\ ,\ \rangle$ represents the pairing between W and W'. This is well-defined since the sum on the right hand side of (4.5) is a finite sum. Now if $\mathcal{B} \subseteq W^{\mathbf{Z}}$ is a linear behavior $\mathcal{B}^{\perp} = \{w' | (w, w') = 0,\ \forall\, w \in \mathcal{B}\}$ is a $K[z, z^{-1}]$ submodule of $W'[z, z^{-1}]$. Conversely if M is a $K[z, z^{-1}]$ submodule of $W'[z, z^{-1}]$ then $M^{\perp} = \{\tilde{w} | (\tilde{w}, w') = 0,\ \forall\, w' \in W'[z, z^{-1}]\}$ is a linear behaviour on W.

Based on the results of [5], the following theorem is easily proved.

Theorem 4.1 *Let $\mathcal{B} \subseteq W^{\mathbf{Z}}$ be a linear behavior. Then*

(a) $(B^{\perp})^{\perp} = \mathcal{B}$.

(b) *$\mathcal{B}$ is N-memory for some $N \in \mathbf{N}$.*

(c) *The following are equivalent:*

- *i)* *$\mathcal{B}$ is controllable.*
- *ii)* *$\mathcal{B}$ is M-controllable for some $M \in \mathbf{N}$.*
- *iii)* $\bar{\mathcal{B}}_f = B$

(d) *There exists a K-finite dimensional behavior $\mathcal{C} \subseteq \mathcal{B}$ such that*

$$\mathcal{B} = \bar{\mathcal{B}}_f \bigoplus \mathcal{C}$$

(e) *There exists two non-negative integers $m(\mathcal{B})$ and $n(\mathcal{B})$ such that*

- *(i)* $\text{dimension}_K(\mathcal{B}_{[1,t]}) = m(\mathcal{B})t + n(\mathcal{B})$.

(ii) *dimension*$_K(\mathcal{B}_{[1,t]}) = m(\mathcal{B})t - n(\bar{\mathcal{B}}_f)$.

Remark 4.1 *$m(\mathcal{B})$ is the number of input variables in any input-output representation of $\mathcal{B}$. $n(\mathcal{B})$ is called the McMillan degree of $\mathcal{B}$ and is the dimension of the state space of any minimal input-state-output representation of $\mathcal{B}$.*

4.1 Adjoint Behaviour

Let V be a finite dimensional K-vector space. We have a canonical embedding

$$V[z,z^{-1}] \hookrightarrow V^Z, \text{ given by}$$

$$\sum_{i=-N}^{N} v_i z^i \longmapsto (0,0,\cdots,v_{-N},\cdots,v_{-1},v_0,v_1,\cdots,v_N,0,0,\cdots). \tag{4.6}$$

Clearly with this identification $V[z,z^{-1}] \simeq (V^Z)_f$.

Let $\mathcal{B}^\perp \subseteq W'[z,z^{-1}]$ as a subspace of $(W')^Z$. Clearly $\mathcal{B}^\perp$ is shift-invariant.

Definition 4.5 *The adjoint behaviour $\mathcal{B}^*$ of $\mathcal{B}$ is defined as*

$$\mathcal{B}^* = \overline{\mathcal{B}^\perp} \subseteq (W')^Z. \tag{4.7}$$

Since $\mathcal{B}^\perp \subseteq (\mathcal{B}^*)_f$ it follows that

$$\mathcal{B}^* = \bar{\mathcal{B}}^\perp \subseteq (\overline{\mathcal{B}^*})_f$$

Hence $\mathcal{B}^* = (\overline{\mathcal{B}^*}_f)$ which implies from Theorem 4.1 that $\mathcal{B}^*$ is controllable.

By identifying $(W')'$ and W we can consider $(\mathcal{B}^*)^* \subseteq W^Z$. We have the following:

Proposition 4.1

$$(\mathcal{B}^*)^* = \bar{\mathcal{B}}_f.$$

Proof.

i) Let $I \subseteq Z$ be a finite subset and let $w \in \mathcal{B}_I$. Let $v \in \mathcal{B}^*$. Since $v|_{-I} \in \mathcal{B}^\perp|_{-I}$, it follows that $w \in (\mathcal{B}^*)^\perp$. Hence $\mathcal{B}_I \subseteq (\mathcal{B}^*)^*$, which yields $\bar{\mathcal{B}}_f \subseteq (\mathcal{B}^*)^*$.

ii) In the other direction, it follows from a) of Theorem 4.1 that

$$(\mathcal{B}^*)^\perp \subseteq ((\mathcal{B}^\perp)^\perp)_f = \mathcal{B}_f \subseteq \bar{\mathcal{B}}_f$$

and hence $(\mathcal{B}^*)^* \subseteq \bar{\mathcal{B}}_f$.

∎

Remark 4.2 *It follows from Proposition 4.1 that $\mathcal{B} \mapsto \mathcal{B}^*$ induces a one-to-one corrrespondence between controllable linear behaviors on W and controllable linear behaviors on W'.*

Theorem 4.2 *Let $\mathcal{B} \subseteq W^{\mathbf{Z}}$ be a controllable linear behavior. Then:*

(0) $(\mathcal{B}^*)_f = \mathcal{B}^{\perp}$.

(1) $\mathcal{B}$ *is* N*-memory* $\Leftrightarrow$ $\mathcal{B}^*$ *is* N*-controllable.*

(2) $\dim_{\mathbf{C}} \mathcal{B}|_I + \dim_{\mathbf{C}} (\mathcal{B}^*)_I = |I| \dim_{\mathbf{C}} W$, $\forall\, I \subseteq \mathbf{Z}$ *finite.*

(3) $m(\mathcal{B}) + m(\mathcal{B}^*) = \dim_{\mathbf{C}} W$

(4) $n(\mathcal{B}) = n(\mathcal{B}^*)$

Proof.

$(\ ,\ )_1$ will denote the pairing between $W^{\mathbf{Z}}$ and $W'[z, z^{-1}]$

$(\ ,\ )_2$ the pairing between $W[z, z^{-1}]$ and $(W')^{\mathbf{Z}}$

It is evident that with the usual identifications (4.6),

$$(\ ,\ )_{1|_{W[z,z^{-1}]\times W'[z,z^{-1}]}} = (\ ,\ )_{2|_{W[z,z^{-1}]\times W'[z,z^{-1}]}} \tag{4.8}$$

$\perp_1$ and $\perp_2$ will denote orthogonal complement with respect to pairing $(\ ,\ )_1$ and $(\ ,\ )_2$.

(0) Clearly

$$\mathcal{B}^{\perp_1} \subseteq (\mathcal{B}^*)_f \subseteq \mathcal{B}^*. \tag{4.9}$$

Hence

$$\mathcal{B}^{\perp_1\perp_1} \supseteq ((\mathcal{B}^*)_f)^{\perp_1} \supseteq ((\mathcal{B}^*)_f \supseteq \mathcal{B}^{*\perp_2} \tag{4.10}$$

It follows from (4.10), using Prop. 4.1 and (a) of Theorem 4.1 that

$$\mathcal{B} = ((\mathcal{B}^*)_f)^{\perp_1} \Rightarrow \mathcal{B}^{\perp_1} = (\mathcal{B}^*)_f.$$

Consider

$$\mathcal{C} = \sum_{\substack{I \text{ interval} \\ |I| \leq N+1}} \mathcal{B}_I^*$$

It can be immediately seen that $\mathcal{C}$ is an N controllable linear behavior and $\mathcal{C} \subseteq \mathcal{B}^*$. Hence $\mathcal{C}^* \supseteq \mathcal{B}$. Moreover, it is easy to see that

$$\mathcal{C}_I = (\mathcal{B}^*)_I,\ \forall\, I \subseteq \mathbf{Z},\ I \text{ an interval and } |I| \leq N+1$$

which, using (0) yields

$$\mathcal{C}^*|_I = \mathcal{B}|_I,\ \forall\, I \subseteq \mathbf{Z},\ I \text{ an interval and } |I| \leq N+1.$$

Since $\mathcal{B}$ is N-memory, this implies $\mathcal{C} \subseteq \mathcal{B}$. Hence $\mathcal{C} = \mathcal{B}$ and $\mathcal{C} = \mathcal{B}$. $\mathcal{B}$ is thus N-controllable.
In the other direction, consider

$$\mathcal{C} = \{w \in W^{\mathbf{Z}} |\ w|_I \in \mathcal{B}|_I,\ \forall\ I \subseteq \mathbf{Z},\ I \text{ an interval}, |I| \leq N+1$$

Clearly $\mathcal{C}$ is an N-memory linear behavior and $\mathcal{C} \supseteq \mathcal{B}$. Hence $\mathcal{C}^* \subseteq \mathcal{B}^*$. Obviously $\mathcal{C}|_I = \mathcal{B}|_I\ \forall\ I \subseteq \mathbf{Z}$ such that $|I| \leq N+1$. Using again (0) we thus obtain:

$$(\mathcal{C}^*)_I \supseteq (\mathcal{C}^{\perp})_I = (\mathcal{B}^{\perp})_I = (\mathcal{B}^*)_I,\ \forall\ I \subseteq \mathbf{Z} \text{ such that } |I| \leq N+1.$$

Since $\mathcal{B}$ is N-controllable it follows that $\mathcal{B}^* \subseteq \mathcal{C}^*$. Hence $\mathcal{B}^* = \mathcal{C}^*$ and $\mathcal{B} = \mathcal{C}$. This implies that $\mathcal{B}$ is N-memory.

(2): It is clear that

$$\dim_{\mathbf{C}} \mathcal{B}|_I + \dim_{\mathbf{C}} (\mathcal{B}^{\perp})_I = |I| \dim_{\mathbf{C}} W \tag{4.11}$$

$\forall\ I \subseteq \mathbf{Z}$ finite.

(2): now follows from (0) and (4).

(3) and (4) are immediate consequences of (2) together with part (e) of Theorem 4.1. ∎

V From Dual Linear Behaviours to Dual Linear Input-Output Maps

Let $W = U \times Y$ where U, the input space and Y the output space are finite-dimensional K-vector spaces. Let $\mathcal{B} \subset W^Z$ be a linear behaviour which is controllable and satisfies

i) $P_{U^Z}\mathcal{B} = U^Z$ where P_{U^Z} denotes the projection onto U^Z.

ii) $\mathcal{B} \cap (\{0\} \times Y)$ is a K-finite dimensional vector space.

iii) $\mathcal{B}$ is non-anticipative, that is, if

$$(u, y) \in \mathcal{B} \text{ and } v \in U^Z \text{ and } v|_{(-\infty,0]} = u|_{(-\infty,0]}$$

then $\exists \tilde{y} \in Y^Z$, such that $\tilde{y}|_{(-\infty,0]} = y|_{(-\infty,0]}$ and $(v, \tilde{y}) \in \mathcal{B}$.

Let $\tilde{\mathcal{B}}$ be a linear shift invariant subspace of $\mathcal{B}$ which has finite support on the left of the origin. Since $\mathcal{B}$ is controllable, $\bar{\tilde{\mathcal{B}}} = \mathcal{B}$.

We will identify trajectories with compact support to the left of the origin with the K-vector space

$$W((z^{-1})) = \{\sum_{i=N}^{\infty} w_i z^{-i} | w_i \in K\}$$

$W((z^{-1}))$ has the structure of a $K[z, z^{-1}]$-module. Using (i), (ii) and (iii) we can conclude that $\tilde{\mathcal{B}}$ is the graph of a $K[z, z^{-1}]$-homomorphism $\Psi : U((z^{-1})) \to Y((z^{-1}))$ which is causal, that is

$$\Psi[z^{-i}U((z^{-1}))] \subset \bar{z}^{(i+1)}Y((z^{-1}))\ \forall\ i \in Z.$$

Let $\mathcal{H}_{\mathcal{B}}$ be the Hankel Operator corresponding to Ψ defined as

$$H_{\mathcal{B}} : U[z] \to Y[[z^{-1}]], \text{ where}$$

$$H_{\mathcal{B}} = P_{z^{-1}Y[[z^{-1}]]} \cdot \Psi|_{U[z]}.$$

Note that $U[z]$ and $Y[[z^{-1}]]$ have the structure of a $K[z]$-module and $H_{\mathcal{B}}$ is a $K[z]$-module homomorphism.

Identify W' with $U' \times Y'$ and let $\mathcal{B}^*$ be the linear behavior in $(W')^Z$ defined as in section 4. We may check that $\mathcal{B}^*$ is the graph of an input-output map η which is causal. Let $H_{\mathcal{B}^*}$ be the corresponding Hankel operator. We then have that the following commutative diagram:

$$\begin{array}{ccc} (Y[[z^{-1}]])' & \xrightarrow{H_{\mathcal{B}'}} & (U[z])' \\ \gamma \uparrow & & \downarrow \delta \\ Y'[z] & \xrightarrow[H_{\mathcal{B}^*}]{} & U'[[z^{-1}]]. \end{array}$$

where γ and δ are $K[z]$-module isomorphisms and $H_{\mathcal{B}^*}$ is a $K[z]$-module homomorphism. The details of this construction as well as the corresponding theory of internal representation and thereby obtaining an analogue of Theorem 3.3 will be published elsewhere.

References

[1] G.D. Forney and M. Trott, Private Communication. See also G.D. Forney and M. Trott, Duals of Abelian Group Codes and Systems, Manuscript. January 1994.

[2] R. de B. Johnston, Linear Systems over Various Rings, Ph.D. dissertation, M.I.T., May 1973. Also Technical Report ESL-P-497, Electronic Systems Laboratory, M.I.T., Cambridge, MA 02139.

[3] R.E. Kalman, P.L. Falb & M.A. Arbib, Topics in Mathematical System Theory, McGraw Hill, New York, 1969.

[4] F. Treves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, 1967.

[5] J.C. Willems, "Models for Dynamics" in Dynamics Reported, Vol. 2, U. Kirchgraber and H.O. Walther, ed., Wiley, New York, 1989.

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